COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **10**, Number **4**, July **2011**

doi:10.3934/cpaa.2011.10.1011

pp. **1011–1036**

ON THE STRUCTURE OF SOLUTIONS OF NONLINEAR HYPERBOLIC SYSTEMS OF CONSERVATION LAWS

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Abstract. We are concerned with entropy solutions $\mathbf{u}$ in L^∞ of nonlinear hyperbolic systems of conservation laws. It is shown that, given any entropy function η and any hyperplane $t = const.$, if $\mathbf{u}$ satisfies a vanishing mean oscillation property on the half balls, then $\eta(\mathbf{u})$ has a trace $\mathcal{H}^d$-almost everywhere on the hyperplane. For the general case, given any set E of finite perimeter and its inner unit normal $\boldsymbol{\nu} : \partial^* E \to \mathbb{S}^d$ and assuming the vanishing mean oscillation property of $\mathbf{u}$ on the half balls, we show that the weak trace of the vector field $(\eta(\mathbf{u}), \mathbf{q}(\mathbf{u}))$, defined in Chen-Torres-Ziemer [9], satisfies a stronger property for any entropy pair $(\eta, \mathbf{q})$. We then introduce an approach to analyze the structure of bounded entropy solutions for the isentropic Euler equations.

1. Introduction.

The purpose of this paper is to employ the theory of divergence-measure fields developed in Chen-Torres-Ziemer [9] to obtain traces on hyperplanes of entropy solutions of the following hyperbolic system:

$$\mathbf{u}_t + div_{\mathbf{x}}\mathbf{f}(\mathbf{u}) = 0, \qquad \mathbf{x} \in \mathbb{R}^d,\ \mathbf{u} = (u^1, u^2, \dots, u^m) \in L^\infty(\mathbb{R}^{d+1}_+; \mathbb{R}^m)\ , \tag{1.1}$$

where $\mathbf{f} = (\mathbf{f}^1, \mathbf{f}^2, \dots, \mathbf{f}^m)$ and $\mathbf{f}^i : \mathbb{R}^m \to \mathbb{R}^d$. More precisely, given any entropy function η, we prove that $\eta(\mathbf{u})$ has traces $\mathcal{H}^d$-almost everywhere on any hyperplane $\{(t, \mathbf{x}) : \mathbf{x} \in \mathbb{R}^d\}$ (see Theorem 3.7).

Our results are based on a vanishing mean oscillation property of $\mathbf{u}$ on the half balls (cf. (3.1)), which was shown to be true for $m = 1$ in De-Lellis-Otto-Westdickenberg [11] (cf. (3.2)). Thus, the main goal of this paper is to show that this property can be further improved; see Theorem 3.2. The desired results are achieved by exploiting the connection between the Lax entropy inequality and divergence-measure fields (see §2).

2000 *Mathematics Subject Classification.* Primary: 35L65, 35L50, 35L80, 35L67, 35L05, 76N10; Secondary: 26B12, 28C05, 76J20, 76L05.

Key words and phrases. Entropy solutions, hyperbolic systems, conservation laws, bounded variation, compensated compactness, divergence-measure fields, entropy methods, scalings, blowup arguments, compactness, structure of entropy solutions, behavior of solutions.

The strong trace at $\{t = 0\}$ was first established for the case $m = 1$ and $d = 1$ by using compensated compactness arguments in Chen-Rascle [7]. Vasseur [25] obtained the strong trace of entropy solutions of multidimensional scalar conservation laws on any Lipschitz deformable boundary. The regularity of entropy solutions of multidimensional scalar conservation laws was also studied in De-Lellis-Otto-Westdickenberg [11], where it was shown that u has the structure of a BV-like function in the sense that the *shock waves* are supported on a codimension-one rectifiable set where u has strong traces. In both [11] and [25], the analysis is done within the framework of the kinetic formulation of conservation laws and under the assumption of the following genuine nonlinearity condition on the flux function $\mathbf{f}$:

$$\mathcal{L}^1(\{v \in \mathbb{R} : \tau + \mathbf{f}'(v) \cdot \xi = 0\}) = 0 \qquad \text{for all } (\tau, \xi) \in \mathbb{R}^{d+1} \text{ with } \tau^2 + |\xi|^2 = 1, \tag{1.2}$$

where $\mathcal{L}^1$ is the one-dimensional Lebesgue measure.

For $m = 1$, $d = 1$, and a general flux function f, strong traces for a class of functionals of entropy solutions on any Lipschitz deformable boundary were obtained in Kwon-Vasseur [26]. For $d > 1$ and $m = 1$ with the requirement that the flux vector $\mathbf{f}$ be only continuous, Panov [21] used techniques of H-measures to establish strong traces of entropy solutions on the hyperplane $\{t = 0\}$.

In Section 4, we analyze the general case on a set E of finite perimeter with its inner unit normal $\boldsymbol{\nu} : \partial^* E \to \mathbb{S}^d$. Then, under the assumption of the vanishing mean oscillation property on the half balls, we show that the weak trace of the vector field $(\eta(\mathbf{u}), \mathbf{q}(\mathbf{u}))$ defined in Chen-Torres-Ziemer [9] satisfies a stronger property for any entropy pair $(\eta, \mathbf{q})$ (see Theorem 4.4).

In Section 5, we introduce an approach to analyze the structure of bounded entropy solutions for the isentropic Euler equations in gas dynamics:

$$\begin{cases} \partial_t \rho + \partial_x m = 0, \\ \partial_t m + \partial_x (\frac{m^2}{\rho} + p) = 0, \end{cases} \tag{1.3}$$

where $\rho \geq 0$ denotes the density, m the momentum, and $p(\rho) \geq 0$ the pressure. The physical region for (1.3) is $\{(\rho, m) : |m| \leq C\,\rho\}$ for some $C > 0$, in which the term $\frac{m^2}{\rho}$ in the flux function is only Lipschitz continuous near the vacuum. For $\rho > 0$, $v = \frac{m}{\rho}$ represents the velocity of the fluid. For (1.3), strict hyperbolicity and genuine nonlinearity away from the vacuum require that

$$p'(\rho) > 0, \qquad 2\,p'(\rho) + \rho\, p''(\rho) > 0 \qquad \text{for } \rho > 0. \tag{1.4}$$

Near the vacuum,

$$\frac{p(\rho)}{\rho^\gamma} \to \kappa_1 > 0 \qquad \text{when } \rho \to 0\,, \text{ for some } \gamma > 1. \tag{1.5}$$

More precisely, the eigenvalues of system (1.3) are

$$\lambda_j = \frac{m}{\rho} + (-1)^j \sqrt{p'(\rho)}, \quad j = 1, 2, \tag{1.6}$$

and the corresponding right-eigenvectors are

$$\mathbf{r}_j = (-1)^j \frac{2\rho\sqrt{p'(\rho)}}{\rho p''(\rho) + 2p'(\rho)} (1, \lambda_j)^\top, \tag{1.7}$$

so that

$$\nabla \lambda_j \cdot \mathbf{r}_j = 1 \quad j = 1, 2. \tag{1.8}$$

From (1.5)–(1.6), we have

$$\lambda_2 - \lambda_1 = 2\sqrt{p'(\rho)} \to 0 \qquad \text{when } \rho \to 0.$$

Therefore, system (1.3) is strictly hyperbolic in the nonvacuum states $\{(\rho, v) : \rho > 0, |v| \le C\}$ and, at the vacuum, the two characteristic speeds of (1.3) may coincide and the system be nonstrictly hyperbolic. This system is one of the archetypes of (1.1).

2. **Gauss-Green formula for bounded divergence-measure fields.** In this section we first introduce some definitions and then present the Gauss-Green formula that will be used to obtain our main result, Theorem 3.7.

Definition 2.1. A Radon measure on Ω is a signed regular Borel measure whose total variation on each compact set $K \Subset \Omega$ is finite, i.e., $\|\mu\|(K) < \infty$. The space of Radon measures supported on an open set Ω is denoted by $\mathcal{M}(\Omega)$.

Notation. We will use the notation

$$N := d + 1,$$

and

$$\mathbf{z} := (t, \mathbf{x})$$

where $\mathbf{x} \in \mathbb{R}^d$.

Definition 2.2. Let Ω be an open set. A vector field $\boldsymbol{F} \in L^p(\Omega; \mathbb{R}^N)$, $1 \le p \le \infty$, is called a *divergence-measure field*, written as $\boldsymbol{F} \in \mathcal{DM}^p(\Omega)$, if $div\, \boldsymbol{F}$, in the sense of distributions, is a (signed) Radon measure with finite total variation on Ω. Furthermore, $\boldsymbol{F}$ is called a $\mathcal{DM}^p_{loc}(\mathbb{R}^N)$ field if $\boldsymbol{F} \in \mathcal{DM}^p(D)$, for any bounded open set $D \subset \mathbb{R}^N$.

Definition 2.3. For every $\alpha \in [0, 1]$ and every $\mathcal{L}^N$-measurable set $E \subset \mathbb{R}^N$, define

$$E^\alpha := \{\mathbf{z} \in \mathbb{R}^N : D(E, \mathbf{z}) = \alpha\}, \tag{2.1}$$

where

$$D(E, \mathbf{z}) := \lim_{r \to 0} \frac{|E \cap B_r(\mathbf{z})|}{|B_r(\mathbf{z})|}. \tag{2.2}$$

Then E^α is the set where E *has density* α. We define the *measure-theoretic boundary* of E, $\partial^m E$, as

$$\partial^m E := \mathbb{R}^N \setminus (E^0 \cup E^1).$$

Definition 2.4. Let $E \Subset \Omega$ be an $\mathcal{L}^N$-measurable subset. We say that E is a *set of finite perimeter* if $\chi_{_E}$ is a function of bounded variation, $\chi_{_E} \in BV(\Omega)$. Consequently, if E is a set of finite perimeter, then $\nabla\chi_{_E}$ is a (vector-valued) Radon measure whose total variation is denoted by $\|\nabla\chi_{_E}\|$.

Definition 2.5. Let $E \Subset \Omega$ be a set of finite perimeter. The *reduced boundary* of E, denoted as $\partial^* E$, is the set of all points $\mathbf{z} \in \Omega$ such that

(i) $\|\nabla\chi_{_E}\| (B_r(\mathbf{z})) > 0$ for all $r > 0$;
(ii) The limit $\boldsymbol{\nu}_E(\mathbf{z}) := \lim_{r \to 0} \frac{\nabla\chi_{_E}(B_r(\mathbf{z}))}{\|\nabla\chi_{_E}\|(B_r(\mathbf{z}))}$ exists and $|\boldsymbol{\nu}_E(\mathbf{z})| = 1$.

The unit vector, $\boldsymbol{\nu}_{_E}(\mathbf{z})$, is called the *measure-theoretic interior unit normal to E at* $\mathbf{z}$. The following result is due to Federer (see also [27], Lemma 5.9.5; and [2], Theorem 3.61):

Theorem 2.6. *Let E be a set of finite perimeter and $\mathbf{z} \in \partial^* E$. Let*

$$\Pi^{\pm} := \{(\tau, \mathbf{y}) \in \mathbb{R}^N \,:\, \pm\boldsymbol{\nu}(\mathbf{z}) \cdot \big((\tau, \mathbf{y}) - \mathbf{z}\big) > 0\}.$$

For $r > 0$, define

$$E_r := \{(\tau, \mathbf{y}) \in \mathbb{R}^N \,:\, r\big((\tau, \mathbf{y}) - \mathbf{z}\big) \in E\}.$$

Then

(i) *As $r \to 0$, the set E_r converges to Π^+; moreover, for every set A such that $\mathcal{H}^{N-1}(\partial A \cap \partial \Pi^+) = 0$,*

$$\lim_{r \to 0} \|\nabla \chi_{E_r}\|(A) = \|\nabla \chi_{\Pi^+}\|(A) = \mathcal{H}^{N-1}(A \cap \partial \Pi^+);$$

(ii) $\lim_{r\to 0} \frac{|E \cap B_r(\mathbf{z}) \cap \Pi^-|}{r^N} = 0$*;*

(iii) $\lim_{r\to 0} \frac{|(\mathbb{R}^N \setminus E) \cap B_r(\mathbf{z}) \cap \Pi^+|}{r^N} = 0$*;*

(iv) *The reduced boundary of E, $\partial^* E$, is an $(N-1)$-rectifiable set which means that there exists a countable family of C^1-manifolds M_k of dimension $N-1$ and a set $\mathcal{N}$ of $\mathcal{H}^{N-1}$ measure zero such that*

$$\partial^* E \subset \big(\bigcup_{k=1}^{\infty} M_k\big) \bigcup \mathcal{N};$$

(v) *The generalized gradient of χ_E enjoys the following basic relationship with $\mathcal{H}^{N-1}$:*

$$\|\nabla \chi_E\| = \mathcal{H}^{N-1} \llcorner \partial^* E;$$

(vi) $\lim_{r\to 0} \frac{\|\nabla \chi_E\|(B_r(\mathbf{z}))}{\alpha(N-1) r^{N-1}} = 1$*, where $\alpha(N-1)$ is the Lebesgue measure of the unit ball in $\mathbb{R}^{N-1}$.*

Remark 1. If $E \Subset \Omega$ is a set of finite perimeter, then $\mathcal{H}^{N-1}(\partial^m E) < \infty$. Conversely, it was proved by Federer (see [17], 4.5.11) that, if $\mathcal{H}^{N-1}(\partial^m E) < \infty$, then E is a set of finite perimeter.

