Quiz 4

Problem. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $|f'(x)| \leq \pi/6$ for all $x \in \mathbb{R}$. If $f(0) = \ln(5)$ what are the possible values of $f(e)$?

Solution. It must be that $f(e) \in [-\frac{\pi e}{6} + \ln(5), \frac{\pi e}{6} + \ln(5)]$. If $f(e)$ does not belong to this interval, then

$$\left|\frac{f(e) - f(0)}{e - 0}\right| > \pi/6$$

which means that there is $x \in [0, e]$ such that $|f'(x)| > \pi/6$ by the Mean Value Theorem. Conversely, by picking $m \in [-\pi/6, \pi/6]$ the function $f(x) = \ln(5) + mx$ can be made to achieve at $f(e)$ any value in $[-\frac{\pi e}{6} + \ln(5), \frac{\pi e}{6} + \ln(5)]$.