Exam 1

Work 4 problems for 5 points each.

Problem 1. State the pigeonhole principle. Show that if A and B are two finite sets, then there is no surjective map $f : A \to B$ when the cardinality of B is greater than the cardinality of A.

The pigeonhole principle states there is no injective function $f : A \to B$ if the cardinality of A is greater than the cardinality of B.

Suppose $f : A \to B$ were surjective. Define a function $g : B \to A$ by choosing for each $b \in B$ some $a \in A$ such that $f(a) = b$. Then g is injective since f is a function, which contradicts the pigeonhole principle.

Problem 2. Show that a sequence (x_n) is unbounded if and only if there is a subsequence which has no convergent subsequence.

($\Rightarrow$) Suppose (x_n) is unbounded. Without loss of generality assume (x_n) has no upper bound. For each $k \in \mathbb{N}$ inductively choose $n_k > n_{k-1}$ with $x_{n_k} \geq k$. This can be done since the set of indices for which $x_n \geq k$ is infinite for each $k \in \mathbb{N}$. [Otherwise for some k, $x_n < k$ for all but finitely many $n \in \mathbb{N}$ which means that (x_n) has an upper bound.] The subsequence x_{n_k} diverges to infinity; therefore, so does any further subsequence.

($\Leftarrow$) Suppose (x_n) is bounded. Every subsequence is also bounded. By the Bolzano-Weierstrass theorem every subsequence has a convergent subsequence.

Problem 3. Let (x_n) be a sequence. Define convergence of (x_n) to a real number x. **Without** using the Bolzano-Weierstrass theorem show that any convergent sequence has a monotone subsequence.

The sequence (x_n) converges to x if for every $\epsilon > 0$ there is $N \in \mathbb{N}$ so that $|x_n - x| < \epsilon$ for all $n \geq N$.

Let (x_n) converge to x. Without loss of generality assume that $x_n \geq x$ for infintely many terms in the sequence. If there are infinitely many terms equal to x, choose those as the subsequence. Otherwise, inductively pick $n_k > n_{k-1}$ such that $x_{n_k} \in (x, x_{n_{k-1}})$. This can be done since the subsequence of terms greater than x is infinite and still converges to x.

Problem 4. Define density of a subset $E \subset \mathbb{R}$. Show that the set of irrational numbers is dense in $\mathbb{R}$.

E is dense in $\mathbb{R}$ is for every $a, b \in \mathbb{R}$ with $a < b$, $E \cap (a, b) \neq \emptyset$.

The rationals are dense in $\mathbb{R}$. Pick a rational number $q \in (a - \sqrt{2}, b - \sqrt{2})$. Then $q + \sqrt{2} \in (a, b)$. We have that $q + \sqrt{2}$ is irrational or else $q + \sqrt{2} = r \in \mathbb{Q}$ would imply $\sqrt{2} = r - q \in \mathbb{Q}$.

Problem 5. Let (x_n) and (y_n) be convergent sequences such that $x_n < y_n$ for all $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$. Hint: $(y_n - x_n)$ is a sequence with positive values.

By limit laws $\lim(y_n - x_n) = \lim y_n - \lim x_n$ so the sequence $(y_n - x_n)$ converges. By Quiz 2 we know that $\lim(y_n - x_n) \geq 0$. Thus $\lim y_n - \lim x_n \geq 0$ or $\lim y_n \geq \lim x_n$.

Problem 6. Let $E \subset \mathbb{R}$ be non-empty with an upper bound. Show that $x = \sup E$ if and only if x is an upper bound for E and for every $\epsilon > 0$ there is $y \in E$ with $y > x - \epsilon$.

($\Rightarrow$) By definition $x = \sup(E)$ is an upper bound for E. By contradiction suppose that there is $\epsilon > 0$ such that $y \leq x - \epsilon$ for all $y \in E$. Then $x - \epsilon$ is a smaller upper bound for E than x contradicting that x is the least upper bound.

($\Leftarrow$) Suppose x is an upper bound and let $z = \sup(E)$. Given $\epsilon > 0$, find $y \in E$ so that $x - \epsilon < y$. Now we have that $x - \epsilon < y \leq z \leq x$ for all $\epsilon > 0$, but this means that $x = z$.