1. (10 points) Find the maximum value of the function

$$f(x,y) = 3x^2 - 6x + 3y^2 - 12y + 17$$

subject to the constraint $x^2 + y^2 = 5$.

A. 2
B. 32
C. 62
D. 92
E. 122

We use Lagrange multipliers.
Set $g(x,y) = x^2+y^2$. Then
$\nabla F = \langle 6x-6,\ 6y-12 \rangle$
$\nabla g = \langle 2x, 2y \rangle$

so we have the system

$$\begin{cases} 6x-6 = 2\lambda x \\ 6y-12 = 2\lambda y \\ x^2+y^2 = 5. \end{cases}$$

First, $x=0 \overset{\text{Eqn 1}}{\Rightarrow} -6=0$, so $x \neq 0$. We solve Eqn 1 for λ:

$$\frac{3x-3}{x} = \lambda$$

Plugging this into Eqn 2,

$$6y-12 = \frac{6x-6}{x} y$$
$$6xy-12x = 6xy-6y$$
$$-12x = -6y$$
$$y = 2x.$$

Plugging this into Eqn 3,

$$x^2+4x^2 = 5$$
$$5x^2 = 5$$
$$x = \pm 1.$$

So the solutions for x & y are $(1,2)$ and $(-1,-2)$.

$F(1,2) = 2$

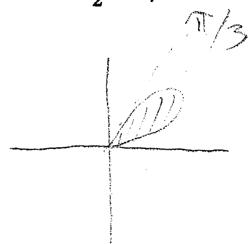
$F(-1,-2) = \boxed{62}$

2. (10 points) Compute the area of one leaf of the graph of the polar function

$$r = \sin(3\theta).$$

(Recall that $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$)

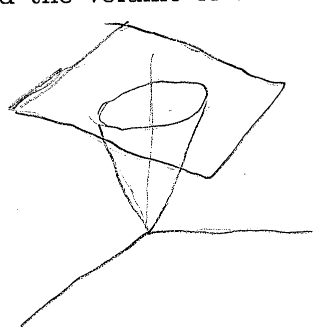
A. $\pi/12$
B. $1/12$
C. $2\pi/3$
D. $2/3$
E. $\pi/3$



$$\int_0^{\pi/3}\int_0^{\sin(3\theta)} r\,dr\,d\theta$$
$$= \int_0^{\pi/3} \left(\frac{r^2}{2}\right)\Big|_{r=0}^{r=\sin(3\theta)} d\theta$$
$$= \frac{1}{2}\int_0^{\pi/3} \sin^2(3\theta)\,d\theta$$
$$= \frac{1}{4}\int_0^{\pi/3} (1-\cos(6\theta))\,d\theta$$
$$= \frac{1}{4}\left(\theta - \frac{1}{6}\sin(6\theta)\right)\Big|_0^{\pi/3}$$
$$= \frac{1}{4}\left[\left(\frac{\pi}{3} - \frac{1}{6}\sin(2\pi)\right) - \left(0 - \frac{1}{6}\sin 0\right)\right]$$
$$= \frac{1}{4}\left(\frac{\pi}{3}\right) = \frac{\pi}{12}.$$

3. (10 points) Find the volume of the solid bounded by the surfaces $z = \sqrt{x^2+y^2}$ and $z = 2$.

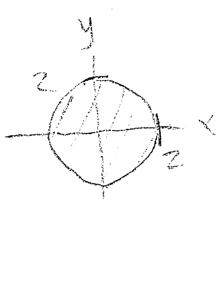
A. 2π
B. $8\pi/3$
C. $10\pi/3$
D. 4π
E. $14\pi/3$



The solid is below the plane and inside the cone. The intersection has equation

$$2 = \sqrt{x^2+y^2}$$
$$4 = x^2+y^2$$

in the plane $z=2$, so the solid lies above the disk $\{(x,y) \mid x^2+y^2 \le 4\}$ in the xy-plane. In cylindrical coordinates,



$$V = \int_0^{2\pi}\int_0^2\int_r^2 r\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 (2r - r^2)\,dr\,d\theta$$
$$= \int_0^{2\pi} \left(4 - \frac{8}{3}\right) d\theta$$
$$= \frac{4}{3}\int_0^{2\pi} d\theta$$
$$= \frac{8\pi}{3}.$$

