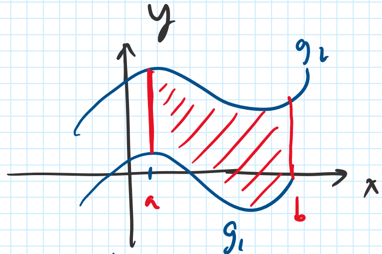
Math 261, Lecture 22, 10/15/18

Today: §15.3, Next §15.4, §15.5

Recap: $\iint_D f(x,y)\,dA$

$$= \int_a^b \int_{y=g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

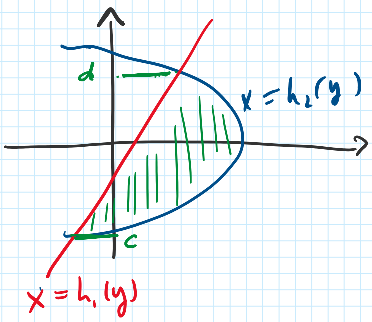


"Type I"

$a \le x \le b$

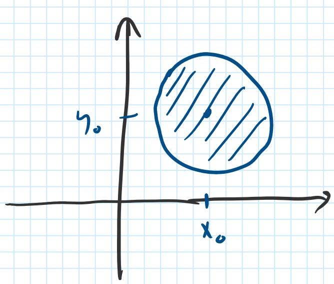
$g_1(x) \le y \le g_2(x)$

"Type II"

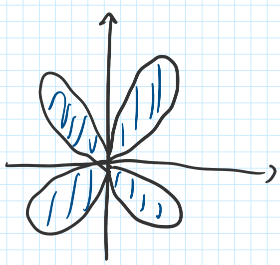


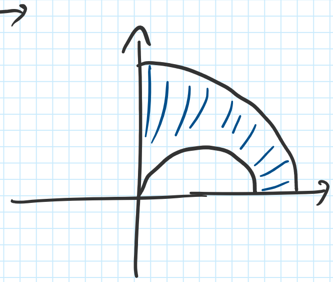
$$\int_c^d \int_{x=h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

§15.3 Double Integrals over Polar Coordinates



D has a circular or trig component to its definition



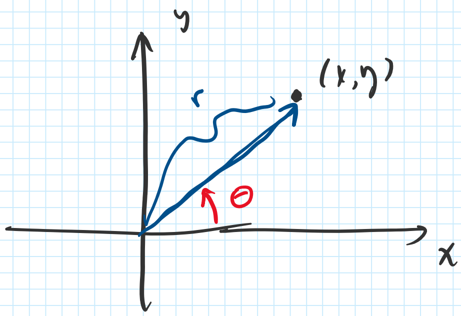


Review of Polar Coords.

y

$\sqrt{x^2+y^2}$

Review of Polar Coords.



$r = \sqrt{x^2+y^2}$

or $r^2 = x^2+y^2$

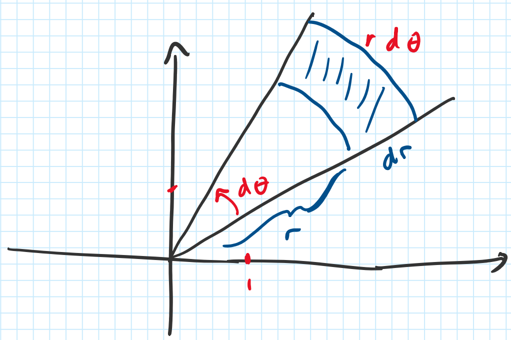
$x = r\cos\theta$
$y = r\sin\theta$

$r \geq 0$
$0 \leq \theta \leq 2\pi$

How to compute $\iint_D f(x,y)\, dA$ in terms of (r,θ)?

