MA 34100 Fall 2016, HW 6

October 13, 2016

1 $4.7.4 \cdots [3\ pts]$

In many applications of open sets and closed sets we wish to work just inside some other set A. It is convenient to have a language for this. A set $E \subset A$ is said to be open relative to A if $E = A \cap G$ for some set $G \subset \mathbb{R}$ that is open. A set $E \subset A$ is said to be closed relative to A if $E = A \cap F$for some set $F \subset \mathbb{R}$ that is closed. Answer the following questions.

(a) Let $A = [0, 1]$ describe, if possible, sets that are open relative to A but not open as subsets of R.

Set E that is open relative to A but not open as subsets of R could be written as $E = G \cap A$ where G is an open set in $\mathbb{R}$ and $G \cap \{0, 1\} \neq \emptyset$.

(b) Let $A = [0, 1]$ describe, if possible, sets that are closed relative to A but not closed as subsets of R.

Impossible, $\forall E$ is closed relative to A, $\exists F$ closed in R, s.t., $E = A \cap F$, since both of A and F are closed in R, thus $E = A \cap F$ is closed in R.

(c) Let $A = (0, 1)$ describe, if possible, sets that are open relative to A but not open as subsets of R.

Impossible, $\forall E$ is open relative to A, $\exists G$ open in R, s.t., $E = A \cap G$, since both of A and G are open in R, thus E is open in R.

(d) Let $A = (0, 1)$ describe, if possible, sets that are closed relative to A but not closed as subsets of R.

Set E that is closed relative to A but not closed as subsets of R could be written as $E = F \cap A$ where F is a closed set in $\mathbb{R}$ and $F' \cap \{0, 1\} \neq \emptyset$.

2 $4.7.5 \cdots [2\ pts]$

Let $A = \mathbb{Q}$. Give examples of sets that are neither open nor closed but are both relative to $\mathbb{Q}$.

For example: $E = \{\frac{1}{n} : n \in \mathbb{N}\}$. $E = (a, b) \cap Q = [a, b] \cap Q, \forall a, b \in \mathbb{R} \setminus \mathbb{Q}$ and $a < b$.

3 $4.7.6 \cdots [2\ pts]$

Show that all the subsets of $\mathbb{N}$ are both open and closed relative to $\mathbb{N}$.

Proof. for any subset E of $\mathbb{N}$, Let $E = \{a_i : a_i \in \mathbb{N}\}$. Let $G = \cup(a_i - 0.25, a_i + 0.25)$, then G is an open set in R and $E = G \cap \mathbb{N}$, so E is open relative to $\mathbb{N}$. Let $F = \cup[a_i - 0.25, a_i + 0.25]$, since

F is the union of disjoint closed sets, thus F is closed, and $E = F \cap \mathbb{N}$, so E is closed relative to $\mathbb{N}$. Thus any subset E of $\mathbb{N}$ is both open and closed relative to $\mathbb{N}$. $\square$

4 $\quad 4.7.8 \cdots [3\ pts]$

Let E be a nonempty set of real numbers and define the function

$$f(x) = \inf\{|x - e| : e \in E\}.$$

(a) Show that $f(x) = 0$ for all $x \in E$.
Since $x \in E$, then $0 \leq f(x) = \inf\{|x - e| : e \in E\} \leq |x - x| = 0$, thus $f(x) = 0$.

(b) Show that $f(x) = 0$ if and only if $x \in \overline{E}$.
$\Leftarrow$, for $x \in \overline{E} = E \cap E'$, then if $x \in E$, by (a), $f(x) = 0$; if $x \in E'$, then $\forall \epsilon > 0, \exists y \in E$ and $y \neq x$,s.t., $|x - y| < \epsilon$, then $f(x) \leq |x - y| < \epsilon$, so $f(x) = 0$.
$\Rightarrow$, Assume $x \notin \overline{E}$, then $\exists \delta > 0$, define $B := \{y \in \mathbb{R}, |x - y| < \delta\}$, then $B \cap E = \emptyset$. which means $f(x) > \delta$, however, it is contradictive with $f(x) = 0$. Thus, $x \in \overline{E}$.

(c) Show for any nonempty closed set E that $\{x \in \mathbb{R} : f(x) > 0\} = (R \setminus E)$.
Define $S = \{x : f(x) = 0\}, P = \{x : f(x) > 0\}$, then $P = \mathbb{R} \setminus S$. By (b), $S = \overline{E} = E$, thus $P = \mathbb{R} \setminus E$.