Exam 3

Work 4 problems for 5 points each.

Problem 0. State the Mean Value Theorem. State the Ratio test and Raabe's test for convergence for the series $\sum_{k=1}^{\infty} a_k$.

If $f : [a, b] \to \mathbb{R}$ is continuous and differentiable on (a, b), then there is $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

If $\limsup_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} < 1$ then $\sum_{k=1}^{\infty} a_k$ converges. If $\liminf_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

If $\lim_{k\to\infty} k(\frac{a_k}{a_{k+1}} - 1) > 1$, then $\sum_{k=1}^{\infty} a_k$ converges. If $\lim_{k\to\infty} k(\frac{a_k}{a_{k+1}} - 1) < 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.

Problem 1. Give a proof of the fact that a series with positive terms $\sum_{k=1}^{\infty} a_k$ converges if and only if the sequence of partial sums is bounded.

Since the terms are positive, the sequence of partial sums is monotone increasing, therefore it converges if it is bounded. Conversely any convergent sequence is bounded.

Problem 2. Define what it means for a function to be convex on (a, b). Let $f : (0, 1) \to \mathbb{R}$ be a convex function. What are all possible sets where f achieves a minimum value? Explain.

The function is convex if for all $x, y \in (a, b)$ and $t \in [0, 1]$, it holds that $f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$.
Such a set must be (relatively) closed, connected subset of (a, b), possibly empty. If $f(x) = m = f(y)$ is the minimum value, then for $z \in (x, y)$, $z = tx+(1-t)y$, so $f(z) \leq tm+(1-t)m = m$, so $f(z) = m$. Since such an f is continuous, then if $f(x_k) = m$, $x_k \to x$, $f(x_k) \to f(x)$, so $f(x) = m$.

Problem 3. You have forgotten about Taylor series. Show that $e^x \geq 1 + x$ for $x \in [0, 1]$.

Let $f(x) = e^x - 1 - x$. Since $f'(x) = e^x - 1 \geq 0$, then $f(x)$ is increasing and $f(0) = 0$.

Problem 4. For which $\alpha, \beta > 0$ does $\sum_{n=1}^{\infty} n^{\alpha} \sin(n^{-\beta})$ converge?

$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ so $\lim_{n\to\infty} \frac{n^{\alpha}\sin(n^{-\beta})}{n^{\alpha-\beta}} = 1$. Thus $\sum_{n=1}^{\infty} n^{\alpha} \sin(n^{-\beta})$ converges if and only if $\sum_{n=1}^{\infty} n^{\alpha-\beta}$ converges, i.e., $\alpha < \beta - 1$.

Problem 5. Let $f : (0, 1) \to \mathbb{R}$ be a differentiable function such that for every sequence of distinct points (x_k) in $[0, 1]$, $\sum_{k=1}^{\infty} \frac{f'(x_k)}{k}$ converges. What can you conclude about f? Explain.

If f is constant, i.e., the derivative is always zero, then the condition is satisfied. If f' takes two values, then by the intermediate value theorem for derivatives, it takes all values in some interval so there is a sequence of distinct points such that $f'(x_k) \geq c > 0$ or $f'(x_k) \leq c < 0$ which means $\sum_{k=1}^{\infty} \frac{f'(x_k)}{k}$ diverges by comparison to the harmonic series. Therefore f' takes at most one value, which must be zero again by the divergence of the harmonic series. So f is constant.