"$dA = dx\,dy$"

Write dA in terms of $dr\, d\theta$

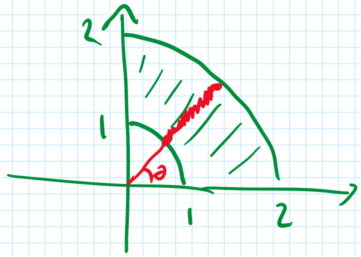


* $\boxed{dA = r\, dr\, d\theta}$ *

General form

$$\iint_D f(x,y)\, dA = \iint_D \underbrace{f(r\cos\theta, r\sin\theta)\, r}_{\text{new integrand}}\, dr\, d\theta$$

Ex. Find the volume of the solid bounded by the elliptic paraboloid $x^2+2y^2+z=8$ over D over D



$$z = f(x,y) = 8 - x^2 - 2y^2$$
$$= 8 - (x^2+y^2) - y^2$$
$$= 8 - r^2 - r^2\sin^2\theta$$

$0 \le \theta \le \pi/2 \qquad 1 \le r \le 2$

$$\iint_D f(x,y)\,dA = \int_{\theta=0}^{\pi/2}\int_{r=1}^{2} \overbrace{(8-r^2-r^2\sin^2\theta)\,r}^{8r - r^3 - r^3\sin^2\theta}\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[4r^2 - \frac{1}{4}r^4 - \frac{1}{4}r^4\sin^2\theta\right]\Bigg|_{r=1}^{2} d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[16 - 4 - 4\sin^2\theta - 4 + \frac{1}{4} + \frac{1}{4}\sin^2\theta\right] d\theta$$

$$= \int_0^{\pi/2} \frac{33}{4} - \frac{15}{4}\sin^2\theta \; d\theta$$

$$\boxed{\begin{aligned} * \quad & \\ * \quad & \sin^2\theta = \frac{1-\cos(2\theta)}{2} \\ * \quad & \cos^2\theta = \frac{1+\cos(2\theta)}{2} \end{aligned}}$$

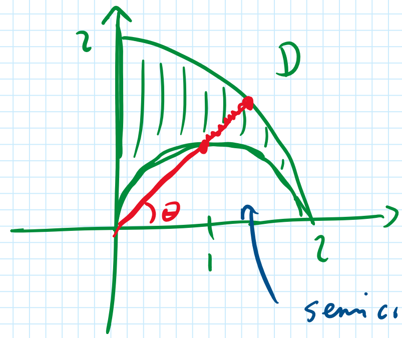
$$= \int_0^{\pi/2} \frac{33}{4} - \frac{15}{8}(1-\cos(2\theta))\,d\theta$$

$$\left(\frac{1}{2}\sin(2\theta)\right)' = \cos 2\theta$$

$$= \frac{51}{\ldots}\theta + \frac{15}{\ldots}\cos(2\theta)\Big|^{\pi/2} = \frac{51}{16}\pi - \frac{15}{8} \approx 8.13883$$

$$= \frac{51}{8}\theta + \frac{15}{16}\cos(2\theta)\Big|_0^{\pi/2} = \frac{51}{16}\pi - \frac{15}{8} \approx 8.13883$$

Ex. $z = x$



$$(x-1)^2 + (y-0)^2 = 1^2$$
$$x^2 - 2x + \cancel{1} + y^2 = \cancel{1}$$
$$x^2 + y^2 = 2x$$

semicircle radius = 1
centered at (1,0)

$$r^{\cancel{2}} = 2\cancel{r}\cos\theta$$
$$r = 2\cos\theta$$

$$z = r\cos\theta$$

$$0 \le \theta \le \pi/2 \qquad 2\cos\theta \le r \le 2$$

$$\iint_D f(x,y)\,dA = \int_0^{\pi/2}\int_{2\cos\theta}^{2} r\cos\theta \; r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_{2\cos\theta}^{2} r^2\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{1}{3}r^3\cos\theta\right]_{r=2\cos\theta}^{2} d\theta$$

$$= \int_0^{\pi/2} \frac{8}{3}\cos\theta - \frac{8}{3}\cos^4\theta \; d\theta$$

$$= \int_0^{\pi/2} \frac{8}{3}\cos\theta - \frac{8}{3}\left(\frac{1+\cos(2\theta)}{2}\right)^2 d\theta$$

$$= \int^{\pi/2} \frac{8}{3}\cos\theta - \frac{8}{3}\left(\frac{1}{4} + \frac{1}{2}\cos(2\theta) + \frac{\cos(2\theta)^2}{4}\right) d\theta$$

$$= \int_0 \frac{8}{3}\cos\theta - \frac{8}{3}\left(\frac{1}{4} + \frac{1}{2}\cos(2\theta) + \frac{\cos(4\theta)}{4}\right) d\theta$$

