Blame It On The Analyst

von Neumann's proof of Radon-Nikodym

23/06/2010 in Analysis, For Mathematicians, Probability | Tags: absolutely continuous, lebesgue decomposition, measure theory, radon-nikodym

There is a remarkably nice proof of the Lebesgue decomposition theorem (described below) by von Neumann. This leads immediately to the Radon-Nikodym theorem.

Theorem:

> *If μ and ν are two finite measures on $(\Omega, \mathcal{F})$ then there exists a non-negative (w.r.t. both measures) measurable function f and a μ-null set B such that*
>
> $$\nu(A) = \int_A f\, d\mu + \nu(A \cap B)$$
>
> *for each $A \in \mathcal{F}$.*

Proof:

Let $\pi := \mu + \nu$ and consider the operator

$$T(f) := \int f\, d\nu.$$

It is obvious that the operator is linear and moreover for any $f \in L^2(\pi)$ we have

$$|T(f)| \leq ||f||_{L^2}$$

so that T is a linear functional on $L^2(\pi)$. By the Reisz representation theorem for Hilbert spaces there exists a $h \in L^2(\pi)$ such that

$$T(f) = \int f d\nu = \int fh\, d\pi = \int fh\, d\mu + \int fh\, d\nu. \quad (*)$$

Now consider the following sets;

$$N := \{h < 0\}, \quad M := \{0 \leq h < 1\}, \quad B := \{h \geq 1\}.$$

First by (*)

$$0 \geq \int_N h\, d\pi = \int \mathbf{1}_N h\, d\pi = \nu(N) = \int_N h\, d\mu + \int_N h\, d\nu$$

which gives that $\nu(N) = \mu(N) = 0$.

Next we have that

$$\nu(B) = T(\mathbf{1}_B) = \int_B h\, d\mu + \int_B h\, d\nu \geq \nu(B) + \mu(B)$$

so that $\mu(B) = 0$.

For the last set consider $M_n := \{0 \leq h \leq 1 - 1/n\}$, then by rearranging (*) we have,

$\int(1-h)f\,d\nu = \int hf\,d\mu$ in particular

$\nu(M_n) = \int \frac{\mathbf{1}_{M_n}}{1-h}(1-h)\,d\nu = \int h\frac{\mathbf{1}_{M_n}}{1-h}\,d\mu$.

Let $f = \frac{h}{1-h}$ then by applying monotone convergence and recalling that $\mu(B) = \mu(N) = 0$ we have,

$\nu(M \cap A) = \int_A f\,d\mu$.

Thus putting this all together, for any $A \in \mathcal{F}$ we have

$\nu(A) = \nu(A \cap N) + \nu(A \cap M) + \nu(A \cap B) = \int_A f\,d\mu + \nu(A \cap B)$

as claimed.

Q.E.D.

There is a rather obvious extension of this to σ-finite measures but the theorem does not hold for infinite measures.

We say that a measure ν is absolutely continuous with respect to μ (written $\mu >> \nu$) if whenever $\mu(A) = 0$ then $\nu(A) = 0$. The Radon-Nikodym theorem deals with this. Essentially it says that we ν has a density with respect to μ. For instance many people know the p.d.f. of a Gaussian distribution but the reason that the p.d.f. exists in because the Gaussian measure is absolutely continuous with respect to the Lebesgue measure.

Corollary:

> *A (sigma) finite measures ν is absolutely continuous with respect to an other (sigma) finite measure if and only if there exists a measurable function f such that*
>
> $\nu(A) = \int_A f\,d\mu$.

Proof:

Clearly if two measures are related by the given formula then $\nu << \mu$.

The converse follows from the theorem above as $\nu(A \cap B) = 0$.

Q.E.D.

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