We will use the following Gauss-Green formula proved in Chen-Torres [8] and Chen-Torres-Ziemer [9] (see also Silhavy [24]):

Theorem 2.7. *Let $\boldsymbol{F} \in \mathcal{DM}^{\infty}_{loc}(\Omega, \mathbb{R}^N)$ and let $E \Subset \Omega$ be a bounded set of finite perimeter. Then there exist functions $\mathcal{F}_i \cdot \boldsymbol{\nu} \in L^\infty(\partial^* E)$ and $\mathcal{F}_e \cdot \boldsymbol{\nu} \in L^\infty(\partial^* E)$ such that*

$$\int_{E^1} \operatorname{div}(\varphi \boldsymbol{F}) = -\int_{\partial^* E} \varphi (\mathcal{F}_i \cdot \boldsymbol{\nu})(\mathbf{z}) d\mathcal{H}^{N-1}$$

and

$$\int_{E} \operatorname{div}(\varphi \boldsymbol{F}) = -\int_{\partial^* E} \varphi (\mathcal{F}_e \cdot \boldsymbol{\nu})(\mathbf{z}) d\mathcal{H}^{N-1},$$

for every $\varphi \in C_0^\infty(\Omega)$. Moreover, $\|\mathcal{F}_i \cdot \boldsymbol{\nu}\|_\infty \le \|\boldsymbol{F}\|_\infty$ and $\|\mathcal{F}_e \cdot \boldsymbol{\nu}\|_\infty \le \|\boldsymbol{F}\|_\infty$.

We have the following product rule for bounded divergence-measure fields ([9]):

Theorem 2.8. *Let $\boldsymbol{F} \in \mathcal{DM}^\infty(\Omega)$ and $g \in BV(\Omega)$ bounded with compact support. Then*

$$\operatorname{div}(g\boldsymbol{F}) = g^* \operatorname{div} \boldsymbol{F} + \overline{\boldsymbol{F} \cdot \nabla g}, \tag{2.3}$$

where g^* is the precise representative of g, $\overline{\boldsymbol{F} \cdot \nabla g}$ is the weak* limit of the measures $\boldsymbol{F} \cdot \nabla g_k$, and g_k is a sequence of mollifications of g.

Remark 2. If $\boldsymbol{F} \in \mathcal{DM}^\infty(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$, then $div\,(\varphi \boldsymbol{F}) = \varphi div\, \boldsymbol{F} + \boldsymbol{F} \cdot \nabla \varphi$. If $\boldsymbol{F} \in \mathcal{DM}^\infty_{loc}(\Omega)$, then $div\, \boldsymbol{F} \ll \mathcal{H}^{N-1}$ (see [9], Lemma 2.25).

Consider the hyperbolic system of conservation laws (1.1).

Definition 2.9. Let $\mathcal{P}$ denote the set of all pairs $(\eta, \mathbf{q})$ such that $\eta : \mathbb{R}^m \to \mathbb{R}$ is convex, $\mathbf{q} \in C^{2,1}(\mathbb{R}^m, \mathbb{R}^d)$ and

$$\nabla \mathbf{q}_k(\mathbf{u}) = \nabla \eta(\mathbf{u}) \nabla \mathbf{f}_k(\mathbf{u}), \quad k = 1, 2, ..., d. \tag{2.4}$$

The pair $(\eta, \mathbf{q})$ is called a convex entropy pair of system (1.1).

A bounded entropy solution $\mathbf{u} \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R}^m)$ of (1.1) is characterized by the entropy inequality

$$\eta(\mathbf{u})_t + div_{\mathbf{x}} \mathbf{q}(\mathbf{u}) \le 0 \qquad \text{in } \mathcal{D}'_{t,\mathbf{x}} \tag{2.5}$$

for any convex entropy pair. If we define

$$\boldsymbol{F}^\eta_{\mathbf{u}}(t, \mathbf{x}) := (\eta(\mathbf{u}(t, \mathbf{x})), \mathbf{q}(\mathbf{u}(t, \mathbf{x}))),$$

then the entropy inequality (2.5) and the Riesz representation theorem imply (see [16], Corollary 1, page 53) that there exists a measure $\mu_\eta \in \mathcal{M}(\mathbb{R}_+ \times \mathbb{R}^d)$ such that

$$div_{(t,\mathbf{x})} \boldsymbol{F}^\eta_{\mathbf{u}} = \mu_\eta.$$

Remark 3. Through this paper, we consider an entropy solution $\mathbf{u}$ of (1.1) defined in the whole space $\mathbb{R}^{d+1}$. This can be done in view of the extension theorems for divergence-measure fields proved in Chen-Torres-Ziemer [9] (Section 8). Indeed, setting $\mathbf{u} = 0$ on $\mathbb{R}_- \times \mathbb{R}^d$, we obtain that $\mathbf{u}$ is defined in the whole space $\mathbb{R}^{d+1}$, and

$$\boldsymbol{F}^\eta_{\mathbf{u}}(t, \mathbf{x}) := (\eta(\mathbf{u}(t, \mathbf{x})), \mathbf{q}(\mathbf{u}(t, \mathbf{x}))), \quad (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d,$$

satisfies

$$\boldsymbol{F}^\eta_{\mathbf{u}} \in \mathcal{DM}^\infty_{loc}(\mathbb{R}^{d+1}).$$

3. **Traces for hyperbolic systems of conservation laws on hyperplanes.**

In this section we define traces for $\eta(\mathbf{u})$ on any hyperplane parallel to $\{t = 0\}$, if $\mathbf{u}$ satisfies the vanishing mean oscillation property (3.1) below. We begin with some definitions.

Definition 3.1. We denote the open ball of radius r and center $(\tau, \mathbf{y})$ as $B_r(\tau, \mathbf{y})$. For every $(\tau, \mathbf{y})$, we denote by $B^+_r(\tau, \mathbf{y})$ the intersection of $B_r(\tau, \mathbf{y})$ with the set $\Pi^\tau := \{(t, \mathbf{x}) : t > \tau\}$. We also define the cylinder

$$C^+_r(\tau, \mathbf{y}) := \mathcal{B}_r(\tau, \mathbf{y}) \times (0, r),$$

where $\mathcal{B}_r(\tau, \mathbf{y})$ denotes the intersection of $B_r(\tau, \mathbf{y})$ with the set $\partial \Pi^\tau$.

Remark 4. Since $B^+_r(\tau, \mathbf{y})$ can be inscribed in $C^+_r(\tau, \mathbf{y})$, then the results in this section can be stated for cylinders or balls.

Definition 3.2. We denote by $\overline{\mathbf{u}}_r(\tau, \mathbf{y})$ the vector in $\mathbb{R}^m$ which is the average of $\mathbf{u}$ over the half ball $B^+_r(\tau, \mathbf{y})$.

Definition 3.3. We say that $\mathbf{u}$ satisfies the vanishing mean oscillation property on $\partial \Pi^\tau$ for half balls if, for any continuous $\mathbf{q} \in C(\mathbb{R}^m, \mathbb{R}^d)$,

$$\lim_{r \to 0} \frac{1}{r^{d+1}} \int_{B^+_r(\tau, \mathbf{y})} |\mathbf{q}(\mathbf{u}(t, \mathbf{x})) - \overline{\mathbf{q}(\mathbf{u})}_r(\tau, \mathbf{y})| \, dt d\mathbf{x} = 0 \tag{3.1}$$

for $\mathcal{H}^d$-almost every $(\tau, \mathbf{y}) \in \partial \Pi^\tau$, where $\overline{\mathbf{q}(\mathbf{u})}_r(\tau, \mathbf{y})$ is the vector in $\mathbb{R}^d$ which is the average of $\mathbf{q}(\mathbf{u})$ over the half ball $B^+_r(\tau, \mathbf{y})$.

Remark 5. For the scalar case (i.e. $m = 1$) and for any hyperplane Π^τ, property (3.1) follows from De Lellis-Otto-Westdickenberg [11], as we proceed to show next.

Theorem 3.4 (De Lellis-Otto-Westdickenberg [11])**.** *Let $\tau \in \mathbb{R}$. If $\mathbf{f} \in C^{2,1}$ satisfies (1.2) and if $u \in L^\infty(\mathbb{R}^{d+1}, \mathbb{R})$ satisfies the entropy inequality (2.5), then, for $\mathcal{H}^d$-almost every $(\tau, \mathbf{y}) \in \partial\Pi^\tau$,*

$$\lim_{r\to 0} \frac{1}{r^{d+1}} \int_{B_r^+(\tau,\mathbf{y})} |u(t,\mathbf{x}) - \overline{u}_r(\tau,\mathbf{y})|\, dtd\mathbf{x} = 0. \tag{3.2}$$

Then we have

Lemma 3.5. *Let $(\eta, \mathbf{q}) \in \mathcal{P}$ be any convex entropy pair, and let $\tau \in \mathbb{R}$. Then, for $\mathcal{H}^d$-almost every $(\tau, \mathbf{y}) \in \partial\Pi^\tau$,*

$$\lim_{r\to 0} \frac{1}{r^{d+1}} \int_{B_r^+(\tau,\mathbf{y})} |\mathbf{q}(u(t,\mathbf{x})) - \overline{\mathbf{q}(u)}_r(\tau,\mathbf{y})| dtd\mathbf{x} = 0, \tag{3.3}$$

where $\overline{\mathbf{q}(u)}_r(\tau,\mathbf{y})$ is the average of $\mathbf{q}(u)$ over the half ball $B_r^+(\tau,\mathbf{y})$.

Proof. Fix $(\tau,\mathbf{y}) \in \partial\Pi^\tau$ for which (3.2) holds. Given any $r_k \to 0$, there exists a subsequence (denoted again as r_k) and a constant $u_\infty^{\tau,\mathbf{y}}$ (that depends on r_k) such that (cf. [11])

$$u_{r_k}^{\tau,\mathbf{y}} \to u_\infty^{\tau,\mathbf{y}} \qquad \text{in } L^1_{loc}(\Pi^\tau), \tag{3.4}$$

where

$$u_{r_k}^{\tau,\mathbf{y}}(t,\mathbf{x}) := u(\tau + r_k t, \mathbf{y} + r_k\mathbf{x}), \quad (t,\mathbf{x}) \in \Pi^\tau.$$

Therefore, for a further subsequence,

$$u_{r_k}^{\tau,\mathbf{y}}(t,\mathbf{x}) \to u_\infty^{\tau,\mathbf{y}} \qquad \text{for } \mathcal{L}^{d+1}\text{-a.e. } (t,\mathbf{x}) \in B_1^+(\mathbf{0}),$$

which yields that, for any $(\eta,\mathbf{q}) \in \mathcal{P}$,

$$\mathbf{q}(u_{r_k}^{\tau,\mathbf{y}}(t,\mathbf{x})) \to \mathbf{q}(u_\infty^{\tau,\mathbf{y}}) \quad \text{for } \mathcal{L}^{d+1}\text{-a.e. } (t,\mathbf{x}) \in B_1^+(\mathbf{0}) \tag{3.5}$$

since $\mathbf{q}$ is continuous.

From (3.5) and the dominated convergence theorem (recall that u is bounded), and performing the change of variables $\alpha = \tau + r_k t$, $\boldsymbol{\xi} = \mathbf{y} + r_k\mathbf{x}$, we obtain

$$\frac{1}{r_k^{d+1}} \int_{B_{r_k}^+(\tau,\mathbf{y})} |\mathbf{q}(u(\alpha,\boldsymbol{\xi})) - \mathbf{q}(u_\infty^{\tau,\mathbf{y}})| d\alpha d\boldsymbol{\xi} \to 0 \quad \text{as } r_k \to 0. \tag{3.6}$$

For simplicity, we write $u_\infty^{\tau,\mathbf{y}} := u_\infty$ and $\overline{\mathbf{q}(u)}_r(\tau,\mathbf{y}) := \overline{\mathbf{q}(u)}_r$ in the rest of the proof. We find that, for any $r > 0$,

$$\begin{aligned} &\frac{1}{r^{d+1}} \int_{B_r^+(\tau,\mathbf{y})} |\mathbf{q}(u(t,\mathbf{x})) - \overline{\mathbf{q}(u)}_r| dtd\mathbf{x} \\ \le &\frac{1}{r^{d+1}} \int_{B_r^+(\tau,\mathbf{y})} |\mathbf{q}(u(t,\mathbf{x})) - \mathbf{q}(u_\infty)| dtd\mathbf{x} + \frac{1}{r^{d+1}} \int_{B_r^+(\tau,\mathbf{y})} |\overline{\mathbf{q}(u)}_r - \mathbf{q}(u_\infty)| dtd\mathbf{x}. \end{aligned} \tag{3.7}$$

On the other hand, if $\omega(d+1)$ denotes the $\mathcal{L}^{d+1}$-measure of the unit ball in $\mathbb{R}^{d+1}$, we have

$$\begin{aligned}
&\frac{1}{r^{d+1}}\int_{B_r^+(\tau,\mathbf{y})} |\overline{\mathbf{q}(u)}_r - \mathbf{q}(u_\infty)|dtd\mathbf{x} \\
&= \frac{\omega(d+1)}{2}|\overline{\mathbf{q}(u)}_r - \mathbf{q}(u_\infty)| \\
&= \frac{1}{r^{d+1}}\left|\int_{B_r^+(\tau,\mathbf{y})} \mathbf{q}(u(t,\mathbf{x})) - \mathbf{q}(u_\infty)dtd\mathbf{x}\right| \\
&\le \frac{1}{r^{d+1}}\int_{B_r^+(\tau,\mathbf{y})} |\mathbf{q}(u(t,\mathbf{x})) - \mathbf{q}(u_\infty)|dtd\mathbf{x}. \qquad (3.8)
\end{aligned}$$

From (3.7) and (3.8), we conclude

$$\frac{1}{r^{d+1}}\int_{B_r^+(\tau,\mathbf{y})} |\mathbf{q}(u(t,\mathbf{x})) - \overline{\mathbf{q}(u)}_r|dtd\mathbf{x} \le \frac{2}{r^{d+1}}\int_{B_r^+(\tau,\mathbf{y})} |\mathbf{q}(u(t,\mathbf{x})) - \mathbf{q}(u_\infty)|dtd\mathbf{x},$$

and therefore, (3.6) yields

$$\frac{1}{r_k^{d+1}}\int_{B_{r_k}^+(\tau,\mathbf{y})} |\mathbf{q}(u(t,\mathbf{x})) - \overline{\mathbf{q}(u)}_{r_k}|dtd\mathbf{x} \to 0 \qquad \text{as } r_k \to 0,$$

which implies (3.3). The dependence of (3.3) on the subsequence is illusory. The reason is that, if there were a subsequence $r_k \to 0$ such that

$$\frac{1}{r_k^{d+1}}\int_{B_{r_k}^+(\tau,\mathbf{y})} |\mathbf{q}(u(t,\mathbf{x})) - \overline{\mathbf{q}(u)}_{r_k}|dtd\mathbf{x} \to l \neq 0,$$

then one could appeal to the previous argument to conclude that, for some further subsequence,

$$\frac{1}{r_k^{d+1}}\int_{B_{r_k}^+(\tau,\mathbf{y})} |\mathbf{q}(u(t,\mathbf{x})) - \overline{\mathbf{q}(u)}_{r_k}|dtd\mathbf{x} \to 0,$$

which is contrary to our assertion that $l \neq 0$. □

We will need the following result (see Giusti [18], Lemma 2.3):

Lemma 3.6. *Let μ be a positive Radon measure in $\mathbb{R}_+ \times \mathbb{R}^d$. Then, for $\mathcal{H}^d$-almost every $\mathbf{y} \in \mathbb{R}^d$,*

$$\lim_{r\to 0} \frac{\mu(C_r^+(0,\mathbf{y}))}{r^d} = 0.$$

The following theorem shows that $\eta(\mathbf{u})$ has traces $\mathcal{H}^d$-almost everywhere on hyperplanes under assumption (3.1).

