Notes on Type III von Neumann Algebras

Thomas Sinclair

Mathematics Department, Purdue University, 150 N. University Street, West Lafayette, IN 47907-2067

Email address: tsincla@purdue.edu

URL: http://www.math.purdue.edu/~tsincla/

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Abstract. The following are course notes compiled for the class "Introduction to Type III von Neumann Algebras" which I taught in the spring 2024 semester at Purdue University.

Contents

CHAPTER 1

Operators on Hilbert Space

1. Basic Theory

Throughout H will denote a complex Hilbert space with inner product $\langle \xi \,, \eta \rangle$. The inner product is a *sequilinear form* which we always take to be conjugate linear in the second argument. We write $\|\xi\| := \langle \xi \,, \xi \rangle^{1/2}$.

Proposition 1.1 (Cauchy–Schwarz Inequality). *For all $\xi, \eta \in H$ we have that*

$$|\langle \xi \,, \eta \rangle| \leq \|\xi\|\|\eta\|. \tag{1}$$

Definition 1.2. Let $T : H \to K$ be a linear map between Hilbert spaces. We say that T is *bounded* if there is a constant C so that $\|T\xi\| \leq C\|\xi\|$ for all $\xi \in H$.

In this case, we define the operator norm $\|T\|$ to be the infimum over all such C. Equivalently,

$$\|T\| = \sup_{\xi \neq 0} \frac{\|T\xi\|}{\|\xi\|} = \sup_{\|\xi\|=1} \|T\xi\| \tag{2}$$

Note that by linearity, boundedness is the same as T being Lipschitz.

Lemma 1.3. *Let $T : X \to Y$ be a linear map between normed spaces (not necessarily complete). The following are equivalent.*

1. *T is bounded.*
2. *T is norm continuous.*
3. *$\|\xi_n\| \to 0$ implies that $\|T\xi_n\| \to 0$.*

Definition 1.4. A linear map $T : H \to H$ is said to be a *linear operator*. We denote by $\mathcal{B}(H)$ the collection of all bounded linear operators.

Lemma 1.5. *For $T, S \in \mathcal{B}(H)$ we have that $\|\lambda T + S\| \leq |\lambda| \, \|T\| + \|S\|$ and $\|TS\| \leq \|T\|\|S\|$.*

Lemma 1.6. *For $T \in \mathcal{B}(H)$ we have that*

$$\|T\| = \sup_{\|\xi\|=\|\eta\|=1} |\langle T\xi \,, \eta \rangle|. \tag{3}$$

Proof. Fixing ξ, $|\langle T\xi \,, \eta \rangle|$ is optimized when $\eta = \frac{T\xi}{\|T\xi\|}$ over $\|\eta\| = 1$ by Cauchy–Schwarz. □

PROPOSITION 1.7 (Riesz Representation Theorem). *Let $\phi : H \to \mathbb{C}$ be a bounded linear map (that is, a* linear functional*). There is a* unique *$\eta \in H$ with $\|\eta\| = \|\phi\|$ so that*

$$\phi(\xi) = \langle \xi , \eta \rangle . \tag{4}$$

DEFINITION 1.8. We call a sesquilinear form $\kappa : H \times H \to \mathbb{C}$ *bounded* if there is a C so that $|\kappa(\xi, \eta)| \leq C\|\xi\|\|\eta\|$. The infimum of all such C is denoted by $\|\kappa\|$.

LEMMA 1.9. *κ is a bounded sesquilinear form on H if and only if there is a (unique) bounded operator $T \in \mathcal{B}(H)$ so that $\kappa(\xi, \eta) = \langle T\xi , \eta \rangle$. Moreover $\|\kappa\| = \|T\|$.*

PROOF. Clearly $\langle T\xi , \eta \rangle$ is a bounded sesquilinear form of norm $\|T\|$ by Lemma 1.6. In the other direction, fixing ξ, we have that $\eta \mapsto \overline{\kappa(\xi, \eta)}$ is a bounded linear functional, so $\kappa(\xi, \eta) = \langle \xi' , \eta \rangle$ uniquely by the Riesz Representation Theorem (Proposition 1.7). Define $T\xi := \xi'$. It is an exercise to check that T is linear. $\|T\| = \|\kappa\|$ again by Lemma 1.6. □

LEMMA 1.10. *Let (T_n) be a net in $\mathcal{B}(H)$ with $\sup_n \|T_n\| \leq M$. If $(\langle T_n\xi , \eta \rangle)$ converges for every $\xi, \eta \in H$, then $\langle T\xi , \eta \rangle := \lim_n \langle T_n\xi , \eta \rangle$ defines a bounded linear operator with $\|T\| \leq M$.*

PROOF. It is straightforward to see that $\lim_n \langle T_n\xi , \eta \rangle$ defines a bounded sequilinear form with norm at most M. The result then follows by Lemma 1.9. □

DEFINITION 1.11. Let (T_n) be a net in $\mathcal{B}(H)$ we say that (T_n) converges to $T \in \mathcal{B}(H)$ in the *weak operator topology* (WOT) if $\lim_n \langle T_n\xi , \eta \rangle = \langle T\xi , \eta \rangle$ for all $\xi, \eta \in H$. Note by the previous lemmas such a T is unique, so the WOT is Hausdorff.

