MATH 490, WORKSHEET #8
WEDNESDAY, APRIL 3

Problem 1, ICMC 2018. Let $a_1, \ldots, a_n$ be positive real numbers. If $a_1^x + a_2^x + \cdots + a_n^x \geqslant n$ for all real numbers x, show that $a_1 a_2 \cdots a_n = 1$.

Problem 2, ICMC 1995. If $n \geqslant 1$ is an integer, show that

$$1 \cdot 2^2 \cdot 3^3 \cdots n^n < \left(\frac{2n+1}{3}\right)^{n(n+1)/2}.$$

Problem 3, Putnam 2016. Let $f : \mathbb{R} \to \mathbb{R}$ be a function so that

$$f(x) + f(1 - \frac{1}{x}) = \arctan(x).$$

Compute $\int_0^1 f(x)dx$.

Problem 4, VTRMC 2002. Find rational numbers a, b, c, d, e so that

$$\sqrt{7 + \sqrt{40}} = a + b\sqrt{2} + c\sqrt{5} + d\sqrt{7} + e\sqrt{10}.$$

Problem 5, VTRMC 2009. Is there a twice differentiable function such that $f'(x) = f(x+1) - f(x)$ for all x and $f''(0) > 0$?

Problem 6, Putnam 2014. Show that every coefficient of the Taylor expansion of $(1 - x + x^2)e^x$ about 0 is rational and the numerator is either 1 or a prime number.

Problem 7, ICMC 2011. Find a function $f : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ so that for each $y \in \mathbb{R} \setminus \{0\}$ there is exactly one $x \in \mathbb{R}$ with $f(x) = y$.

Problem 8, ICMC 2015. Let $f : \mathbb{R} \to \mathbb{R}$ so that $f^{(n)}$ exists for all n. If f has infinitely many zeroes in $[0, 1]$ show that there is a point $x \in [0, 1]$ so that $f^{(n)}(x) = 0$ for all n. Give an example of such a function which is nonconstant on every interval.