Theorem 3.7. *Let $(\eta, \mathbf{q})$ be any convex entropy pair and let $\tau \in \mathbb{R}$. If $\mathbf{u} \in L^\infty(\mathbb{R}^{d+1}; \mathbb{R}^m)$ satisfies the entropy inequality (2.5) and condition (3.1) on $\partial\Pi^\tau$, then $\eta(\mathbf{u})$ has a trace at $\partial\Pi^\tau$; that is, there exists a function $\eta(\mathbf{u})_{tr} \in L^\infty(\partial\Pi^\tau)$ such that, for $\mathcal{H}^d$-almost every $(\tau,\mathbf{y}) \in \partial\Pi^\tau$,*

$$\lim_{r\to 0} \frac{1}{r^{d+1}}\int_{B_r^+(\tau,\mathbf{y})} \eta(\mathbf{u}(t,\mathbf{x}))\, dtd\mathbf{x} = \eta(\mathbf{u})_{tr}(\tau,\mathbf{y}). \qquad (3.9)$$

In particular, if we choose $\eta = u^i$, $i = 1, ..., m$, we obtain the trace for each component of $\mathbf{u}$.

Proof. We divide the proof into three steps.

Step 1: We apply the Gauss-Green formula given by Theorem 2.7 to

$$\boldsymbol{F}^{\eta}_{\mathbf{u}}(t,\mathbf{x}) := (\eta(\mathbf{u}(t,\mathbf{x})), \mathbf{q}(\mathbf{u}(t,\mathbf{x}))),$$

which is a divergence-measure field. Indeed, as explained in §2, we have

$$\boldsymbol{F}^{\eta}_{\mathbf{u}} \in \mathcal{DM}^{\infty}_{loc}(\mathbb{R}^N). \tag{3.10}$$

Without loss of generality and to simplify the exposition, we prove (3.9) for the hyperplane $\Pi := \{t = 0\}$. Theorem 2.7 gives the existence of a function $\mathcal{F}\cdot\boldsymbol{\nu} \in L^{\infty}(\Pi)$ (that depends on η) which is the weak normal trace of the vector field $\boldsymbol{F}^{\eta}_{\mathbf{u}}$ on $\partial\Pi$. Let $\mathcal{G} \subset \Pi$ be the set of all Lebesgue points of $\mathcal{F}\cdot\boldsymbol{\nu}$ for which Lemma 3.6 and property (3.1) hold. We have $\mathcal{H}^d(\Pi \setminus \mathcal{G}) = 0$. For the rest of the proof, we identify $(0,\mathbf{x}) \in \Pi$ with $\mathbf{x}$. Also, to simplify our exposition and without loss of generality, we can assume $\mathbf{0} \in \mathcal{G}$. We define

$$C_r^+ := C_r^+(\mathbf{0}) = \mathcal{B}_r(\mathbf{0}) \times (0,r).$$

From Theorem 2.7, we have

$$\int_{C_r^+} div_{(t,\mathbf{x})}(\Phi \boldsymbol{F}^{\eta}_{\mathbf{u}}(t,\mathbf{x}))dtd\mathbf{x} = -\int_{\partial C_r^+} \Phi\mathcal{F}\cdot\boldsymbol{\nu}d\mathcal{H}^d \tag{3.11}$$

for any $\Phi \in C^1_0(\mathbb{R}^{d+1})$. The following functions:

$$\Phi_r(t,\mathbf{x}) = \varphi(\frac{\mathbf{x}}{r})(r-t), \quad (t,\mathbf{x}) \in \mathbb{R}^{d+1}, 0 \le \varphi \le 1, \text{ supp } \varphi \subset \mathcal{B}_1(\mathbf{0}), \tag{3.12}$$

will be used as test functions in (3.11). We recall that the point $(0,\mathbf{x})$ has been identified with $\mathbf{x}$ and, for simplicity of notation, Φ_r will be denoted simply as Φ. After substituting Φ, the right hand side of (3.11) becomes

$$\int_{C_r^+} div_{(t,\mathbf{x})}(\Phi \boldsymbol{F}^{\eta}_{\mathbf{u}}(t,\mathbf{x}))dtd\mathbf{x} = -\int_{\mathcal{B}_r(\mathbf{0})} r\varphi(\frac{\mathbf{x}}{r})\mathcal{F}\cdot\boldsymbol{\nu}(\mathbf{x})d\mathcal{H}^d(\mathbf{x}).$$

The product rule for divergence-measure fields (2.3) yields

$$div\,(\Phi \boldsymbol{F}^{\eta}_{\mathbf{u}}) = \boldsymbol{F}^{\eta}_{\mathbf{u}}\cdot\nabla\Phi + \Phi div\,\boldsymbol{F}^{\eta}_{\mathbf{u}},$$

and hence (using the notation $\mu_\eta := div\,\boldsymbol{F}^{\eta}_{\mathbf{u}}$):

$$\int_{C_r^+} \Phi d\mu_\eta + \int_{C_r^+} \boldsymbol{F}^{\eta}_{\mathbf{u}}\cdot\nabla\Phi dtd\mathbf{x} = -\int_{\mathcal{B}_r(\mathbf{0})} r\varphi(\frac{\mathbf{x}}{r})\mathcal{F}\cdot\boldsymbol{\nu}(\mathbf{x})d\mathcal{H}^d.$$

Therefore,

$$\int_{C_r^+} \Phi d\mu_\eta + \int_{C_r^+} \big(\eta(\mathbf{u})\Phi_t + \mathbf{q}(\mathbf{u})\cdot\nabla_{\mathbf{x}}\Phi\big)dtd\mathbf{x} = -\int_{\mathcal{B}_r(\mathbf{0})} r\varphi(\frac{\mathbf{x}}{r})\mathcal{F}\cdot\boldsymbol{\nu}(\mathbf{x})d\mathcal{H}^d,$$

and hence

$$\begin{aligned}&\frac{1}{r^{d+1}}\int_{C_r^+} \Phi d\mu_\eta + \frac{1}{r^{d+1}}\int_{C_r^+} \big(\eta(\mathbf{u})\Phi_t + \mathbf{q}(\mathbf{u})\cdot\nabla_{\mathbf{x}}\Phi\big)dtd\mathbf{x}\\ &= -\frac{1}{r^{d+1}}\int_{\mathcal{B}_r(\mathbf{0})} r\varphi(\frac{\mathbf{x}}{r})\mathcal{F}\cdot\boldsymbol{\nu}(\mathbf{x})d\mathcal{H}^d = -\frac{1}{r^d}\int_{\mathcal{B}_r(\mathbf{0})} \varphi(\frac{\mathbf{x}}{r})\mathcal{F}\cdot\boldsymbol{\nu}(\mathbf{x})d\mathcal{H}^d.\end{aligned}$$

Using the definition of Φ, we obtain

$$\frac{1}{r^{d+1}}\int_{C_r^+} \eta(\mathbf{u})\Phi_t dtd\mathbf{x} = -\frac{1}{r^{d+1}}\int_{C_r^+} \varphi(\frac{\mathbf{x}}{r})\eta(\mathbf{u}(t,\mathbf{x}))dtd\mathbf{x}.$$

Therefore, the following equation holds:

$$\begin{aligned}&\frac{1}{r^{d+1}}\int_{C_r^+}(r-t)\varphi(\frac{\mathbf{x}}{r})d\mu_\eta-\frac{1}{r^{d+1}}\int_{C_r^+}\varphi(\frac{\mathbf{x}}{r})\eta(\mathbf{u}(t,\mathbf{x}))dtd\mathbf{x}\\&\quad+\frac{1}{r^{d+1}}\int_{C_r^+}\mathbf{q}(\mathbf{u})\cdot\nabla_{\mathbf{x}}\Phi dtd\mathbf{x}\\&=-\frac{1}{r^d}\int_{\mathcal{B}_r(\mathbf{0})}\varphi(\frac{\mathbf{x}}{r})\mathcal{F}\cdot\boldsymbol{\nu}(\mathbf{x})d\mathcal{H}^d.\end{aligned}\tag{3.13}$$

Step 2: We now show in (3.13) that, as $r\to 0$,

$$\frac{1}{r^{d+1}}\int_{C_r^+}\mathbf{q}(\mathbf{u})\cdot\nabla_{\mathbf{x}}\Phi dtd\mathbf{x}\to 0,\tag{3.14}$$

and

$$\frac{1}{r^{d+1}}\int_{C_r^+}(r-t)\varphi\left(\frac{\mathbf{x}}{r}\right)d\mu_\eta\to 0.\tag{3.15}$$

We have

$$\begin{aligned}&\frac{1}{r^{d+1}}\int_{\mathcal{B}_r(\mathbf{0})\times(0,r)}\mathbf{q}(\mathbf{u})\cdot\nabla_{\mathbf{x}}\Phi dtd\mathbf{x}\\&=\frac{1}{r}\cdot\frac{1}{r^{d+1}}\int_{\mathcal{B}_r(\mathbf{0})\times(0,r)}(r-t)\mathbf{q}(\mathbf{u}(t,\mathbf{x}))\cdot\nabla_{\mathbf{x}}\varphi\left(\frac{\mathbf{x}}{r}\right)dtd\mathbf{x}\\&=\frac{1}{r}\int_{\mathcal{B}_1(\mathbf{0})\times(0,1)}(r-r\alpha)\mathbf{q}(\mathbf{u}(r\alpha,r\boldsymbol{\xi}))\cdot\nabla_{\mathbf{x}}\varphi(\boldsymbol{\xi})d\alpha d\boldsymbol{\xi}\\&=\int_{\mathcal{B}_1(\mathbf{0})\times(0,1)}(1-\alpha)\mathbf{q}(\mathbf{u}(r\alpha,r\boldsymbol{\xi}))\cdot\nabla_{\mathbf{x}}\varphi(\boldsymbol{\xi})d\alpha d\boldsymbol{\xi},\end{aligned}$$

where the following change of variables has been performed: $t=r\alpha$ and $\mathbf{x}=r\boldsymbol{\xi}$. If $\overline{\mathbf{q}(\mathbf{u})}_r$ denotes the average of $\mathbf{q}(\mathbf{u})$ in the cylinder $\mathcal{B}_r(\mathbf{0})\times(0,r)$, then

$$\overline{\mathbf{q}(\mathbf{u})}_r\cdot\int_0^1(1-\alpha)\Big(\int_{\mathcal{B}_1(\mathbf{0})}\nabla_{\mathbf{x}}\varphi(\boldsymbol{\xi})d\boldsymbol{\xi}\Big)d\alpha=0,$$

since φ has compact support in $\mathcal{B}_1(\mathbf{0})$. Therefore, with $C_1^+=\mathcal{B}_1(\mathbf{0})\times(0,1)$ and using (3.1), we compute

$$\begin{aligned}&\Big|\frac{1}{r^{d+1}}\int_{\mathcal{B}_r(\mathbf{0})\times(0,r)}\mathbf{q}(\mathbf{u})\cdot\nabla_{\mathbf{x}}\Phi dtd\mathbf{x}\Big|\\&=\Big|\int_{C_1^+}(1-\alpha)\mathbf{q}(\mathbf{u}(r\alpha,r\boldsymbol{\xi}))\cdot\nabla_{\mathbf{x}}\varphi(\boldsymbol{\xi})d\alpha d\boldsymbol{\xi}-\overline{\mathbf{q}(\mathbf{u})}_r\cdot\int_{C_1^+}(1-\alpha)\nabla_{\mathbf{x}}\varphi(\boldsymbol{\xi})d\alpha d\boldsymbol{\xi}\Big|\\&=\Big|\int_{C_1^+}(1-\alpha)(\mathbf{q}(\mathbf{u}(r\alpha,r\boldsymbol{\xi}))-\overline{\mathbf{q}(\mathbf{u})}_r)\cdot\nabla_{\mathbf{x}}\varphi(\boldsymbol{\xi})d\alpha d\boldsymbol{\xi}\Big|\\&\le\int_{C_1^+}|\mathbf{q}(\mathbf{u}(r\alpha,r\boldsymbol{\xi}))-\overline{\mathbf{q}(\mathbf{u})}_r||\nabla_{\mathbf{x}}\varphi(\boldsymbol{\xi})|d\alpha d\boldsymbol{\xi}\\&\le C\int_{C_1^+}|\mathbf{q}(\mathbf{u}(r\alpha,r\boldsymbol{\xi}))-\overline{\mathbf{q}(\mathbf{u})}_r|d\alpha d\boldsymbol{\xi}\\&=\frac{C}{r^{d+1}}\int_{C_r^+}|\mathbf{q}(\mathbf{u}(t,\mathbf{x}))-\overline{\mathbf{q}(\mathbf{u})}_r|dtd\mathbf{x}\\&\to 0\qquad\text{as } r\to 0,\end{aligned}$$

which arrives at (3.14). We now proceed to show (3.15). Using that $0 \le \varphi \le 1$, we compute

$$\Big|\frac{1}{r^{d+1}}\int_{C_r^+}(r-t)\varphi(\frac{\mathbf{x}}{r})d\mu_\eta\Big| \le \frac{1}{r^{d+1}}\int_{C_r^+} r|\varphi(\frac{\mathbf{x}}{r})|d\mu_\eta \le \frac{1}{r^d}\|\mu_\eta\|(C_r^+) \to 0$$

as $r \to 0$ due to Lemma 3.6.

