MA 34100 Fall 2016, HW 8

October 27, 2016

1 5.6.8 $\cdots$ [3 *pts*]

Let f be a uniformly continuous function on a set E. Show that if $\{x_n\}$ is a Cauchy sequence in E then $\{f(x_n)\}$ is a Cauchy sequence in $f(E)$. Show that this need not be true if f is continuous but not uniformly continuous.

Proof. (1): To prove $\{f(x_n)\}$ is a Cauchy sequence just need to prove $\forall\epsilon > 0, \exists N$, s.t.,$\forall n, m > N$, have $|f(x_n) - f(x_m)| < \epsilon$. Since f is uniformly continuous on set E, thus $\forall\epsilon > 0, \exists\delta > 0$, s.t., $\forall x, y \in E$, if $|x - y| < \delta$,then $|f(x) - f(y)| < \epsilon$.as $\{x_n\}$ is a Cauchy sequence, then $\exists N$, s.t., $\forall n, m > N$, $|x_n - x_m| < \delta$, thus $|f(x_n) - f(x_m)| < \epsilon$ which proves $\{f(x_n)\}$ is a Cauchy sequence.
(2): for example $f(x) = \frac{1}{x}$, $x \in (0, 2)$ which is continuous but not uniformly continuous. $\{\frac{1}{n}\}$ is a Cauchy sequence, however, $\{f(x_n)\}$ does not converge which proves that it is not a Cauchy sequence. □

2 5.8.7 $\cdots$ [3 *pts*]

Let $f : [a, b] \to [a, b]$ be continuous. Define a sequence recursively by $z_1 = x_1, z_n = f(z_{n-1}) where x_1 \in [a, b]$. Show that if the sequence $\{z_n\}$ is convergent, then it must converge to a fixed point of f.

Proof. Assume $\{z_n\}$ converges to $z \in [a, b]$. Then to prove $\forall\epsilon > 0,\ |f(z) - z| < \epsilon$. Since f is continuous in $[a, b]$, thus $\exists\delta > 0$, $\forall x$,s.t.,$|x - z| < \delta$, $|f(x) - f(z)| < \epsilon$. Since $\{z_n\}$ converges to z, $\Rightarrow \exists N > 0$,s.t.,if $n > N$, $|z_n - z| < \min\{\delta, \epsilon\}$. Then $|f(z) - z| \le |f(z) - f(z_n)| + |f(z_n) - z| = |f(z) - f(z_n)| + |z_{n+1} - z| \le 2\epsilon$. □

3 5.10.13 $\cdots$ [4 *pts*]

Is there a continuous function $f : R \to R$ such that for every real y there are precisely three solutions to the equation $f(x) = y$?

Proof. There exists a continuous function $f : R \to R$ such that for every real y there are precisely three solutions to the equation $f(x) = y$. for example, $f(x) = x - 2(n + 1), \quad x \in [3n,\ 3n + 2]$ and $f(x) = 4n - x + 2, \quad x \in [3n + 2, 3n + 3]$, for $n \in \mathbb{Z}$. □