# MA 34100 Fall 2016, HW 10

November 16, 2016

## 1 $7.9.1 \cdots [4\ pts]$

Let $f$ be differentiable on $[a,b]$ and let $R(f')$ denote the range of $f'$ on $[a,b]$. Give examples to illustrate that $R(f')$ can be
(a) a closed interval.
(b) an open interval.
(c) a half-open interval.
(d) an unbounded interval.

*Proof.*
(a) $f(x) = x^2, x \in [0,1], \quad R(f') = [0,2].$
(b)

$$f(x) = \begin{cases} x^4 e^{-\frac{x^2}{4}} \sin(\frac{8}{x^3}), & x \in [-1,1], x \neq 0, \\ 0, & x = 0. \end{cases} \qquad R(f') = (-24, 24)$$

Proof can be found on page 37-38, "Bernard R. Gelbaum, John M. H. Olmsted. Counterexamples in Analysis, Dover Books on Mathematics, 2003." □

(c) $\exists \beta > 0$, define

$$f(x) = \begin{cases} x^2 + x + x^2 \sin\frac{1}{x}, & x \neq 0, x \in [0,\beta] \\ 0, & x = 0. \end{cases}$$

then

$$f'(x) = \begin{cases} 1 - \cos\frac{1}{x} + 2x(1 + \sin\frac{1}{x}), & x \neq 0, x \in [0,\beta] \\ 1, & x = 0. \end{cases}$$

s.t., $Rf'(x) \in (0,3]$.

The tricky part is to prove the range is open at 0. For $\forall x_k = \frac{1}{2k\pi}, k = 1, 2, ..., f'(x_k) = \frac{1}{k\pi} > 0$, for $x \neq x_k, f'(x) = 1 - \cos\frac{1}{x} + 2x(1 + \sin\frac{1}{x}) \geq 1 - \cos\frac{1}{x} > 0$. So $0 \notin Rf'(x)$. Moreover, $\lim_{k\to\infty} f'(x_k) = 0$, which proves 0 is a limit point. So, it's open at 0.

(d)

$$f(x) = x^2 \sin\frac{1}{x^2}, x \in [-1,1], \quad R(f') = (-\infty, \infty).$$

## 2 $7.10.9 \cdots [3\ pts]$

Corollary 7.35: Let $f$ be defined on an open interval I.

(i) If $f$ is differentiable on I, then $f$ is convex on I if and only if $f'$ is nondecreasing on I.

(ii) If $f$ is twice differentiable on I, then f is convex on I if and only if $f'' \geq 0$ on I.

*Proof.* (i),$\Rightarrow$ Claim, if $f$ is differentiable on I, and $f$ is convex on I, then

$$\begin{aligned}
&\frac{f(x_1+h_1)-f(x)}{h_1} \leq \frac{f(x_1+h_2)-f(x)}{h_2}, \\
&\frac{f(x')-f(x_1)}{x'-x_1} \leq \frac{f(x_2)-f(x')}{x_2-x'}, \\
&\frac{f(x_2)-f(x_2-h_3)}{h_3} \leq \frac{f(x_2)-f(x_2-h_4)}{h_4}, \\
&\text{for } h_1 < h_2, x_1 < x' < x_2, h_3 > h_4,
\end{aligned}$$

For the first inequality, since $f(x_1+h_1) = f(\frac{h_1}{h_2}(x_1+h_2)+\frac{h_2-h_1}{h_2}x_1) \leq \frac{h_1}{h_2}f(x+h_2)+\frac{h_2-h_1}{h_2}f(x_1) \Rightarrow h_2(f(x_1+h_1)-f(x_1)) \leq h_1(f(x_1+h_2)-f(x_1)) \Rightarrow \frac{f(x_1+h_1)-f(x)}{h_1} \leq \frac{f(x_1+h_2)-f(x)}{h_2}$. For the second inequality, $f(x') = f(\frac{x'-x_1}{x_2-x_1}x_2+\frac{x_2-x'}{x_2-x_1}x_1) \leq \frac{x'-x_1}{x_2-x_1}f(x_2)+\frac{x_2-x'}{x_2-x_1}f(x_1) \Rightarrow (x_2-x')[f(x')-f(x_1)] \leq (x'-x_1)[f(x_2)-f(x')] \Rightarrow \frac{f(x')-f(x_1)}{x'-x_1} \leq \frac{f(x_2)-f(x')}{x_2-x'}$. Following same procedure can prove inequality 3. Then,