Step 3: From (3.13)–(3.15), we obtain that, for any $\varphi \in C_0^\infty(\mathcal{B}_1(\mathbf{0}))$,

$$\lim_{r\to 0}\frac{1}{r^{d+1}}\int_{C_r^+}\varphi(\frac{\mathbf{x}}{r})\eta(\mathbf{u}(t,\mathbf{x}))dtd\mathbf{x} = \lim_{r\to 0}\frac{1}{r^d}\int_{\mathcal{B}_r(\mathbf{0})}\varphi(\frac{\mathbf{x}}{r})\mathcal{F}\cdot\boldsymbol{\nu}(\mathbf{x})d\mathcal{H}^d. \tag{3.16}$$

Performing the change of variables $t = r\alpha$ and $x = r\boldsymbol{\xi}$, we obtain that, for any $\varphi \in C_0^\infty(\mathcal{B}_1(\mathbf{0}))$,

$$\lim_{r\to 0}\int_{C_1^+}\varphi(\boldsymbol{\xi})\eta(\mathbf{u}(r\alpha, r\boldsymbol{\xi}))d\alpha d\boldsymbol{\xi} = \lim_{r\to 0}\int_{\mathcal{B}_1(\mathbf{0})}\varphi(\boldsymbol{\xi})\mathcal{F}\cdot\boldsymbol{\nu}(r\boldsymbol{\xi})d\mathcal{H}^d. \tag{3.17}$$

Both limits in (3.17) exist because $\mathbf{0}$ is a Lebesgue point of the normal trace function $\mathcal{F}\cdot\boldsymbol{\nu}$. Since (3.17) holds for any test function φ with compact support, we can choose a sequence $\varphi_k \in C_0^\infty(\mathcal{B}_1(\mathbf{0}))$ such that $\varphi_k \to 1$ pointwise. Therefore, the following limits exist for each k:

$$\lim_{r\to 0}\int_{C_1^+}\varphi_k(\boldsymbol{\xi})\eta(\mathbf{u}(r\alpha, r\boldsymbol{\xi}))d\alpha d\boldsymbol{\xi} = \lim_{r\to 0}\int_{\mathcal{B}_1(\mathbf{0})}\varphi_k(\boldsymbol{\xi})\mathcal{F}\cdot\boldsymbol{\nu}(r\boldsymbol{\xi})d\mathcal{H}^d. \tag{3.18}$$

We define

$$h_k(r) := \int_{C_1^+}\varphi_k(\boldsymbol{\xi})\eta(\mathbf{u}(r\alpha, r\boldsymbol{\xi}))d\alpha d\boldsymbol{\xi}$$

and note that $h_k \to h$ uniformly on r, where

$$h(r) := \int_{C_1^+}\eta(\mathbf{u}(r\alpha, r\boldsymbol{\xi}))d\alpha d\boldsymbol{\xi}.$$

Also, from (3.18), the following limit exists:

$$A_k := \lim_{r\to 0}\int_{C_1^+}\varphi_k(\boldsymbol{\xi})\eta(\mathbf{u}(r\alpha, r\boldsymbol{\xi}))d\alpha d\boldsymbol{\xi}.$$

Therefore, we conclude that (see, for example, Rudin [23], Theorem 7.11):

$$\lim_{k\to\infty} A_k \quad \text{exists}$$

and

$$\lim_{r\to 0}\lim_{k\to\infty} h_k(r) = \lim_{k\to\infty}\lim_{r\to 0} h_k(r).$$

Proceeding in the same way with the right hand side of (3.18), we conclude

$$\lim_{k\to\infty}\lim_{r\to 0}\int_{\mathcal{B}_1(\mathbf{0})}\varphi_k(\boldsymbol{\xi})\mathcal{F}\cdot\boldsymbol{\nu}(r\boldsymbol{\xi})d\mathcal{H}^d = \lim_{r\to 0}\lim_{k\to\infty}\int_{\mathcal{B}_1(\mathbf{0})}\varphi_k(\boldsymbol{\xi})\mathcal{F}\cdot\boldsymbol{\nu}(r\boldsymbol{\xi})d\mathcal{H}^d,$$

which yields

$$\begin{aligned}\lim_{r\to 0}\int_{C_1^+}\eta(\mathbf{u}(r\alpha,r\boldsymbol{\xi}))d\alpha d\boldsymbol{\xi} &= \lim_{r\to 0}\lim_{k\to\infty}\int_{C_1^+}\varphi_k(\boldsymbol{\xi})\eta(\mathbf{u}(r\alpha,r\boldsymbol{\xi}))d\alpha d\boldsymbol{\xi}\\ &= \lim_{k\to\infty}\lim_{r\to 0}\int_{C_1^+}\varphi_k(\boldsymbol{\xi})\eta(\mathbf{u}(r\alpha,r\boldsymbol{\xi}))d\alpha d\boldsymbol{\xi}\\ &= \lim_{k\to\infty}\lim_{r\to 0}\int_{\mathcal{B}_1(\mathbf{0})}\varphi_k(\boldsymbol{\xi})\mathcal{F}\cdot\boldsymbol{\nu}(r\boldsymbol{\xi})d\mathcal{H}^d\\ &= \lim_{r\to 0}\lim_{k\to\infty}\int_{\mathcal{B}_1(\mathbf{0})}\varphi_k(\xi)\mathcal{F}\cdot\boldsymbol{\nu}(r\boldsymbol{\xi})d\mathcal{H}^d\\ &= \lim_{r\to 0}\int_{\mathcal{B}_1(\mathbf{0})}\mathcal{F}\cdot\boldsymbol{\nu}(r\boldsymbol{\xi})d\mathcal{H}^d.\end{aligned}$$

Hence,

$$\lim_{r\to 0}\int_{C_1^+}\eta(\mathbf{u}(r\alpha,r\boldsymbol{\xi}))d\alpha d\boldsymbol{\xi} = \lim_{r\to 0}\int_{\mathcal{B}_1(\mathbf{0})}\mathcal{F}\cdot\boldsymbol{\nu}(r\boldsymbol{\xi})d\mathcal{H}^d. \tag{3.19}$$

Changing the variables back in (3.19) yields

$$\lim_{r\to 0}\frac{1}{r^{d+1}}\int_{\mathcal{B}_r(\mathbf{0})\times(0,r)}\eta(\mathbf{u}(t,\mathbf{x}))dtd\mathbf{x} = \lim_{r\to 0}\frac{1}{r^d}\int_{\mathcal{B}_r(\mathbf{0})}\mathcal{F}\cdot\boldsymbol{\nu}(\mathbf{x})d\mathcal{H}^d.$$

Since $\mathbf{0}$ is a Lebesgue point of $\mathcal{F}\cdot\boldsymbol{\nu}$, we conclude

$$\lim_{r\to 0}\frac{1}{r^{d+1}}\int_{C_r^+}\eta(\mathbf{u}(t,\mathbf{x}))dtd\mathbf{x} = \mathcal{F}\cdot\boldsymbol{\nu}(\mathbf{0}).$$

We conclude that the desired function is $\eta(\mathbf{u})_{tr} := \mathcal{F}\cdot\boldsymbol{\nu}$. □

Remark 6. For the one-dimensional system of isentropic Euler equations, the compensated compactness results (see Section 5) imply that, given any $r_k \to 0$, there exists a subsequence of r_k (denoted again as r_k) such that

$$\mathbf{u}^{\tau,y}_{r_k}(t,x) \to \mathbf{u}^{\tau,y}_\infty(t,x) \qquad \text{a.e. } (t,x)\in\Pi^\tau \tag{3.20}$$

with $\mathbf{u}^{\tau,y}_\infty \in L^\infty(\Pi^\tau)$. If $\mathbf{u}^{\tau,y}_\infty$ is a constant, then proceeding as in Lemma 3.5, we obtain (3.1) for this system. Thus, (3.20) and (3.9) imply the existence of the strong trace:

$$\lim_{r\to 0}\frac{1}{r^{d+1}}\int_{B_r^+(\tau,y)}\left|\eta(\mathbf{u}(t,x)) - \frac{2}{w(d+1)}\mathcal{F}\cdot\boldsymbol{\nu}(\tau,y)\right| dtdx = 0$$

for $\mathcal{H}^d$-a.e. $(\tau,y)\in\partial\Pi^\tau$, and also

$$\lim_{r\to 0}\frac{1}{r^{d+1}}\int_{B_r^+(\tau,y)}\left|\eta(\mathbf{u}(t,x))\right| dtdx = \left|\mathcal{F}\cdot\boldsymbol{\nu}(\tau,y)\right|$$

for $\mathcal{H}^d$-a.e. $(\tau,y)\in\partial\Pi^\tau$. A Liouville-type result for the one-dimensional system of isentropic Euler equations (which would yield that $\mathbf{u}^{\tau,y}_\infty$ is a constant) is discussed in Section 5.

4. **Traces on sets of finite perimeter.** We first introduce the following definitions.

Definition 4.1. If E is a set of finite perimeter, we define

$$\mathbf{T}(\mathbf{z}) := \{(t,\mathbf{x}) : ((t,\mathbf{x})-\mathbf{z})\cdot\boldsymbol{\nu}(\mathbf{z}) = 0\}$$

for $\mathcal{H}^d$-almost every $\mathbf{z} \in \partial^* E$, where $\boldsymbol{\nu}(\mathbf{z})$ is the inner unit normal at $\mathbf{z}$. We also define the cylinder

$$C_r^+(\mathbf{z}) := (B_r(\mathbf{z}) \cap \mathbf{T}(\mathbf{z})) \times (0,r),$$

and

$$C_r(\mathbf{z}) := (B_r(\mathbf{z}) \cap \mathbf{T}(\mathbf{z})) \times (-r,r).$$

Remark 7. Since the ball $B_r(\mathbf{z})$ can be inscribed in the cylinder $C_r(\mathbf{z})$, the results in this section can be stated with balls or cylinders equivalently.

Definition 4.2. Let $E \subset \mathbb{R}^{d+1}$ be a set of finite perimeter. We say that $\mathbf{u}$ satisfies the vanishing mean oscillation property on the half balls if, for any continuous $\mathbf{q} \in C(\mathbb{R}^m, \mathbb{R}^d)$ and $\mathcal{H}^d$-almost every $\mathbf{z} \in \partial^* E$,

$$\lim_{r\to 0} \frac{1}{r^{d+1}} \int_{B_r^+(\mathbf{z})} |\mathbf{q}(\mathbf{u}(t,\mathbf{x})) - \overline{\mathbf{q}(\mathbf{u})}_r(\mathbf{z})|\, dt d\mathbf{x} = 0, \tag{4.1}$$

where $\overline{q(\mathbf{u})}_r(\mathbf{z})$ is the vector in $\mathbb{R}^d$ which is the average of $\mathbf{q}(\mathbf{u})$ over the half ball $B_r^+(\mathbf{z}) := B_r(\mathbf{z}) \cap \{(t,\mathbf{x}) : ((t,\mathbf{x})-\mathbf{z})\cdot\boldsymbol{\nu}(\mathbf{z}) > 0\}$.

Remark 8. Condition (4.1) holds for the scalar case (i.e. $m = 1$) due to the rectifiability of the set of shock waves, established by De-Lellis-Otto-Westdickenberg [11] (recall from Theorem 2.6 that $\partial^* E$ is also a d-rectifiable set).

We now proceed to extend Lemma 3.6 to the case of sets of finite perimeter.

Lemma 4.3. *Let E be a bounded set of finite perimeter, and let μ be a positive Radon measure in $\mathbb{R}^{d+1}$ such that $\mu \ll \mathcal{H}^d$. Then, for $\mathcal{H}^d$-almost every $\mathbf{z} \in \partial^* E$,*

$$\lim_{r\to 0} \frac{\mu(D_r^{\mathbf{z}})}{r^d} = 0,$$

where $D_r^{\mathbf{z}} = E^1 \cap C_r(\mathbf{z})$.

Proof. Since $\mu \ll \mathcal{H}^d$, Corollary 4.4 in Chen-Torres-Ziemer [9] gives the existence of a sequence of smooth sets A_l such that

$$\lim_{l\to\infty} \mu(A_l \Delta E^1) = 0 \tag{4.2}$$

and

$$\lim_{l\to\infty} \mathcal{H}^d(\partial A_l \cap (\partial^* E \cup E^0)) = 0. \tag{4.3}$$

Therefore, given any $\varepsilon > 0$, there exists a smooth set A_ε such that

$$\mu(E^1 \setminus A_\varepsilon) < \varepsilon. \tag{4.4}$$

Let

$$S_k = \{\mathbf{z} \in \partial^* E : \limsup_{r\to 0} \frac{\mu(D_r^{\mathbf{z}})}{r^d} > \frac{1}{k}\}.$$

It suffices to show that $\mathcal{H}^d(S_k) = 0$ for each k.

