MA 34100 Fall 2016, HW 4

September 28, 2016

1 $\quad 2.4.5 \cdots [2\ pts]$

Show that Definition 2.6 is equivalent to the following slight modification:

We write $\lim_{n\to\infty} s_n = L$ provided that for every positive integer m there is a real number N so that $|s_n - L| < \frac{1}{m}$ whenever $n \geq N$.

Proof.
1),$\Rightarrow$: By the modified definition , $\forall m \in N$, there is an integer N_m such that if $n \geq N_m$, then $|s_n - L| < \frac{1}{m}$. Then $\forall \epsilon > 0$, by Archimedean theorem, $\exists m$ s.t.,$\frac{1}{m} < \epsilon$. Then $\forall n > N_m, |s_n - L| < \frac{1}{m} < \epsilon$, satisfies the condition of definition 2.6.
2),$\Leftarrow$: For any integer m, choose $\epsilon = \frac{1}{m}$, then by definition of 2.6, $\exists N_1 \in N$, if $n > N_1$, we get $|s_n - L| < \epsilon = \frac{1}{m}$,satisfies the condition of the modified definition. □

2 $\quad 2.4.14 \cdots [2\ pts]$

Show that the statement "$\{s_n\}$ converges to L" is false if and only if there is a positive number c so that the inequality $|s_n - L| > c$ holds for infinitely many values of n.

Proof.
$\Rightarrow$:if $\exists c > 0$, there are infinitely many values of n makes $|s_n - L| > c$, then if suppose $\{s_n\}$ converges to L, means for $c, \exists N_1 \in N, \forall n > N_1, |s_n - L| < c$, since N_1 is finite, it is impossible to have infinity many values of n makes $|s_n - L| > c$.
$\Leftarrow$: if $\{s_n\}$ does not converge to L, then it means $\exists c > 0$, for $\forall N_1 \in N$, $\exists n > N_1$, such that $|s_n - L| > c$. Then if suppose only finite many values of $\{n_i\}$ makes $|s_{n_i} - L| > c$, then let $N_1 = \max\{n_i\}$, then $\forall n > N_1, |s_n - L| < c$, it contradictive with $\{s_n\}$ does not converge to L. □

3 $\quad 2.7.5 \cdots [3.5\ pts]$

Which statements are true?
(a) If $\{s_n\}$ and $\{t_n\}$ are both divergent then so is $\{s_n + t_n\}$.[False]. For example $s_n = (-1)^n$, $t_n = (-1)^{n+1}, s_n + t_n = 0$.
(b) If $\{s_n\}$ and $\{t_n\}$ are both divergent then so is $\{s_n t_n\}$.[False]. For example $s_n = t_n = (-1)^n, s_n t_n = 1$.
(c) If $\{s_n\}$ and $\{s_n + t_n\}$ are both convergent then so is $\{t_n\}$.[True].
(d) If $\{s_n\}$ and $\{s_n t_n\}$ are both convergent then so is $\{t_n\}$.[False]. For example $s_n = \frac{1}{n}, t_n = (-1)^n$.
(e) If $\{s_n\}$ is convergent so too is $\{\frac{1}{s_n}\}$.[False]. For example $s_n = \frac{1}{n}$.

(f) If $\{s_n\}$ is convergent so too is $\{s_n^2\}$. [True].
(g) If $\{s_n^2\}$ is convergent so too is $\{s_n\}$.[False]. For example $s_n = (-1)^n$.

4 2.11.28 $\cdots$ [2 *pts*]

Let $\{x_n\}$ be a bounded sequence that diverges. Show that there is a pair of convergent subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ so that $\lim_{k\to\infty} |x_{n_k} - x_{m_k}| > 0$.

Proof. Since $\{x_n\}$ is a bounded sequence, by Bolzano-Weierstrass, there exists a convergent subsequence, let it be $\{x_{n_k}\}$ and assume it converges to L. Since $\{x_n\}$ does not converge to L, by problem 2.4.14, $\exists c > 0$ there are infinity number n, such that $|x_n - L| > c$, let them be $\{\hat{x_n}\}$, since $\{\hat{x_n}\}$ are bounded and infinity number, thus by Bolzano-Weierstrass, there exists a convergent subsequence $\{x_{m_k}\}$ of $\{\hat{x_n}\}$ and $|x_{m_k} - L| > c$. Thus $c < |x_{m_k} - L| \leq |x_{m_k} - x_{n_k}| + |x_{n_k} - L| \Rightarrow |x_{m_k} - x_{n_k}| \geq c - |x_{n_k} - L|$. Then $\lim_{n\to\infty} |x_{m_k} - x_{n_k}| \geq c - \lim_{n\to\infty} |x_{n_k} - L| = c$. □