MATH 490, WORKSHEET #10
WEDNESDAY, APRIL 17

Problem 1, Putnam 1954. Let N be a set with an odd number of elements. Let $f : N \times N \to N$ satisfy $f(x, y) = f(y, x)$ and $\{f(x, y) : y \in N\} = N$. Show that $\{f(x, x) : x \in N\} = N$.

Problem 2, Putnam 1984. Let $f(n) = 1! + 2! + \cdots + n!$. Show $f(n + 1) = a(n)f(n) + b(n)f(n - 1)$ for some polynomials a, b.

Problem 3, Putnam 2014. Let $A = (A_{ij})$ be the $n \times n$ matrix

$$A_{ij} = \frac{1}{\min(i, j)}$$

for $1 \leqslant i, j \leqslant n$. Compute $\det(A)$.

Problem 4, VTRMC 2017. Let P be an interior point of a triangle of area T. Through the point P, draw lines parallel to the three sides, partitioning the triangle into three triangles and three parallelograms. Let a, b, and c be the areas of the three triangles. Prove that $\sqrt{T} = \sqrt{a} + \sqrt{b} + \sqrt{c}$.

Problem 5, VTRMC 2015. Consider the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. Prove that every positive rational number can be obtained as an unordered partial sum of this series. (An unordered partial sum may skip some of the terms $\frac{1}{k}$.)

Problem 6, Putnam 1969. If G is a finite group, show that G cannot be written as the union of two proper subgroups. Can G be written as the union of three proper subgroups?

Problem 7, ICMC 2016. Let G be a group which satisfies $(gh)^3 = g^3h^3$ and such that $g^3 = e$ implies $g = e$. Show that $g \mapsto g^3$ is a bijection if G is finite.