Let S_k^ε denote the set of all $\mathbf{z} \in S_k$ such that there exists $r_{\mathbf{z}}$ so that

$$D_{r_{\mathbf{z}}}^{\mathbf{z}} \subset E^1 \setminus A_\varepsilon.$$

Due to (4.2) and (4.3), we find that $S_k^{\varepsilon_i}$ is an increasing sequence of sets when $\varepsilon_i \to 0$ and $S_k = \cup S_k^{\varepsilon_i}$. Hence

$$\mathcal{H}^d(S_k) = \lim_{i\to\infty} \mathcal{H}^d(S_k^{\varepsilon_i}).$$

We now proceed to show that $\mathcal{H}^d(S_k^{\varepsilon_i}) < c(d)\varepsilon_i$, where $c(d)$ is a constant that depends on dimension d. For each $\mathbf{z} \in S_k^{\varepsilon_i}$ (choosing smaller $r_{\mathbf{z}}$ if necessary), the definition of S_k implies

$$\frac{\mu(D^{\mathbf{z}}_{r_{\mathbf{z}}})}{r_{\mathbf{z}}^d} > \frac{1}{2k}. \tag{4.5}$$

By choosing even smaller $r_{\mathbf{z}}$ if necessary, we can also assume (see Giusti [18], Lemma 3.5, page 45):

$$\mathcal{H}^d(\partial^* E \cap C_{r_{\mathbf{z}}}(\mathbf{z})) \le c(d) r_{\mathbf{z}}^d. \tag{4.6}$$

A covering argument yields a countable sequence $\mathbf{z}_j \subset S_k^{\varepsilon_i}$ such that the sets $F_j := \partial^* E \cap C_{r_j}(\mathbf{z}_j), r_j := r_{\mathbf{z}_j}$, are pairwise disjoints and $S_k^{\varepsilon_i} \subset \cup G_j$, where $F_j \subset G_j$ and $\mathcal{H}^d(G_j) \le c(d)\mathcal{H}^d(F_j)$. Thus, from (4.4)–(4.6), we can estimate $\mathcal{H}^d(S_k^{\varepsilon_i})$ in terms of $\mu(E^1 \setminus A_{\varepsilon_i})$:

$$\begin{aligned}
\mathcal{H}^d(S_k^{\varepsilon_i}) &\le c(d)\sum \mathcal{H}^d(F_j)\\
&= c(d)\sum \mathcal{H}^d(\partial^* E\cap C_{r_j}(\mathbf{z}_j))\\
&\le c(d)\sum r_j^d\\
&< 2kc(d)\sum \mu(D^{\mathbf{z}_j}_{r_j})\\
&\le c(d)\mu(E^1\setminus A_\varepsilon) < c(d)\varepsilon_i.
\end{aligned}$$

This yields that $\mathcal{H}^d(S_k) = \lim_{\varepsilon_i\to 0}\mathcal{H}^d(S_k^{\varepsilon_i}) \le \lim_{\varepsilon_i\to 0}\big(c(d)\varepsilon_i\big) = 0$. ☐

Given any bounded set of finite perimeter, $E \subset \mathbb{R}^{d+1}$, Theorem 2.7 yields the existence of the weak interior normal trace $\mathcal{F}\cdot\boldsymbol{\nu} \in L^\infty(\partial^* E)$ of the vector field

$$\boldsymbol{F}^\eta_{\mathbf{u}}(t,\mathbf{x}) := (\eta(\mathbf{u}(t,\mathbf{x})), \mathbf{q}(\mathbf{u}(t,\mathbf{x})))$$

on $\partial^* E$. We note that $\mathcal{H}^d$-almost every $\mathbf{z} \in \partial^* E$ is a Lebesgue point of $\mathcal{F}\cdot\boldsymbol{\nu}$. Assuming (4.1), we show next that the weak trace of the vector field $\boldsymbol{F}^\eta_{\mathbf{u}}$ satisfies the stronger property:

Theorem 4.4. *Let $E \subset \mathbb{R}^{d+1}$ be a bounded set of finite perimeter. If $\mathbf{u} \in L^\infty(\mathbb{R}^{d+1};\mathbb{R}^m)$ satisfies the entropy inequality (2.5) and property (4.1), then there exists a function $\mathcal{F}\cdot\boldsymbol{\nu} \in L^\infty(\partial^* E)$ such that, for every convex entropy pair $(\eta,\mathbf{q}) \in \mathcal{P}$ and for $\mathcal{H}^d$-almost every $\mathbf{z} \in \partial^* E$,*

$$\lim_{r\to 0}\frac{1}{r^{d+1}}\int_{E^1\cap B_r(\mathbf{z})} (\eta(\mathbf{u}(t,\mathbf{x})), \mathbf{q}(\mathbf{u}(t,\mathbf{x})))\cdot\boldsymbol{\nu}(\mathbf{z})\,dtd\mathbf{x} = \mathcal{F}\cdot\boldsymbol{\nu}(\mathbf{z}). \tag{4.7}$$

Proof. Denote $\mathcal{F}\cdot\boldsymbol{\nu}$ the weak interior normal trace of the vector field

$$\boldsymbol{F}^\eta_{\mathbf{u}} := (\eta(\mathbf{u}), \mathbf{q}(\mathbf{u}))$$

on $\partial^* E$ given by Theorem 2.7. Let $\mathcal{G} \subset \partial^* E$ be the set of all Lebesgue points of $\mathcal{F}\cdot\boldsymbol{\nu}$ that satisfy Lemma 4.3 and property (4.1). We obtain that $\mathcal{H}^d(\partial^* E \setminus \mathcal{G}) = 0$. Then we divide the proof in two steps:

Step 1: We first consider the case that $\mathbf{0} \in \mathcal{G}$ and $\boldsymbol{\nu}(\mathbf{0}) = (1, 0, ..., 0)$. Proceeding as in §3, we can show

$$\begin{aligned}
&\lim_{r\to 0} \frac{1}{r^{d+1}} \int_{E^1\cap C_r(\mathbf{0})} \eta(\mathbf{u}(t,\mathbf{x}))dtd\mathbf{x} \\
&= \lim_{r\to 0} \frac{1}{r^{d+1}} \int_{E^1\cap C_r(\mathbf{0})} (\eta(\mathbf{u}(t,\mathbf{x})), \mathbf{q}(\mathbf{u}(t,\mathbf{x}))) \cdot \boldsymbol{\nu}(\mathbf{0})dtd\mathbf{x} \\
&= \mathcal{F}\cdot \boldsymbol{\nu}(\mathbf{0}).
\end{aligned} \tag{4.8}$$

Indeed, if we apply Theorem 2.7 to

$$D_r := E \cap C_r,$$

where (to simplify notation):

$$C_r := C_r(\mathbf{0}),$$

we obtain

$$\int_{(D_r)^1} div_{(t,\mathbf{x})}(\Phi \boldsymbol{F}^\eta_{\mathbf{u}}(t,\mathbf{x}))dtd\mathbf{x} = -\int_{\partial^* D_r} \Phi\mathcal{F}\cdot\boldsymbol{\nu} d\mathcal{H}^d \tag{4.9}$$

for any $\Phi \in C^1_0(\mathbb{R}^{d+1})$. Choose Φ as in (3.12) in §3 and note that

$$\mathcal{H}^d([\partial^* D_r]\Delta[(\partial^* E\cap C_r)\cup(\partial^* C_r \cap E)]) = 0$$

and

$$(D_r)^1 = E^1 \cap C_r.$$

Then (4.9) becomes

$$\int_{E^1\cap C_r} div_{(t,\mathbf{x})}(\Phi \boldsymbol{F}^\eta_{\mathbf{u}}(t,\mathbf{x}))dtd\mathbf{x} = -\int_{\partial^* E\cap C_r} \Phi\mathcal{F}\cdot\boldsymbol{\nu} d\mathcal{H}^d. \tag{4.10}$$

From (4.10) and proceeding as in §3, we obtain

$$\begin{aligned}
&\frac{1}{r^{d+1}}\int_{E^1\cap C_r} (r-t)\varphi(\frac{\mathbf{x}}{r})d\mu_\eta - \frac{1}{r^{d+1}}\int_{E^1\cap C_r}\varphi(\frac{\mathbf{x}}{r})\eta(\mathbf{u}(t,\mathbf{x}))dtd\mathbf{x} \\
&+\frac{1}{r^{d+1}}\int_{E^1\cap C_r} \mathbf{q}(\mathbf{u})\cdot\nabla_{\mathbf{x}}\Phi dtd\mathbf{x} \\
&= -\int_{\partial^* E\cap C_r} \frac{r-t}{r^{d+1}}\varphi(\frac{\mathbf{x}}{r})\mathcal{F}\cdot\boldsymbol{\nu}(t,\mathbf{x})d\mathcal{H}^d.
\end{aligned} \tag{4.11}$$

The difference between this case and the one considered in §3 is that $\partial^* E$ is not flat, but this can be overcome by using the regularity of the reduced boundary in Theorem 2.6. In particular, Theorem 2.6 states that

$$\frac{|(E^1\cap C_r)\Delta C_r^+|}{r^{d+1}} \to 0 \qquad \text{as } r\to 0.$$

Therefore, using property (4.1) and proceeding as in §3, we obtain

$$\frac{1}{r^{d+1}}\int_{E^1\cap C_r} \mathbf{q}(\mathbf{u})\cdot\nabla_{\mathbf{x}}\Phi\, dtd\mathbf{x} \to 0 \qquad \text{as } r\to 0. \tag{4.12}$$

Also, since $\mu_\eta \ll \mathcal{H}^d$ (see Remark 2), Lemma 4.3 implies

$$\frac{1}{r^{d+1}}\int_{E^1\cap C_r} (r-t)\varphi(\frac{\mathbf{x}}{r})d\mu_\eta \le \frac{2\,\|\mu_\eta\|\,(E^1\cap C_r)}{r^d} \to 0 \qquad \text{as } r\to 0. \tag{4.13}$$

Therefore, (4.11) reduces to

$$\lim_{r\to 0}\frac{1}{r^{d+1}}\int_{E^1\cap C_r}\varphi(\frac{\mathbf{x}}{r})\eta(\mathbf{u}(t,\mathbf{x}))dtd\mathbf{x}=\lim_{r\to 0}\int_{\partial^* E\cap C_r}\frac{t-r}{r^{d+1}}\varphi(\frac{\mathbf{x}}{r})\mathcal{F}\cdot\boldsymbol{\nu}(t,\mathbf{x})d\mathcal{H}^d \tag{4.14}$$

for any $\varphi\in C_0^\infty(B_1(\mathbf{0}))$. If we proceed now as in §3 by invoking Theorem 2.6 (i), we obtain

$$\lim_{r\to 0}\frac{1}{r^{d+1}}\int_{E^1\cap C_r}\eta(\mathbf{u}(t,\mathbf{x}))dtd\mathbf{x}=\lim_{r\to 0}\frac{1}{r^d}\int_{\partial^* E\cap C_r}\mathcal{F}\cdot\boldsymbol{\nu}d\mathcal{H}^d,$$

and, since $\mathbf{0}$ is a Lebesgue point of $\mathcal{F}\cdot\boldsymbol{\nu}$, we conclude

$$\lim_{r\to 0}\frac{1}{r^{d+1}}\int_{E^1\cap C_r}\eta(\mathbf{u}(t,\mathbf{x}))dtd\mathbf{x}=\mathcal{F}\cdot\boldsymbol{\nu}(\mathbf{0}).$$

Step 2: We now fix any $\mathbf{z}\in\mathcal{G}$. We perform the change of variables

$$(s,\mathbf{y}):=(y_0,y_1,...,y_N)=T(t,\mathbf{x}) \tag{4.15}$$

so that $T(\boldsymbol{\nu}(\mathbf{z}))=(1,0,...,0)$ (this change of variables was used in [25] and [26] in the scalar case) and, without loss of generality, we assume that $T(\mathbf{z})=\mathbf{0}$. The equation in the new coordinates is

$$div_{(s,\mathbf{y})}\tilde{\mathbf{f}}(\tilde{\mathbf{u}}(s,\mathbf{y}))=0,$$

where

$$\tilde{\mathbf{u}}(s,\mathbf{y})=\mathbf{u}(t,\mathbf{x}),\quad (t,\mathbf{x})=T^{-1}(s,\mathbf{y}),$$

and

$$\tilde{\mathbf{f}}^i(\xi_1,...,\xi_m)=T(\xi_i,\mathbf{f}^i(\xi_1,...,\xi_m)),\quad i=1,...,m.$$

For any entropy pair $(\eta,\mathbf{q})$ (recall $\eta:\mathbb{R}^m\to\mathbb{R}$ and $\mathbf{q}:\mathbb{R}^m\to\mathbb{R}^d$), we also define

$$\tilde{\mathbf{g}}^\eta(\xi_1,...,\xi_m)=T(\eta(\xi_1,...,\xi_m),\mathbf{q}(\xi_i,...,\xi_m))$$

and

$$\tilde{C}_r=T(C_r(\mathbf{z})),\ \tilde{E}=T(E).$$

We define the vector field

$$\tilde{\boldsymbol{F}}^\eta_{\mathbf{u}}(s,\mathbf{y}):=\tilde{\mathbf{g}}^\eta(\tilde{\mathbf{u}}(s,\mathbf{y})),$$

which is also a divergence-measure field in the new coordinates. In order to see this, we note that, since T is a rotation, we have

$$\begin{aligned}\int_{\mathbb{R}^N}\boldsymbol{F}^\eta_{\mathbf{u}}(t,\mathbf{x})\cdot\nabla_{(t,\mathbf{x})}\varphi(t,\mathbf{x})d\mathbf{x}dt&=\int_{\mathbb{R}^N}(\eta(\mathbf{u}(t,\mathbf{x})),\mathbf{q}(\mathbf{u}(t,\mathbf{x})))\cdot\nabla_{(t,\mathbf{x})}\varphi(t,\mathbf{x})d\mathbf{x}dt\\&=\int_{\mathbb{R}^N}T(\eta(\mathbf{u}(t,\mathbf{x})),\mathbf{q}(\mathbf{u}(t,\mathbf{x})))\cdot T\nabla_{(t,\mathbf{x})}\varphi(t,\mathbf{x})d\mathbf{x}dt\\&=\int_{\mathbb{R}^N}\tilde{\mathbf{g}}^\eta(\tilde{\mathbf{u}}(s,\mathbf{y}))\cdot\nabla_{(s,\mathbf{y})}\varphi(s,\mathbf{y})dsd\mathbf{y}\\&=\int_{\mathbb{R}^N}\tilde{\boldsymbol{F}}^\eta_{\mathbf{u}}(s,\mathbf{y})\cdot\nabla_{(s,\mathbf{y})}\varphi(s,\mathbf{y})dsd\mathbf{y}\end{aligned}$$

for any $\varphi\in C_0^\infty(\mathbb{R}^N)$. Therefore, since $\boldsymbol{F}^\eta_{\mathbf{u}}\in\mathcal{DM}^\infty_{loc}(\mathbb{R}^N)$, it follows that, for each bounded open set $D\subset\mathbb{R}^N$,

$$\sup\{\tilde{\boldsymbol{F}}^\eta_{\mathbf{u}}\cdot\nabla_{(s,\mathbf{y})}\varphi:\varphi\in C_0^\infty(D),|\varphi|\le 1,spt(\varphi)\subset D\}<\infty;$$

that is, $\tilde{\boldsymbol{F}}^\eta_{\mathbf{u}} \in \mathcal{DM}^\infty_{loc}(\mathbb{R}^N)$. We denote the normal trace of the divergence-measure vector field $\tilde{\boldsymbol{F}}^\eta_{\mathbf{u}}(s,\mathbf{y})$ on $\partial\tilde{E}$ as $\tilde{\mathcal{F}}\cdot\tilde{\boldsymbol{\nu}}$. Let $\tilde{\boldsymbol{\nu}}$ denote the normal to $\tilde{E}$. Since T is a rotation, we find that $T(\mathbf{z})$ is a Lebesgue point for $\tilde{\mathcal{F}}\cdot\tilde{\boldsymbol{\nu}}$ and $\tilde{\mathbf{u}}$ satisfies (4.1) on $\tilde{E}$. Therefore, from Step 1, we obtain

$$\lim_{r\to 0}\frac{1}{r^{d+1}}\int_{\tilde{E}^1\cap\tilde{C}_r}\tilde{\mathbf{g}}^\eta(\tilde{\mathbf{u}}(s,\mathbf{y}))\cdot\tilde{\boldsymbol{\nu}}(\mathbf{0})dsd\mathbf{y} = \tilde{\mathcal{F}}\cdot\tilde{\boldsymbol{\nu}}(\mathbf{0}). \tag{4.16}$$

