MA 34100 Fall 2016, HW 12

December 4, 2016

1 8.3.6 $\cdots$ [3 *pts*]

If f is continuous on an interval $[a, b]$ and

$$\int_a^b f(x)g(x)dx = 0$$

for every continuous function g on $[a, b]$ show that f is identically equal to zero there.

Proof. Let $g = f$, then $\int_a^b f^2(x) = 0$ if and only if $f(x) \equiv 0$ in $[a, b]$. $\square$

2 8.5.6 $\cdots$ [4 *pts*]

Let f be a continuous function on $[1, \infty)$ such that the integral $\int_1^\infty f(x)dx$ converges. Can you conclude that $\lim_{x\to+\infty} f(x) = 0$?

Proof. No, for example $f(x) = \sin(x^2)$. since $\int_1^\infty \sin x^2 dx = \int_1^\infty \frac{1}{2}\frac{\sin t}{\sqrt{t}}dt = \int_1^{2\pi} \frac{1}{2}\frac{\sin t}{\sqrt{t}}dt + \int_{2\pi}^\infty \frac{1}{2}\frac{\sin t}{\sqrt{t}}dt$, first part is finite, for the second part,

$$\int_{2\pi}^\infty \frac{1}{2}\frac{\sin t}{\sqrt{t}}dt = \sum_{k=1}^\infty \int_{2k\pi}^{2k\pi+2\pi} \frac{1}{2}\frac{\sin t}{\sqrt{t}}dt = \sum_{k=1}^\infty b_k,$$

$b_k = \int_{2k\pi}^{2k\pi+2\pi} \frac{1}{2}\frac{\sin t}{\sqrt{t}}dt$. Since

$$\begin{aligned} b_k &= \int_{2k\pi}^{2k\pi+\pi} \frac{1}{2}\frac{\sin t}{\sqrt{t}}dt + \int_{2k\pi+\pi}^{2k\pi+2\pi} \frac{1}{2}\frac{\sin t}{\sqrt{t}}dt \\ &< \frac{1}{2}\frac{1}{\sqrt{2k\pi}}\int_{2k\pi}^{2k\pi+\pi} \sin t dt + \frac{1}{2}\frac{1}{\sqrt{2k\pi+2\pi}}\int_{2k\pi+\pi}^{2k\pi+2\pi} \sin t dt \\ &= \frac{1}{\sqrt{2k\pi}} - \frac{1}{\sqrt{2k\pi+2\pi}} < \frac{1}{2k\sqrt{2k\pi}}, \end{aligned}$$

and obviously that $b_k > 0$, so $\int_{2\pi}^\infty \frac{1}{2}\frac{\sin t}{\sqrt{t}}dt = \sum_{k=1}^\infty b_k$ converges. However, $\lim_{x\to\infty} \sin x^2 \neq 0$. $\square$

3 8.5.11 $\cdots$ [3 *pts*]

(Cauchy Criterion for Convergence) Let $f : [a, \infty] \to R$ be a continuous function. Show that the integral $\int_a^\infty f(x)dx$ converges if and only if for every $\epsilon > 0$ there is a number M so that, for all $M < c < d$,

$$|\int_c^d f(x)dx| < \epsilon.$$

Proof. Define $F(x) := \int_a^x f(s)ds$, then the integral $\int_a^\infty f(x)dx$ converges, if and only if $F(x)$ converges for $x \to \infty$, if and only if $\forall \epsilon > 0$, $\exists M > 0$, $\forall d > c > M$, $|F(d) - F(c)| < \epsilon \Leftrightarrow |\int_c^d f(x)dx| < \epsilon$. $\square$