4. (10 points) Compute

$$\int_0^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dy\, dx.$$

A. $\pi/4$

B. $\pi/2$

C. π

D. 2π

E. 4π

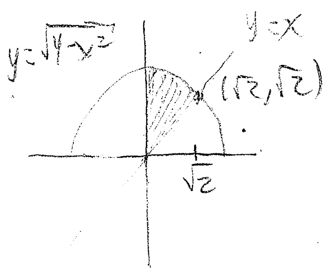
The integral is simpler in spherical coordinates.

The z limits are from $z=0$ to $z=\sqrt{4-x^2-y^2}$.

$$\Updownarrow$$

$$x^2+y^2+z^2=4,\ z\geq 0,$$

the upper hemisphere of the sphere of radius 2 centered at $(0,0,0)$.



The region determined by the x and y limits is shown to the left. So the integral, in spherical coordinates, is

$$\int_0^{\pi/2}\int_{\pi/4}^{\pi/2}\int_0^2 \rho\cdot\rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$$

$$= 4\int_0^{\pi/2}\int_{\pi/4}^{\pi/2} \sin\varphi\, d\theta\, d\varphi$$

$$= \pi \int_0^{\pi/2} \sin\varphi\, d\varphi$$

$$= \pi(-\cos\varphi)\Big|_0^{\pi/2}$$

$$= \pi(1-0)$$

$$= \pi$$

5. (10 points) Suppose a room has temperature

$$T(x, y, z) = x^2 \cos z + xze^y$$

at a point (x, y, z) in the room. Find the unit vector which gives the direction in which T increases most rapidly at the point $(1, 0, 0)$.

the unit vector should be in the direction of $\nabla T(1,0,0)$.

$\nabla T = \langle 2x\cos z + ze^y, xze^y, -x^2 \sin z + xe^y \rangle$

$\nabla T(1,0,0) = \langle 2, 0, 1 \rangle$

$$\boxed{\frac{1}{\sqrt{5}} \langle 2, 0, 1 \rangle}$$

6. (10 points) Classify all critical points of the function

$$f(x, y) = 2x^2 + 2yx^2 - y^2.$$

We need to find the critical points.

$f_x = 4x + 4xy$, $f_y = 2x^2 - 2y$

$$\begin{cases} 0 = 4x + 4xy \\ 0 = 2x^2 - 2y \end{cases}$$

Eqn 2 says $y = x^2$. Plugging this into eqn 1 gives

$0 = 4x + 4x^3 = 4x(1 + x^2)$

$\Rightarrow x = 0$

This gives 1 critical point: $(0,0)$.

$f_{xx} = 4 + 4y$, $f_{yy} = -2$, $f_{xy} = 4x$.

$D(x,y) = (4+4y)(-2) - (4x)^2$

$= -8 - 8y - 16x^2$.

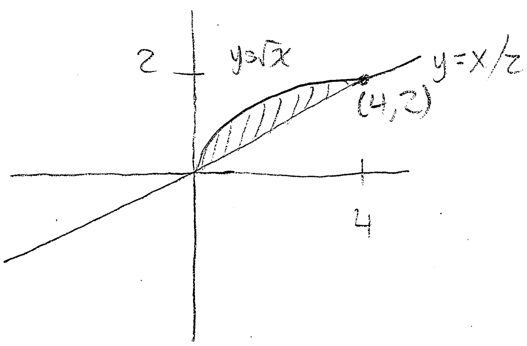
Using the second derivatives test,

$D(0,0) = -8 < 0 \Rightarrow (0,0)$ is a saddle point.