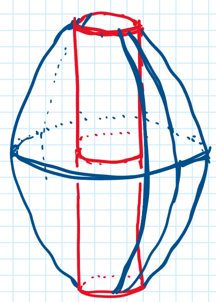
$$\cos(2\theta)^2 = \frac{1+\cos(4\theta)}{2}$$

$$= \int_0^{\pi/2} \frac{8}{3}\cos\theta - \frac{2}{3} - \frac{4}{3}\cos(2\theta) - \frac{1}{3} - \frac{1}{3}\cos(4\theta)\, d\theta$$

$$= \frac{8}{3}\sin\theta - \frac{2}{3}\sin(2\theta) - \frac{1}{12}\sin(4\theta) - \theta \Big|_{\theta=0}^{\pi/2}$$

$$= \frac{8}{3} - \frac{2}{3}\cdot 0 - \frac{1}{12}\cdot 0 - \pi/2 - 0 + 0 + 0 + 0 = \frac{8}{3} - \frac{\pi}{2} \approx 1.09587$$

Ex. Find the volume of ellipsoid $2x^2 + 2y^2 + z^2 = 8$ outside of the cylinder $x^2 + y^2 = 1$

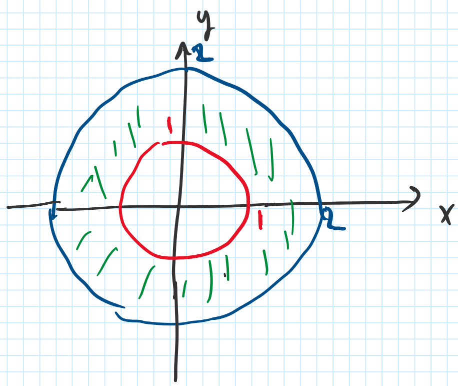


$2 \times$ Volume above xy-plane

$$z^2 = 8 - 2x^2 - 2y^2$$

$$z = \sqrt{8 - 2x^2 - 2y^2}$$

$$z = \sqrt{8 - 2r^2}$$



$z = 0$

$2x^2 + 2y^2 = 8$

$x^2 + y^2 = 4$

$$\text{Volume} = 2\int_{\theta=0}^{2\pi}\int_{r=1}^{2} \sqrt{8 - 2r^2}\; r\, dr\, d\theta$$

$u = r^2$

$du = 2r$

$\theta=0 \quad r=1$

$u = r^2$
$du = 2r$

$$= 2\int_0^{2\pi}\left[-\frac{1}{6}(8-2r^2)^{3/2}\right]\Bigg|_{r=1}^{2} d\theta$$

$$= 2\int_0^{2\pi}\left[-\frac{1}{6}(8-8)^{3/2}+\frac{1}{6}(8-2)^{3/2}\right] d\theta$$

$$= 2\int_0^{2\pi}\sqrt{6}\,d\theta = 2\sqrt{6}\,\theta\Big|_0^{2\pi} = 4\sqrt{6}\pi$$

Bonus Ex. $f(x,y) = e^{-x^2-y^2}$ over the disk of radius R centered at $(0,0)$

$$z = e^{-r^2}$$

$$\iint_D f(x,y)\,dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{R} e^{-r^2}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{-1}{2}e^{-r^2}\right]_{r=0}^{R} d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2}e^{-R^2}+\frac{1}{2}\,d\theta$$

$$= \frac{1}{2}\left(1-e^{-R^2}\right)\theta\Big|_0^{2\pi}$$

$$= \pi\left(1-e^{-R^2}\right)$$

As $R \to \infty$ This converges to π

Refer to Exercise 40 on page 1015 in Stewart

Refer to Exercise 40 on page 1015 in Stewart to see how this shows $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$!

This integral can be evaluated even though you cannot write an antiderivative for e^{-x^2}!