Since T is a rotation, we have

$$\tilde{\mathbf{g}}^\eta(\tilde{\mathbf{u}}(s,\mathbf{y}))\cdot\tilde{\boldsymbol{\nu}}(T(\mathbf{z})) = \mathbf{g}^\eta(\mathbf{u}(t,\mathbf{x}))\cdot\boldsymbol{\nu}(\mathbf{z}), \qquad (t,\mathbf{x}) = T^{-1}(s,\mathbf{y}), \tag{4.17}$$

and

$$\tilde{\mathcal{F}}\cdot\tilde{\boldsymbol{\nu}}(T(\mathbf{z})) = \mathcal{F}\cdot\boldsymbol{\nu}(\mathbf{z}). \tag{4.18}$$

Changing the variables in (4.16) and using that $|\det T| = 1$, we conclude

$$\lim_{r\to 0}\frac{1}{r^{d+1}}\int_{E^1\cap C_r(\mathbf{z})}\mathbf{g}^\eta(\mathbf{u}(t,\mathbf{x}))\cdot\boldsymbol{\nu}(\mathbf{z})dtd\mathbf{x} = \mathcal{F}\cdot\boldsymbol{\nu}(\mathbf{z}),$$

which is our desired result:

$$\lim_{r\to 0}\frac{1}{r^{d+1}}\int_{E^1\cap C_r(\mathbf{z})}(\eta(\mathbf{u}(t,\mathbf{x})),\mathbf{q}(\mathbf{u}(t,\mathbf{x})))\cdot\boldsymbol{\nu}(\mathbf{z})dtd\mathbf{x} = \mathcal{F}\cdot\boldsymbol{\nu}(\mathbf{z}).$$

□

5. **Structure of entropy solutions.** In this section, we present an approach through the prototype, the isentropic Euler equations (1.3), to analyze the structure of entropy solutions in L^∞ for hyperbolic systems of conservation laws (1.1). We assume that system (1.1) is endowed with at least one strictly convex entropy.

5.1. **Rescaling of the Entropy Solution $\mathbf{u}(t,\mathbf{x})$ of (1.1).**

Definition 5.1. For fixed $(s,\mathbf{y})\in\mathbb{R}^{d+1}$, we define, for every $r>0$, the rescalings of μ_η and $\mathbf{u}$ as

$$\mathbf{u}^{(s,\mathbf{y}),r}(t,\mathbf{x}) = \mathbf{u}((s,\mathbf{y})+r(t,\mathbf{x})), \tag{5.1}$$

$$\mu_\eta^{(s,\mathbf{y}),r}(A) = \frac{1}{r^d}\mu_\eta((s,\mathbf{y})+rA) \tag{5.2}$$

for all Borel sets $A\subset\mathbb{R}^{d+1}$, where

$$\mu_\eta = -\operatorname{div}_{(t,\mathbf{x})}(\eta(\mathbf{u}),\mathbf{q}(\mathbf{u}))$$

is the nonnegative entropy dissipation measure.

Definition 5.2. The upper and lower densities of μ_η are defined as

$$\theta^{*d}(\mu_\eta;(t,\mathbf{x})) := \limsup_{r\to 0}\frac{\mu_\eta(B_r(t,\mathbf{x}))}{r^d}, \qquad \theta_*^d(\mu_\eta;(t,\mathbf{x})) := \liminf_{r\to 0}\frac{\mu_\eta(B_r(t,\mathbf{x}))}{r^d}.$$

For any $(t,\mathbf{x})\in\mathbb{R}^{d+1}$ and $r>0$, the Gauss-Green formula for bounded divergence-measure fields (Theorem 2.7) yields

$$\mu_\eta(B_r(t,\mathbf{x})) = \int_{\partial B_r(t,\mathbf{x})}(\eta(\mathbf{u}),\mathbf{q}(\mathbf{u}))\cdot\boldsymbol{\nu}d\mathcal{H}^d \le \|\boldsymbol{F}^\eta_{\mathbf{u}}\|_\infty\,\alpha(d)r^d, \tag{5.3}$$

where $\boldsymbol{F}^\eta_{\mathbf{u}} = (\eta(\mathbf{u}),\mathbf{q}(\mathbf{u}))$. Therefore, we have

$$\theta^{*d}(\mu_\eta;(t,\mathbf{x})) < \infty \qquad \text{for every } (t,\mathbf{x})\in\mathbb{R}^{d+1}. \tag{5.4}$$

We now define

$$D_0 = \{(t, \mathbf{x}) \in \mathbb{R}^{d+1} : \theta^{*d}(\mu_\eta; (t, \mathbf{x})) = 0\}, \tag{5.5}$$

$$J_\eta = \{(t, \mathbf{x}) \in \mathbb{R}^{d+1} : 0 < \theta_*^d(\mu_\eta; (t, \mathbf{x})) \le \theta^{*d}(\mu_\eta; (t, \mathbf{x})) < \infty\}, \tag{5.6}$$

$$D_\infty = \{(t, \mathbf{x}) \in \mathbb{R}^{d+1} : \theta^{*d}(\mu_\eta; (t, \mathbf{x})) = \infty\}. \tag{5.7}$$

Remark 9. We note that (5.4) implies $D_\infty = \emptyset$.

Definition 5.3. Let

$$J := \cup_\eta J_\eta \tag{5.8}$$

for all convex entropy functions η with $\nabla^2 \eta > 0$ in the region where the entropy solution lies.

Lemma 5.4. *Assume that the $\mathcal{H}^d$-rectifiable set S is a shock wave. Then S is contained in J.*

Proof. Since the set S is a shock wave, for the strictly convex entropy η_*, S belongs to the support of the measure μ_{η^*}. Therefore, for every $(t, \mathbf{x}) \in S$, we have $\theta^{*d}(\mu_{\eta^*}; (t, \mathbf{x})) > 0$. However, Remark 9 implies $\theta^{*d}(\mu_{\eta^*}; (t, \mathbf{x})) < \infty$. Thus, we have $(t, \mathbf{x}) \in J_{\mu_{\eta^*}}$. □

Remark 10. The rectifiability of the entropy dissipation measures μ_η would imply the rectifiability of the shock location J.

5.2. **Compactness of the rescaling sequence.** Our approach consists in performing blow up arguments directly in the equation given by the entropy inequality:

$$\partial_t \eta(\mathbf{u}) + \nabla_{\mathbf{x}} \cdot \mathbf{q}(\mathbf{u}) = -\mu_\eta. \tag{5.9}$$

Consider the rescaling sequences $\mathbf{u}^{(s,\mathbf{y}),r}$ and $\mu_\eta^{(s,\mathbf{y}),r}$ defined in (5.1) and (5.2). We notice that they satisfy

$$\partial_t \eta(\mathbf{u}^{(s,\mathbf{y}),r}) + \nabla_{\mathbf{x}} \cdot \mathbf{q}(\mathbf{u}^{(s,\mathbf{y}),r}) = -\mu_\eta^{(s,\mathbf{y}),r}. \tag{5.10}$$

That is, (5.9) is invariant under the chosen rescalings.

Then we have the following compactness theorems.

Proposition 5.5. *Given $r_k \to 0$, the sequence of measures $\mu_\eta^{(s,\mathbf{y}),r_k}$ has a locally weakly* converging subsequence to a Radon measure $\mu_{\eta,\infty}^{(s,\mathbf{y})}$ in $\mathbb{R}^{d+1}$.*

Proof. From (5.3), we find that, for any compact set $K \subset \mathbb{R}^{d+1}$,

$$\|\mu_\eta^{(s,\mathbf{y}),r_k}\|(K) \le C\|\boldsymbol{F}_\eta\|_\infty.$$

This uniform boundedness implies the existence of a subsequence (still denoted as r_k) such that

$$\mu_\eta^{(s,\mathbf{y}),r_k} \rightharpoonup \mu_{\eta,\infty}^{(s,\mathbf{y})} \qquad \text{in the sense of measures.}$$

□

Proposition 5.6. *For the scaling sequence $\mathbf{u}^{(s,\mathbf{y}),r_k}$,*

$$\partial_t \eta(\mathbf{u}^{(s,\mathbf{y}),r_k}) + \nabla_{\mathbf{x}} \cdot \mathbf{q}(\mathbf{u}^{(s,\mathbf{y}),r_k}) \qquad \textit{is compact in } H^{-1}_{loc}.$$

Proof. The scaling sequence $\mathbf{u}^{(s,\mathbf{y}),r_k}$ satisfies (5.10). Proposition 5.5 implies that the scaling measure sequence $\mu_\eta^{(s,\mathbf{y}),r_k}$ is uniformly bounded, which implies its compactness in $W^{-1,p}_{loc}$, $p<2$. On the other hand, since $\mathbf{u}(t,\mathbf{x})\in L^\infty$,

$$\eta(\mathbf{u}^{(s,\mathbf{y}),r_k})_t+\nabla_{\mathbf{x}}\cdot\mathbf{q}(\mathbf{u}^{(s,\mathbf{y}),r_k})\qquad \text{is bounded in } W^{-1,\infty}_{loc}.$$

Then the compactness interpolation theorem implies that

$$\eta(\mathbf{u}^{(s,\mathbf{y}),r_k})_t+\nabla_{\mathbf{x}}\cdot\mathbf{q}(\mathbf{u}^{(s,\mathbf{y}),r_k})\qquad \text{is compact in } H^{-1}_{loc}.$$

□

For the isentropic Euler equations (1.3), the compactness result still holds even though the strict hyperbolicity fails near the vacuum. More precisely, assume that the pressure function $p=p(\rho)\in C^4(0,\infty)$ satisfies condition (1.4) (i.e., strict hyperbolicity and genuine nonlinearity) away from the vacuum and, near the vacuum, $p(\rho)$ is only asymptotic to the γ-law pressure (as real gases): there exists a sequence of exponents

$$1<\gamma:=\gamma_1<\gamma_2<\ldots<\gamma_N\le(3\gamma-1)/2<\gamma_{N+1}\tag{5.11}$$

and a sufficiently smooth function $P=P(\rho)$ such that

$$p(\rho)=\sum_{n=1}^{N}\kappa_n\,\rho^{\gamma_n}+\rho^{\gamma_{N+1}}\,P(\rho),\tag{5.12}$$

$$P(\rho)\ \text{ and }\ \rho^3\,P'''(\rho)\ \text{ are bounded as } \rho\to 0,\tag{5.13}$$

for some coefficients $\kappa_n\in\mathbb{R}$ with $\kappa_1>0$. The solutions under consideration will remain in a bounded subset of $\{\rho\ge 0\}$ so that the behavior of $p(\rho)$ for large ρ is irrelevant. This means that the pressure law $p(\rho)$ has the same singularity as $\sum_{n=1}^N\kappa_n\,\rho^{\gamma_n}$ near the vacuum. Observe that $p(0)=p'(0)=0$, but, for $k>\gamma_1$, the higher derivative $p^{(k)}(\rho)$ is unbounded near the vacuum with different orders of singularity.

Consider the Cauchy problem for (1.3) with the initial data:

$$(\rho,m)|_{t=0}=(\rho_0(x),m_0(x)),\qquad 0\le\rho_0(x)\le C_0,\ |m_0(x)|\le C_0\rho_0(x).\tag{5.14}$$

The main difficulty of this system is that strict hyperbolicity fails, and the flux function is only Lipschitz continuous at the vacuum state $\rho=0$. Nevertheless, a compactness theorem has been established by using only weak entropy pairs consisting of those η vanishing on the vacuum $\rho=0$ for any fixed $\frac{m}{\rho}\in\mathbb{R}$. For example, the mechanical energy-energy flux pair

$$\eta_*=\frac{1}{2}\frac{m^2}{\rho}+\rho\int_0^\rho\frac{p(r)}{r^2}dr,\quad q_*=\frac{m^3}{2\rho^2}+m\int_0^\rho\frac{p'(r)}{r}dr\tag{5.15}$$

is a convex weak entropy pair. One can prove that, for $0\le\rho\le C$, $|\frac{m}{\rho}|\le C$,

$$|\nabla\eta(\rho,m)|\le C_\eta,\qquad |\nabla^2\eta(\rho,m)|\le C_\eta\nabla^2\eta_*(\rho,m),$$

for any weak entropy η, with C_η independent of (ρ,m).