$\boxed{\text{The only critical point, } (0,0), \text{ is a saddle point.}}$

7. (10 points) Let D be the region bounded by the curves $y = \sqrt{x}$ and $y = x/2$. Write down (but do not evaluate) **two** iterated integrals with **different** orders of integration that can be used to compute

$$\iint_D \sin(xy)\, dA.$$



Treating y as a function of x:

$D = \{(x,y) \mid 0 \le x \le 4,\ \frac{x}{2} \le y \le x^2\}$

Treating x as a function of y:

$D = \{(x,y) \mid y^2 \le x \le 2y,\ 0 \le y \le 2\}$

$$\boxed{\int_0^4 \int_{x/2}^{\sqrt{x}} \sin(xy)\, dy\, dx}$$

$$\boxed{\int_0^2 \int_{y^2}^{2y} \sin(xy)\, dx\, dy}$$

8. (10 points) Compute

$$\int_0^1 \int_0^{2y^{2/3}} x^2\sqrt{x^3+y^2}\,dx\,dy.$$

Let $u = x^3+y^2$. Then $\frac{du}{3} = x^2\,dx$.

$\frac{1}{3}\int_0^1 \int_{y^2}^{9y^2} u^{1/2}\,du\,dy$

$= \frac{1}{3}\int_0^1 \frac{2}{3}u^{3/2}\Big|_{u=y^2}^{u=9y^2}\,dy$

$= \frac{2}{9}\int_0^1 (27y^3 - y^3)\,dy$

$= \frac{52}{9}\int_0^1 y^3\,dy = \frac{52}{9}\cdot\frac{y^4}{4}\Big|_0^1 = \frac{13}{9}$

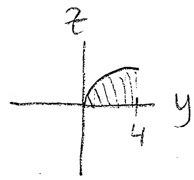
$$\boxed{13/9}$$

9. (10 points) Write down (but do not evaluate) an iterated integral that gives the value of

$$\iiint_E xyz\,dV$$

where E is the solid in the first octant bounded by $z = \sqrt{y}$, $z = x-2$ and $y = 4$.

Projection onto yz-plane:



Regardless of (y,z) in this region, the smallest x-value is 0 and the largest is $z+2$ (From $z = x-2$).

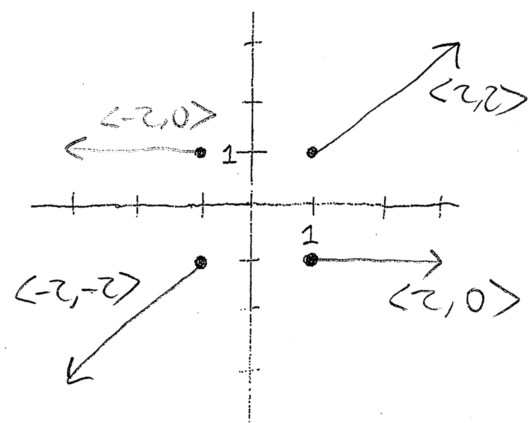
$E = \{(x,y,z) \mid 0 \le x \le z+2,\ 0 \le y \le 4,\ 0 \le z \le \sqrt{y}\}$

$$\boxed{\int_0^4 \int_0^{\sqrt{y}} \int_0^{z+2} xyz\,dx\,dz\,dy}$$

10. Consider the vector field

$$\mathbf{F}(x,y) = \langle 2x, x+y \rangle.$$

(a) (4 points) Sketch the vectors $\mathbf{F}(x,y)$ at each point depicted on the graph below. **Draw directly on the graph.**



(b) (6 points) Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the line segment from (1,1) to (3,2).

A direction vector for the line containing C (in the direction for this orientation) is $\langle 3,2\rangle - \langle 1,1\rangle = \langle 2,1\rangle$, so a vector equation for C is

$\vec{r}(t) = \langle 1,1\rangle + t\langle 2,1\rangle,\ 0 \le t \le 1.$

$= \langle 2t+1, t+1\rangle.$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \langle 2(2t+1), 2t+1+t+1\rangle \cdot \langle 2,1\rangle\, dt$$
$$= \int_0^1 (8t+4+3t+2)\,dt$$
$$= \int_0^1 (11t+6)\,dt$$
$$= \left(\tfrac{11}{2}t^2 + 6t\right)\Big|_0^1$$
$$= \tfrac{11}{2} + 6$$
$$= \tfrac{23}{2}$$

$\boxed{23/2}$