$$\begin{aligned}
f'(x_1) = \lim_{h\to 0}\frac{f(x_1+h)-f(x_1)}{h} &\leq \frac{f(x_1+h_1)-f(x_1)}{h_1} \\
\leq \frac{f(x_2)-f(x_2-h_2)}{h_2} &\leq \lim_{h\to 0}\frac{f(x_2)-f(x_2-h)}{h} = f'(x_2),
\end{aligned}$$

for $h_1 > 0, h_2 > 0, x_2 - h_2 \geq x_1 + h_1$. Hence $f'$ is increasing.

(i), $\Leftarrow$ by mean value theorem, for any $x_1 < x_2, 0 < \alpha < 1$, $\exists \eta_1 \in (x_1, \alpha x_1 + (1-\alpha)x_2), \eta_2 \in (\alpha x_1+(1-\alpha)x_2, x_2)$ s.t., $f'(\eta_1) = \frac{f(\alpha x_1+(1-\alpha)x_2)-f(x_1)}{(1-\alpha)(x_2-x_1)}, f'(\eta_2) = \frac{f(x_2)-f(\alpha x_1+(1-\alpha)x_2)}{\alpha(x_2-x_1)}$, since $\eta_1 < \eta_2$ and $f'$ is nondecreasing, thus $\frac{f(\alpha x_1+(1-\alpha)x_2)-f(x_1)}{(1-\alpha)(x_2-x_1)} = f'(\eta_1) \leq f'(\eta_2) = \frac{f(x_2)-f(\alpha x_1+(1-\alpha)x_2)}{\alpha(x_2-x_1)} \Rightarrow f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha)f(x_2)$. which means $f$ is convex.

(ii), If $f$ is twice differentiable, then $f'$ is nondecreasing if and only if $f'' \geq 0$, then we complete the proof by combining the result in (i). $\square$

## 3 $7.13.3 \cdots [3\ pts]$

Let $p$ be a polynomial of the $nth$ degree that is everywhere nonnegative. Show that

$$p(x) + p'(x) + p''(x) + \cdots + p^{(n)}(x) \geq 0 \quad \text{for all } x.$$

*Proof.* Claim: If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \geq 0$ everywhere, then $a_n > 0$ and $n$ is even.

if $x \neq 0$, then $p(x) = x^n(a_n + a_{n-1}\frac{1}{x} + \cdots + a_0\frac{1}{x^n})$. since $\lim_{x\to\infty} a_{n-1}\frac{1}{x} + \cdots + a_0\frac{1}{x^n} = 0 \Rightarrow \forall \epsilon > 0, \exists N$, s.t., when $|x| > N$, $|a_{n-1}x^{n-1} + \cdots + a_0| < \epsilon < \frac{1}{2}|a_n|$. Then, if $a_n < 0$, let $x > N$, we will have $p(x) < x^n(a_n + \epsilon) < \frac{1}{2}a_n x^n < 0$ which is a contradiction, so $a_n > 0$. If $n$ is odd, then let $x < -N$, we have $p(x) < x^n(a_n - \epsilon) < \frac{1}{2}a_n x^n < 0$ which is a contradiction, so $n$ is even.

For our problem, define $H(x) := p(x) + p'(x) + p''(x) + \cdots + p^{(n)}(x)$ then by same discussion we can get $\exists N > 0$, when $|x| > N$, $H(x) > 0$. Assume $\{x_0, x_1, \ldots, x_{n-1}\}$ are the critical points which means $H'(x_i) = 0$, then choose $N_1$ s.t., $N_1 > N$ and $(-N_1, N_1) \supset \{x_i\}$. Since $H(x_i) = H'(x_i) + p(x_i) = p(x_i) \geq 0$, then

$$H(x) \geq \min\{\{H(x_i)\}, H(N_1), H(-N_1)\} \geq 0, \quad x \in [-N_1, N_1]$$

Combined with $H(x) > 0$ in $(\infty, -N_1] \cup [N_1, \infty)$, we complete the proof.

$\square$