Theorem 5.7 (Chen-LeFloch [6]). *Assume that a sequence of functions $(\rho^\varepsilon,m^\varepsilon)$ satisfies that*

(i) *There exists some $C>0$ independent of ε such that*

$$0\le\rho^\varepsilon(t,x)\le C,\qquad |m^\varepsilon(t,x)|\le C\,\rho^\varepsilon(t,x)\qquad \textit{for a.e. } (t,x);\tag{5.16}$$

(ii) *For any weak entropy pair (η, q) of (1.3), (1.4), and (5.11)–(5.13),*

$$\partial_t \eta(\rho^\varepsilon, m^\varepsilon) + \partial_x q(\rho^\varepsilon, m^\varepsilon) \quad \text{is compact in } H^{-1}_{loc}(\mathbb{R}^2_+). \tag{5.17}$$

Then the sequence $(\rho^\varepsilon, m^\varepsilon)$ is compact in $L^1_{loc}(\mathbb{R}^2_+)$.

Moreover, there exists a global solution $(\rho(t,x), m(t,x))$ of the Cauchy problem (1.3), (1.4), and (5.11)–(5.13), satisfying

$$0 \le \rho(t,x) \le C, \qquad \big|m(t,x)\big| \le C\rho(t,x),$$

for some C depending only on C_0 and γ, and

$$\partial_t \eta(\rho, m) + \partial_x q(\rho, m) \le 0$$

in the sense of distributions for any convex weak entropy pair (η, q).

Furthermore, the bounded solution operator $(\rho, m)(t,\cdot) = S_t(\rho_0, m_0)(\cdot)$ is compact in L^1 for $t > 0$.

For polytropic perfect gases,

$$p(\rho) = \kappa_1 \rho^\gamma, \qquad \gamma > 1, \tag{5.18}$$

the similar results were proved by DiPerna [14] for the case $\gamma = \frac{N+1}{N}, N \ge 5$ odd, for $L^2 \cap L^\infty(\mathbb{R})$ initial data, by Ding-Chen-Luo [12] and Chen [4] for $1 < \gamma \le 5/3$ for usual gases with general L^∞ initial data. The results are also true for $\gamma \ge 3$ due to Lions-Perthame-Tadmor [19] and for $5/3 < \gamma < 3$ due to Lions-Perthame-Souganidis [20].

Proposition 5.8. *For the isentropic Euler equations (1.3) with general pressure law (5.12)–(5.13), the scaling sequence $\mathbf{u}^{(s,y),r_k} = (\rho^{(s,y),r_k}, m^{(s,y),r_k})$ is compact in L^1. That is, given $r_k \to 0$, there exists a subsequence (still denoted as r_k) and a function $\mathbf{u}_\infty^{(s,y)} \in L^\infty(\mathbb{R}^2)$ such that, for $\mathcal{L}^2$-almost every (t,x),*

$$\mathbf{u}^{(s,y),r_k}(t,x) \to \mathbf{u}_\infty^{(s,y)}(t,x) \qquad \text{as } r_k \to 0.$$

Proof. Since $(\rho, m)(t,x) \in L^\infty$,

$$0 \le \rho(t,x) \le C, \quad |m(t,x)| \le C\rho(t,x)$$

for some $C > 0$, the scaling sequence $(\rho^{(s,y),r_k}, m^{(s,y),r_k})$ has the same bound:

$$0 \le \rho^{(s,y),r_k}(t,x) \le C, \quad |m^{(s,y),r_k}(t,x)| \le C\,\rho^{(s,y),r_k}(t,x).$$

Furthermore, from Proposition 5.6, we obtain that, for any weak entropy pair (η, q),

$$\partial_t \eta(\mathbf{u}^{(s,y),r_k}) + \partial_x q(\mathbf{u}^{(s,y),r_k}) \qquad \text{is compact in } H^{-1}_{loc}.$$

The compactness theorem (Theorem 5.7) for the isentropic Euler equations yields that there exists a subsequence (still denoted as r_k) and a function $\mathbf{u}_\infty^{(s,y)} \in L^\infty(\mathbb{R}^2)$ such that, for almost every $(t,x) \in \mathbb{R}^2$,

$$\mathbf{u}^{(s,y),r_k}(t,x) \to \mathbf{u}_\infty^{(s,y)}(t,x) \qquad \text{as } r_k \to 0.$$

□

5.3. **The limit of the rescalings.** We consider the set of all limits of the rescaling sequence $\mathbf{u}^{(s,\mathbf{y}),r}$; that is, we define

$$\mathbb{L}^{(s,\mathbf{y})} := \{\mathbf{u}_\infty^{(s,\mathbf{y})} : \mathbf{u}^{(s,\mathbf{y}),r_k} \to \mathbf{u}_\infty^{(s,\mathbf{y})} \text{ in } L^1_{loc} \text{ for some sequence } r_k \to 0\}.$$

Any $\mathbf{u}_\infty^{(s,\mathbf{y})} \in \mathbb{L}^{(s,\mathbf{y})}$ and $\mu_{\eta,\infty}^{(s,\mathbf{y})}$ obtained from the same subsequence $r_k \to 0$ satisfies the equation:

$$\partial_t \eta(\mathbf{u}_\infty^{(s,\mathbf{y})}) + \nabla_{\mathbf{x}} \cdot \mathbf{q}(\mathbf{u}_\infty^{(s,\mathbf{y})}) = \mu_{\eta,\infty}^{(s,\mathbf{y})}.$$

We now introduce *an approach* to show that the limit of the rescalings is constant on $\mathbb{R}^2$ (a Liouville-type theorem) through the isentropic Euler equations (1.3) under the following framework:

Framework (A): Each $\mathbf{u}_\infty^{(s,y)} \in \mathbb{L}^{(s,y)}$ satisfies the following condition: $(\eta(\mathbf{u}_\infty^{(s,y)}(t,x)), q(\mathbf{u}_\infty^{(s,y)}(t,x)))\cdot\boldsymbol{\nu}(\xi),\ \xi = \frac{x}{t},$ is self-similar for any unit vector $\boldsymbol{\nu}(\xi)$. That is, there exists $\alpha_\eta \in L^\infty$ such that, for almost every ξ,

$$(\eta(\mathbf{u}_\infty^{(s,y)}(t,x)), q(\mathbf{u}_\infty^{(s,y)}(t,x))) \cdot \boldsymbol{\nu}(\xi) = \alpha_\eta(\xi), \qquad \xi = \frac{x}{t},$$

for any entropy pair (η, q).

1. We first have

Lemma 5.9. *Consider the isentropic Euler equations* (1.3) *with* (5.18). *If* $\mathbf{u}_\infty^{(s,y)}(t,x)$ *satisfies Framework* (A) *for any entropy pair* (η,q), *then* $\mathbf{u}_\infty^{(s,y)}(t,x)$ *is self-similar. That is, there exists* $\mathbf{v} \in L^\infty$ *such that, for almost every* ξ,

$$\mathbf{u}_\infty^{(s,y)}(t,x) = \mathbf{v}(\xi), \qquad \xi = \frac{x}{t}.$$

This can be achieved as follows. For simplicity of notation, we drop the index (s,y) below. Set $\xi = \frac{x}{t}$. If we write $\boldsymbol{\nu}(\xi) = (\nu_1(\xi), \nu_2(\xi))$, Framework (A) yields

$$\nu_1(\xi)\eta(\mathbf{u}_\infty(t,x)) + \nu_2(\xi) q(\mathbf{u}_\infty(t,x)) = \alpha_\eta(\xi)$$

for every entropy pair (η,q), where α_η is self-similar (here we have omitted the fixed point (s,y) to simplify notation). Choosing $(\eta,q) = (\rho,m)$, we obtain

$$\nu_1(\xi)\rho_\infty + \nu_2(\xi) m_\infty = \alpha_1(\xi). \tag{5.19}$$

Choosing $(\eta, q) = (m, \frac{m^2}{\rho} + p(\rho))$, we obtain

$$\nu_1(\xi) m_\infty + \nu_2(\xi)\Big(\frac{m_\infty^2}{\rho_\infty} + \frac{\rho_\infty^\gamma}{\gamma}\Big) = \alpha_2(\xi); \tag{5.20}$$

and choosing now

$$(\eta, q) = (\frac{1}{2}\frac{m^2}{\rho} + \frac{p}{\gamma-1}, \frac{m}{\rho}(\frac{1}{2}\frac{m^2}{\rho} + \frac{\gamma p}{\gamma - 1})),$$

we obtain

$$\nu_1(\xi)\Big(\frac{1}{2}\frac{m_\infty^2}{\rho_\infty} + \frac{\rho_\infty^\gamma}{\gamma(\gamma-1)}\Big) + \nu_2(\xi)\frac{m_\infty}{\rho_\infty}\Big(\frac{1}{2}\frac{m_\infty^2}{\rho_\infty} + \frac{\rho_\infty^\gamma}{\gamma-1}\Big) = \alpha_3(\xi), \tag{5.21}$$

where $\alpha_i(\xi), i = 1,2,3,$ are self-similar. From (5.19)–(5.21), we obtain the following equation for ρ_∞:

$$a(\xi)\rho_\infty^2 + b(\xi)\rho_\infty + c(\xi) = 0, \tag{5.22}$$

where

$$a(\xi) = -\frac{3\alpha_1\tilde{\nu}^2}{2} + \tilde{\nu} - \alpha_2\tilde{\nu} - \alpha_3, \qquad b(\xi) = -\alpha_1\tilde{\nu} + \frac{\gamma}{\gamma-1}\alpha_1\alpha_2 + \frac{2\gamma-1}{\gamma-1}\alpha_1^2\tilde{\nu},$$

$$c(\xi) = -\frac{(1+\gamma)\alpha_1^3}{2(\gamma-1)}, \qquad \tilde{\nu} = \frac{\nu_1}{\nu_2}.$$

Choosing now

$$\eta = \frac{m^3}{\rho^2} + \frac{6}{\gamma(\gamma-1)}\rho^{\gamma-1}m, \quad q = \frac{m^4}{\rho^3} + \frac{6(\gamma+1)}{\gamma(\gamma-1)}\rho^{\gamma-2}m^2 + \frac{6}{\gamma(\gamma-1)(2\gamma-1)}\rho^{2\gamma-1}, \tag{5.23}$$

we have

$$\nu_1(\xi)\Big(\frac{m^3}{\rho^2} + \frac{6}{\gamma(\gamma-1)}\rho^{\gamma-1}m\Big) + \nu_2(\xi)\Big(\frac{m^4}{\rho^3} + \frac{6(\gamma+1)}{\gamma(\gamma-1)}\rho^{\gamma-2}m^2 + \frac{6}{\gamma(\gamma-1)(2\gamma-1)}\rho^{2\gamma-1}\Big) = \alpha_4(\xi). \tag{5.24}$$

Working now with (5.19)–(5.20) and (5.24), we obtain another quadratic equation for ρ_∞:

$$\tilde{a}(\xi)\rho_\infty^2 + \tilde{b}(\xi)\rho_\infty + \tilde{c}(\xi) = 0. \tag{5.25}$$

From (5.22) and (5.25), since $\nu(\xi)$ is any unit vector, we can solve for ρ_∞ in terms of self-similar functions; that is, ρ_∞ is self-similar. From (5.19), we conclude that m_∞ is also self-similar.

2. Then we have

Lemma 5.10. *If $\mathbf{u}(t,x) \in L^\infty$ is self-similar and satisfies*

$$\partial_t\eta(\mathbf{u}) + \partial_x q(\mathbf{u}) = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^2), \tag{5.26}$$

where (η, q) is an entropy pair, then, for $\mathbf{v}(\xi) := \mathbf{u}(t,x)$ with $\xi = \frac{x}{t}$,

$$q(\mathbf{v}(\xi)) - \xi\eta(\mathbf{v}(\xi)) \tag{5.27}$$

is locally Lipschitz on $\mathbb{R}$, which implies that, in the (t,x)-plane, (5.27) is continuous on $\mathbb{R}^2 \setminus \{t=0\}$.

This can be proved as follows. Since $\mathbf{u}$ is self-similar, then

$$\mathbf{u}(t,x) = \mathbf{v}(\xi) \in L^\infty(\mathbb{R}), \qquad \xi = \frac{x}{t},$$

and $\mathbf{u}$ satisfies

$$\partial_t\eta(\mathbf{v}(\frac{x}{t})) + \partial_x q(\mathbf{v}(\frac{x}{t})) = 0 \qquad \text{in } \mathcal{D}'(\mathbb{R}^2) \tag{5.28}$$

for the entropy pair (η, q). That is, for any test function $\psi(t,\xi)$ with compact support in $\mathbb{R}^2$, we have

$$\int_{\mathbb{R}^2} \big(\eta(\mathbf{v}(\frac{x}{t}))\psi_t + q(\mathbf{v}(\frac{x}{t}))\psi_x\big)dtdx = 0. \tag{5.29}$$

Using $x = \xi t$, we obtain the corresponding Jacobian:

$$\frac{\partial(t,x)}{\partial(t,\xi)} = t. \tag{5.30}$$

We now change the variables in (5.29) and use (5.30) to write

$$\int_{\mathbb{R}^2} \Big(\eta(\mathbf{v}(\xi))(\psi_t - \xi\psi_\xi) + q(\mathbf{v}(\xi))\psi_\xi\Big)dtd\xi = 0. \tag{5.31}$$

Employing the test function $\psi(t,\xi) = \varphi(\xi)\chi(t)$ in (5.31) and using the relation $t\chi'(t) = (t\chi(t))' - \chi(t)$ yield

$$\int_{\mathbb{R}^2} \Big(\eta(\mathbf{v}(\xi))\big(((t\chi(t))' - \chi(t))\varphi(\xi) - \xi\varphi'(\xi)\chi(t)\big) + q(\mathbf{v}(\xi))\varphi'(\xi)\chi(t)\Big)dtd\xi = 0. \tag{5.32}$$

Since

$$\int_{\mathbb{R}} \eta(\mathbf{v}(\xi))\big(\int_{\mathbb{R}} (t\chi(t))'dt\big)d\xi = 0,$$

and (5.32) holds for any test function $\chi(t) \in C_c^\infty(\mathbb{R})$, we obtain

$$\int_{\mathbb{R}^2} \big(q(\mathbf{v}(\xi)) - \xi\eta(\mathbf{v}(\xi))\big)\varphi'(\xi)d\xi = \int_{\mathbb{R}^2} \eta(\mathbf{v}(\xi))\varphi(\xi)d\xi, \tag{5.33}$$

which implies that

$$\big(q(\mathbf{v}(\xi)) - \xi\eta(\mathbf{v}(\xi))\big)' \in L^\infty(\mathbb{R}),$$

since $\eta(\mathbf{v}(\xi)) \in L^\infty(\mathbb{R})$.

