MA 34100 Fall 2016, HW 9

November 3, 2016

1 $7.2.6 \cdots [3\ pts]$

A function f has a symmetric derivative at a point if

$$f_s'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h}$$

exists. Show that $f_s'(x) = f'(x)$ at any point at which the latter exists but that $f_s'(x)$ may exist even when f is not differentiable at x.

Proof. (1). If assume at point x, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists, then

$$\begin{aligned} f_s'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} = \lim_{h\to 0} \frac{f(x+h) - f(x) + f(x) - f(s-h)}{2h} \\ &= \frac{1}{2}\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \frac{1}{2}\lim_{h\to 0} \frac{f(x) - f(x-h)}{h} \\ &= \frac{1}{2}f'(x) + \frac{1}{2}f'(x) = f'(x). \end{aligned}$$

(2). For example

$$f(x) = \begin{cases} x, & x > 0; \\ -x, & x \leq 0; \end{cases} \tag{1.1}$$

Then by definition of symmetric derivative, at point 0, $f_s'(0) = 0$, however, $f'(0)$ does not exist. □

2 $7.2.12 \cdots [4\ pts]$

If $f'(x_0) > 0$ for some point x_0 in the interior of the domain of f show that there is a $\delta > 0$ so that $f(x) < f(x_0) < f(y)$ whenever $x_0 - \delta < x < x_0 < y < x_0 + \delta$. Does this assert that f is increasing in the interval $(x_0 - \delta, x_0 + \delta)$?

Proof. (1). since

$$0 < \alpha = f'(x_0) = \lim_{h\to 0^+} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0^+} \frac{f(x_0) - f(x_0-h)}{h},$$

then $\forall \alpha > \epsilon > 0, \exists \delta > 0$, s.t.,$|f'(x_0) - \frac{f(x_0+h)-f(x_0)}{h}| < \epsilon$, whenever $y = x_0 + h < x_0 + \delta$, then through easy computation we get

$$0 < \alpha - \epsilon < \frac{f(x_0+h) - f(x_0)}{h} < \alpha + \epsilon,$$

thus $f(y) - f(x_0) > 0$ for $0 < y - x_0 < \delta$. For the same reason, we can get $f(x_0) - f(x) > 0$ for $0 < x_0 - x < \delta$.

(2) f may not be increasing in $(x_0 - \delta, x_0 + \delta)$. For example, let

$$f(x) = \begin{cases} x + 2x^2 \sin(\frac{1}{x}) & x \neq 0, \\ 0 & x = 0, \end{cases} \tag{2.2}$$

Then

$$f'(x) = \begin{cases} 1 - 2\cos(\frac{1}{x}) + 4x \sin(\frac{1}{x}) & x \neq 0, \\ 1 & x = 0, \end{cases} \tag{2.3}$$

Then $\forall \delta > 0$, $\exists x \in (x_0 - \delta, x_0 + \delta)$,s.t.,$f'(x) < 0$, thus f is not increasing in $(x_0 - \delta, x_0 + \delta)$. □

3 $7.6.3 \cdots [3\ pts]$

If the nth-degree equation

$$p(x) = a_0 + a_1 x + a_2 x_2 + \cdots + a_n x_n = 0,$$

has n distinct real roots, then how many distinct real roots does the (n-1)st degree equation $p'(x) = 0$ have ?

Proof. $p'(x) = 0$ has n-1 distinct real roots. First, since degree of $p'(x)$ is n-1, then at most it has n-1 real roots. Second, assume $r_1 < r_2 \cdots < r_n$ are n distinct real roots of $p(x)$, then by Rolle's Theorem, $\exists \{c_i\}_{i=1,2,\ldots,n-1}$,s.t.,$c_i \in (r_i, r_{i+1})$ and $p'(c_i) = 0$. so we find n-1 distinct real roots of $p'(x) = 0$, since at most it has n-1 roots, thus it exactly has n-1 distinct real roots. □