MA 34100 Fall 2016, HW 11

November 26, 2016

1 $3.6.3 \cdots [5\ pts]$

Which of these series converge?

$a)$ $\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)^2}$

Divergent. Let $c_k = 1$, by limit comparison test 2, $\lim_{k\to\infty} \frac{a_k}{c_k} = 1 > 0$, since $\sum_{n=1}^{\infty} 1$ divergent, then $\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)^2}$ is divergent.

$b)$ $\sum_{n=1}^{\infty} \frac{3n(n+1)(n+2)}{n^3\sqrt{n}}$

Divergent. Let $c_k = \frac{1}{\sqrt{k}}$, by limit comparison test 2, $\lim_{k\to\infty} \frac{a_k}{c_k} = 3 > 0$, since $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ divergent, then $\sum_{n=1}^{\infty} \frac{3n(n+1)(n+2)}{n^3\sqrt{n}}$ is divergent.

$c)$ $\sum_{n=2}^{\infty} \frac{1}{n^s \log n}$

If $s = 1$, divergent. By Cauchy's Condensation Test, $\sum_{j=2}^{\infty} 2^j a_{2^j} = \frac{1}{\log 2} \sum_{j=2}^{\infty} \frac{1}{j}$ is divergent. So,$\sum_{k=2}^{\infty} \frac{1}{k^s \log k}$ is divergent.

If $s < 1$, divergent. since $\frac{1}{n^s \log n} > \frac{1}{n \log n}$.

If $s > 1$, Convergent. since $0 < \frac{1}{n^s \log n} < \frac{1}{n^s}$.

$d)$ $\sum_{n=1}^{\infty} a^{\frac{1}{n}} - 1$

If $a = 0$, divergent.

If $a = 1$, convergent.

If $a > 1$, divergent. Consider $c_k = e^{\frac{1}{k}} - 1$, by Taylor's expansion, $c_k = e^{\frac{1}{k}} - 1 > 1 + \frac{1}{k} - 1 = \frac{1}{k}$, so $\sum_{k=1}^{\infty} e^{\frac{1}{k}} - 1$ is divergent. by limit comparison test 2, $\lim_{k\to\infty} \frac{a_k}{c_k} = \log a > 0$. So, it is divergent.

If $0 < a < 1$, divergent. $a_k = a^{\frac{1}{k}} - 1 < 0$, so consider $b_k = -a_k$. Let $b = \frac{1}{a}$, $\sum_{n=1}^{\infty} -a^{\frac{1}{n}} + 1 = \frac{b^{\frac{1}{n}} - 1}{b^{\frac{1}{n}}} > 0$, and for sufficient large n, $b^{\frac{1}{n}} < 2$, then $b_k = \frac{b^{\frac{1}{n}} - 1}{b^{\frac{1}{n}}} > \frac{1}{2}(b^{\frac{1}{n}} - 1)$, so it is divergent.

$e)$ $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^t}$

By Cauchy's Condensation Test, $\sum_{j=2}^{\infty} 2^j a_{2^j} = \frac{1}{(\log 2)^t} \sum_{j=2}^{\infty} \frac{1}{j^t}$. Then,

If $t > 1$, it is convergent.

If $t \leq 1$, it is divergent.

$f)$ $\sum_{n=2}^{\infty} \frac{1}{n^s(\log n)^t}$

If $s > 1$, since $0 < \frac{1}{n^s(\log n)^t} < \frac{1}{n^s}$, so it is convergent.

If $s = 1$, just the problem (e), it is convergent for $t > 1$, divergent for $t \leq 1$.

If $s < 1$, then $\exists r > 0, s.t., s + r < 1$, since for sufficient large n, $\frac{1}{n^r} < \frac{1}{(\log n)^t}$, then $\frac{1}{n^s(\log n)^t} > \frac{1}{n^{s+r}} > \frac{1}{n}$, so it is divergent.

$g)$ $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n^2}$

Convergent. By root test, $\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} (1 - \frac{1}{k})^k = \frac{1}{e} < 1$, so it is convergent.

2 3.6.15 $\cdots$ [2 *pts*]

. Let $\{a_n\}$ be a sequence of positive numbers and write

$$L_n = \frac{\log(\frac{1}{a_n})}{\log n}.$$

Show that if $\liminf L_n > 1$, then $\sum a_n$ converges. Show that if $L_n \le 1$ for all sufficient large n, then $\sum a_n$ diverges.

Proof.
a), Let $s_k = \inf_{n\ge k} L_n$. Since s_k increase monotonically and $\lim s_k > 1$, then $\exists r > 0$, s.t., for sufficient large $n, s_n > 1 + r$, then $L_n = \frac{\log(\frac{1}{a_n})}{\log n} \ge s_n > 1 + r \Rightarrow \frac{1}{a_n} > n^{1+r} \Rightarrow a_n < \frac{1}{n^{1+r}} \Rightarrow$ $\sum a_n$ converges.
b), $\exists N > 0$, $\forall n > N$, $L_n \le 1 \Rightarrow a_n \ge \frac{1}{n}$. then $\sum a_n$ diverges. □

3 3.6.16 $\cdots$ [3 *pts*]

Apply the test in Exercise 3.6.15 to obtain convergence of divergence of the following series (x is positive):
a, $\sum_{n=2}^{\infty} x^{\log n}$,
b, $\sum_{n=2}^{\infty} x^{\log\log n}$,
c, $\sum_{n=2}^{\infty} (\log n)^{-\log n}$,

Proof.
a, $L_n = \frac{\log(\frac{1}{x^{\log n}})}{\log n} = -\log x$, then by Exercise 3.6.15, if $x \ge \frac{1}{e}$, it diverges. if $0 < x < \frac{1}{e}$, it converges.
b, $\lim_{n\to\infty} L_n = \lim_{n\to\infty} \frac{\log(\frac{1}{x^{\log\log n}})}{\log n} = \lim_{n\to\infty} \frac{\log\log n}{\log n} \log\frac{1}{x} = 0 < 1$, so it diverges.
c, $L_n = \frac{\log(\log n^{\log n})}{\log n} = \log\log n > 1$. So it converges. □