We conclude that

$$q(\mathbf{v}(\xi)) - \xi\eta(\mathbf{v}(\xi)) \qquad \text{is locally Lipschitz in } \mathbb{R}$$

for the entropy pair (η, q). This implies that, in the (t,x)-plane, $q(\mathbf{v}(\xi)) - \xi\eta(\mathbf{v}(\xi))$ is continuous on $\mathbb{R}^2 \setminus \{t = 0\}$.

3. Show that, if $\mathbf{u} = (\rho, m) \in L^\infty$ is self-similar and satisfies (5.26) in the whole space $\mathbb{R}^2$ in the sense of distributions, then $\mathbf{u}$ is continuous on $\mathbb{R}^2$.

To achieve this, it requires to follow the argument as for Lemma 5.9, employ Lemma 5.10, and use several entropy-entropy flux pairs and the properties of $p(\rho)$.

4. With these, we have

Theorem 5.11 (Liouville-Type Theorem)**.** *Consider system* (1.3) *with general pressure law satisfying* (1.4)*. If* $\mathbf{u} = (\rho, m) \in L^\infty$ *is a continuous self-similar solution of* (1.3) *with* (1.4) *in the whole space* $\mathbb{R}^2$*, then* $\mathbf{u}$ *is constant on* $\mathbb{R}^2$*.*

This can be proved as follows. On the contrary, if the continuous self-similar solution is not constant, then it must contain at least one rarefaction wave on which $\rho > 0$ in the upper-half plane $t > 0$. Since system (1.3) is genuinely nonlinear and strictly hyperbolic when $\rho > 0$, then the solution must contain a corresponding Lax shock wave formed by the compressibility of the characteristics in the lower-half plane $t < 0$, which is a contradiction with the continuity of the solution.

5.4. Regularity of u outside the shock location J.

Theorem 5.12. *Let* $\mathbf{u} \in L^\infty$ *be an entropy solution of* (1.3) *with* (5.18)*. Let* $(s,y) \notin J$ *so that each* $\mathbf{u}_\infty^{(s,y)} \in \mathbb{L}^{(s,y)}$ *is a continuous self-similar solution, then* $\mathbf{u}$ *satisfies the Vanishing Mean Oscillation (VMO) property at* (s,y)*:*

$$\lim_{r\to 0} \frac{1}{r^2}\int_{B_r(s,y)} |\mathbf{u}(t,x) - \bar{\mathbf{u}}_r(s,y)|dtdx = 0, \tag{5.34}$$

where $\bar{\mathbf{u}}_r(s,y)$ *is the average of* $\mathbf{u}$ *in the ball* $B_r(s,y)$*.*

Proof. Fix $(s,y) \notin J$, and let $r_k \to 0$. We perform a blow-up around (s,y) and consider the rescalings $\mathbf{u}^{(s,y),r_k}$. Proposition 5.8 yields a subsequence (still denoted as r_k) such that $\mathbf{u}^{(s,y),r_k} \to \mathbf{u}_\infty^{(s,y)}$ pointwise almost everywhere. Since (s,y) is not in the support of the measures μ_η, then $\mu_{\eta,\infty}^{(s,y)} = 0$, and hence

$$\partial_t \eta(\mathbf{u}_\infty^{(s,y)}) + \partial_x q(\mathbf{u}_\infty^{(s,y)}) = 0 \tag{5.35}$$

in the sense of distributions for any entropy pair (η, q) in the whole space $\mathbb{R}^2$. Then Theorem 5.11 yield

$$\mathbf{u}_\infty^{(s,y)} \qquad \text{is constant.}$$

Since $\mathbf{u}^{(s,y),r_k} \to \mathbf{u}_\infty^{(s,y)}$ pointwise a.e. and $\mathbf{u}^{(s,y),r_k}$ is uniformly bounded, then

$$\mathbf{u}^{(s,y),r_k} \to \mathbf{u}_\infty^{(s,y)} \qquad \text{in } L^1_{loc}(\mathbb{R}^2). \tag{5.36}$$

We have

$$\bar{\mathbf{u}}_{r_k}(s,y) = \frac{1}{|B_{r_k}(s,y)|} \int_{B_{r_k}(s,y)} \mathbf{u}(t,x)\, dtdx.$$

Then

$$\begin{aligned} &\frac{1}{r_k^2} \int_{B_{r_k}(s,y)} |\mathbf{u}(t,x) - \bar{\mathbf{u}}_{r_k}(s,y)|\, dtdx \\ &\le \frac{1}{r_k^2} \int_{B_{r_k}(s,y)} |\mathbf{u}(t,x) - \mathbf{u}_\infty^{(s,y)}|\, dtdx + \frac{1}{r_k^2} \int_{B_{r_k}(s,y)} |\bar{\mathbf{u}}_{r_k}(s,y) - \mathbf{u}_\infty^{(s,y)}|\, dtdx. \end{aligned}$$

We compute

$$\begin{aligned} &\frac{1}{r_k^2} \int_{B_{r_k}(s,y)} |\bar{\mathbf{u}}_{r_k}(s,y) - \mathbf{u}_\infty^{(s,y)}|\, dtdx \\ &= |B_1(\mathbf{0})| |\bar{\mathbf{u}}_{r_k}(s,y) - \mathbf{u}_\infty^{(s,y)}| \\ &= |B_1(\mathbf{0})| \Big| \frac{1}{|B_{r_k}(s,y)|} \int_{B_{r_k}(s,y)} \big(\mathbf{u}(t,x) - \mathbf{u}_\infty^{(s,y)}\big) dtdx \Big| \\ &\le \frac{1}{{r_k}^2} \int_{B_{r_k}(s,y)} |\mathbf{u}(t,x) - \mathbf{u}_\infty^{(s,y)}|\, dtdx. \end{aligned}$$

On the other hand, making the change of variables:

$$t = s + r_k\tau, \qquad x = y + r_k\xi,$$

we have

$$\begin{aligned} \frac{1}{r_k^2} \int_{B_{r_k}(s,y)} |\mathbf{u}(t,x) - \mathbf{u}_\infty^{(s,y)}|\, dtdx &= \int_{B_1(\mathbf{0})} |\mathbf{u}(s + r_k\tau, y + r_k\xi) - \mathbf{u}_\infty^{(s,y)}|\, d\tau d\xi \\ &= \int_{B_1(\mathbf{0})} |\mathbf{u}^{(s,y),r_k}(\tau,\xi) - \mathbf{u}_\infty^{(s,y)}|\, d\tau d\xi \to 0 \end{aligned}$$

as $r_k \to 0$ due to the convergence (5.36). This gives

$$\lim_{r_k \to 0} \frac{1}{r_k^2} \int_{B_{r_k}(s,y)} |\mathbf{u}(t,x) - \bar{\mathbf{u}}_{r_k}(s,y)| dtdx = 0 \tag{5.37}$$

and the desired property (5.34). The dependence of (5.34) on the sequence r_k is illusory. In fact, if there were a sequence $r_k \to 0$ for which

$$\lim_{r_k \to 0} \frac{1}{r_k^2} \int_{B_{r_k}(s,y)} |\mathbf{u}(t,x) - \bar{\mathbf{u}}_{r_k}(s,y)| dtdx = l \ne 0, \tag{5.38}$$

then, proceeding as above with this sequence r_k, we would obtain (5.37) for a further subsequence r_{k_j}, which would contradict (5.38). $\square$

5.5. **Existence of strong traces of $\mathbf{u}$ on hyperplanes.** Fix $s \in \mathbb{R}$ and define

$$\Pi^{\pm} := \{(t, \mathbf{x}) \,:\, \pm(t-s) > 0, \mathbf{x} \in \mathbb{R}^d\}.$$

Therefore, for the blow-up sequence $\mathbf{u}^{(s,\mathbf{y}),r_k}$ around $(s, \mathbf{y}) \in \partial\Pi^+$, if there exist constants $\mathbf{u}_\infty^{(s,\mathbf{y}),+}$ and $\mathbf{u}_\infty^{(s,\mathbf{y}),-}$ and a subsequence (still denoted as) r_k such that

$$\mathbf{u}^{(s,\mathbf{y}),r_k} \to \mathbf{u}_\infty^{(s,\mathbf{y}),+} \text{ in } L^1_{loc}(\Pi^+) \quad \text{and} \quad \mathbf{u}^{(s,\mathbf{y}),r_k} \to \mathbf{u}_\infty^{(s,\mathbf{y}),-} \text{ in } L^1_{loc}(\Pi^-), \quad (5.39)$$

(where, as before, the constants may depend on the blow-up sequence $r_k \to 0$), then, proceeding exactly as in Theorem 5.12, we obtain

$$\lim_{r\to 0} \frac{1}{r^{d+1}} \int_{B_r^{\pm}(s,\mathbf{y})} |\mathbf{u}(t,\mathbf{x}) - \bar{\mathbf{u}}_r^{\pm}(s,\mathbf{y})| dt d\mathbf{x} = 0, \quad (5.40)$$

where $\bar{\mathbf{u}}_r^{\pm}(s,\mathbf{y})$ are the averages of $\mathbf{u}$ in the half balls $B_r^{\pm}(s,\mathbf{y})$, respectively.

The next result is Theorem 3.7 which shows that, if $\mathbf{u}$ satisfies (5.40) (and therefore (3.1) as well), then this regularity can be improved to the existence of traces of $\eta(\mathbf{u})$, for any entropy η.

Theorem 5.13. *Let $\eta : \mathbb{R}^m \to \mathbb{R}$ be any entropy. If $\mathbf{u} \in L^\infty(\mathbb{R}^{d+1}, \mathbb{R}^m)$ is an entropy solution of (1.1) and satisfies (5.40) for $\mathcal{H}^d$-almost every $(s, \mathbf{y}) \in \partial\Pi^+$, then there exist $\eta(\mathbf{u})^+ \in L^\infty(\partial\Pi^+)$ and $\eta(\mathbf{u})^- \in L^\infty(\partial\Pi^-)$ such that, for $\mathcal{H}^d$-almost every $(s, \mathbf{y}) \in \partial\Pi^+$,*

$$\lim_{r\to 0} \frac{1}{r^{d+1}} \int_{B_r^{\pm}(s,\mathbf{y})} \eta(\mathbf{u}(t,\mathbf{x})) dt d\mathbf{x} = \eta(\mathbf{u})^{\pm}(s,\mathbf{y}).$$

We now go back to the one-dimensional system (1.3). The next theorem shows that, if each $\mathbf{u}_\infty^{(s,y)} \in \mathbb{L}^{s,y}$ is a constant that satisfies (5.39), then the solution $\mathbf{u} = (\rho, m)$ has strong traces on any hyperplane $t = s, x \in \mathbb{R}$.

Theorem 5.14. *Let $\mathbf{u} = (\rho, m)$ be an entropy solution of (1.3). Fix any hyperplane $\{(s, x) : x \in \mathbb{R}\}$. Then, if each $\mathbf{u}_\infty^{(s,y)} \in \mathbb{L}^{s,y}$ is a constant that satisfies (5.39) for $\mathcal{H}^1$-almost every $(s, y) \in \partial\Pi^+$, then there exist $\rho^+, m^+ \in L^\infty(\partial\Pi^+)$ and $\rho^-, m^- \in L^\infty(\partial\Pi^-)$ such that*

$$\lim_{r\to 0} \frac{1}{r^2} \int_{B_r^{\pm}(s,y)} |\rho(t,x) - \rho^{\pm}(s,y)| dt dx = 0,$$
$$\lim_{r\to 0} \frac{1}{r^2} \int_{B_r^{\pm}(s,y)} |m(t,x) - m^{\pm}(s,y)| dt dx = 0.$$

Proof. Since (5.39) implies (5.40), then, from Theorem 5.13, there exists $\tilde{\rho}^+ \in L^\infty(\partial\Pi^+)$ such that, for $\mathcal{H}^1$-almost every $(s, y) \in \partial\Pi^+$,

$$\lim_{r\to 0} \frac{1}{r^2} \int_{B_r^{+}(s,y)} \rho(t,x) dt dx = \tilde{\rho}^+(s,y). \quad (5.41)$$

On the other hand, (5.39) yields

$$\lim_{r_k\to 0} \frac{1}{r_k^2} \int_{B_{r_k}^{+}(s,y)} |\rho(t,x) - \rho_\infty^{(s,y),+}| dt dx = 0 \quad (5.42)$$

for some sequence $r_k \to 0$ and a constant $\rho_\infty^{(s,y),+}$. From (5.41) and (5.42), it follows that, for $\mathcal{H}^1$-almost every $(s, y) \in \partial\Pi^+$,

$$\rho_\infty^{(s,y),+} = C\tilde{\rho}^+(s, y), \tag{5.43}$$

where C is a constant independent of (s, y). That is, (5.43) says that the constant $\rho_\infty^{(s,y),+}$ is actually independent of the blow-up sequence $r_k \to 0$ and thus it is unique. Therefore, $\rho^+ := C\tilde{\rho}^+$ satisfies

$$\lim_{r\to 0} \frac{1}{r^2} \int_{B_r^+(s,y)} |\rho(t, x) - \rho^+(s, y)| dtdx = 0.$$

The same reasoning works for ρ^-, m^+, and m^-. □

Acknowledgments. The authors would like to thank Camillo De Lellis for many useful discussions and Constantine Dafermos for helpful comments. Gui-Qiang Chen's research was supported in part by the US National Science Foundation under Grants DMS-0935967 and DMS-0807551, the NSFC under a Joint Project Grant 10728101 (China), the Royal Society-Wolfson Research Merit Award (UK), and the UK EPSRC Science and Innovation Award to the Oxford Centre for Nonlinear PDE (EP/E035027/1). Monica Torres's research was supported in part by the National Science Foundation under grant DMS-0540869.

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Received March 2010; revised December 2010.

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