MATH 490, WORKSHEET #3
WEDNESDAY, FEBRUARY 20

Problem 1. Show the following identities for the Fibonacci numbers $F_0 = 1, F_1 = 1, \dots, F_m, \dots$.

(1) $F_0 + \cdots F_n = F_{n+2} - 1$.
(2) $F_0 - F_1 + F_2 - F_3 + \cdots - F_{2n-1} + F_{2n} = F_{2n-1} - 1$.
(3) $F_0^2 + F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$.
(4) $F_{n-1}F_{n+1} = F_n^2 + (-1)^n$.
(5) $F_{m+n} = F_{m+1}F_n + F_m F_{n-1}$.
(6) If $m|n$ then $F_m|F_n$.

Hint on last two: Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and show $A^n = \begin{pmatrix} Fn & F_{n-1} \\ F_{n-1} & F_{n-2} \end{pmatrix}$.

Problem 2. You are given 7 line segments of lengths at least 1 unit and at most 10 units. Show that three may be found that can form the sides of a triangle.

Problem 3, Putnam 1996. A selfish subset of $\{1, \dots, n\}$ is a subset which contains it cardinality as an element. For instance $\{1, 3, 10\}$ and $\{4, 7, 8, 9\}$ are selfish. Find the number of minimal selfish subsets of $\{1, \dots, n\}$. Minimal means no proper subset is selfish.

Problem 4, Putnam 2000. Show that there are infinitely many integers n such that n, $n + 1$, and $n + 2$ are each sums of two squares.

Problem 5, ICMC 1983. Let p be a prime and $a_1, \dots, a_p$ a list of integers which are not necessarily distinct or arranged in order. Show that there are $1 \leqslant m < n \leqslant p$ so that $p|a_m + \cdots + a_n$.

Problem 6, ICMC 1978. Let k be odd. Define

$$S(n) = \sum_{i=1}^{n} i^k.$$

Show that $(n + 1)|2S(n)$. Hint: use that $(a + b)|(a^k + b^k)$.

Problem 7, ICMC 1976. If $12|(n + 1)$ then $12 \mid \sum_{a|n} a$. Hint: show that there are prime factors p and q of n such that $p = 5 \bmod 6$ and $q = 3 \bmod 4$.

Problem 8, ICMC 2005. If 25 divides $x^5+y^5+z^5$, it divides at least one of x^5+y^5, y^5+z^5 or x^5+z^5.