REMARK 1.12. The WOT is weaker than the norm topology on $\mathcal{B}(H)$ is the sense that norm convergence implies WOT convergence.

NOTATION 1.13. For a normed vector space X, we denote the closed unit ball about 0 by $(X)_1$.

PROPOSITION 1.14. *$(\mathcal{B}(H))_1$ is compact in the WOT.*

PROOF. Let $X = \prod_{(H)_1 \times (H)_1} \mathbb{D}$, where $\mathbb{D}$ is the closed unit disc about 0 in the complex plane. By Tychonoff's Theorem, X is compact in the product topology. Define a map $F : (\mathcal{B}(H))_1 \to X$ by

$$F(T)_{(\xi,\eta)} = \langle T\xi , \eta \rangle .$$

By Lemma 1.10, we have that F has closed image. Moreover, it is straightforward to check the F is an injection and the pullback of the product topology induces the (relative) WOT on $(\mathcal{B}(H))_1$. This established the result. □

LEMMA 1.15. *The $\mathcal{B}(H)$ is norm complete, hence it is a Banach algebra.*

PROOF. A norm Cauchy sequence (T_n) has a WOT accumulation point T by Proposition 1.14. It is an exercise to check that $\|T_n - T\| \to 0$. □

DEFINITION 1.16. We say that $T \in \mathcal{B}(H)$ has an adjoint $T^* \in \mathcal{B}(H)$ if $\langle T\xi\,, \eta\rangle = \langle \xi\,, T^*\eta\rangle$.

REMARK 1.17. It is clear from the definition that $(T^*)^* = T$, $(\lambda T + S)^* = \overline{\lambda}T^* + S^*$ and $(TS)^* = S^*T^*$ if T and S have adjoints.

The proof of the following is an exercise in using the Riesz Representation Theorem (Proposition 1.7).

LEMMA 1.18. *Every $T \in \mathcal{B}(H)$ has a unique adjoint.*

LEMMA 1.19 (The "C*-Identity"). *For $T \in \mathcal{B}(H)$ we have that $\|T^*T\| = \|T\|^2$.*

PROOF. We have that

$$\|T^*T\| = \sup_{\|\xi\|=\|\eta\|=1} |\,\langle T^*T\xi\,, \eta\rangle\,| = \sup_{\|\xi\|=\|\eta\|=1} |\,\langle T\xi\,, T\eta\rangle\,|.$$

By Cauchy–Schwarz, the right hand side is optimized when $T\xi = T\eta$ □

DEFINITION 1.20. For the $T \in \mathcal{B}(H)$, we say that T is *self=adjoint* if $T^* = T$ and *normal* if $T^*T = TT^*$.

REMARK 1.21. If T is self-adjoint, then $\langle T\xi\,, \xi\rangle \in \mathbb{R}$ for all $\xi \in H$. The converse is also true, but left as an exercise.

PROPOSITION 1.22. *If $T \in \mathcal{B}(H)$ is self-adjoint, then*

$$\|T\| = \sup_{\|\xi\|=1} |\,\langle T\xi\,, \xi\rangle\,|.$$

PROOF. We assume $\|\xi\| = \|\eta\| = 1$. Using that $T^* = T$ we have that

$$\langle T(\xi \pm \eta)\,, \xi \pm \eta\rangle = \langle T\xi\,, \xi\rangle \pm 2\,\mathrm{Re}\,\langle T\xi\,, \eta\rangle + \langle T\eta\,, \eta\rangle$$

which yields that

$$4\,\mathrm{Re}\,\langle T\xi\,, \eta\rangle = \langle T(\xi + \eta)\,, \xi + \eta\rangle - \langle T(\xi - \eta)\,, \xi - \eta\rangle$$

By the Parallelogram Identity we have that

$$4 = \|\xi + \eta\|^2 + \|\xi - \eta\|^2.$$

Altogether, this implies that one of the following inequalities must hold:

$$|\,\mathrm{Re}\,\langle T\xi\,, \eta\rangle\,| \le \frac{|\,\langle T(\xi + \eta)\,, \xi + \eta\rangle\,|}{\|\xi + \eta\|^2}$$
$$|\,\mathrm{Re}\,\langle T\xi\,, \eta\rangle\,| \le \frac{|\,\langle T(\xi - \eta)\,, \xi - \eta\rangle\,|}{\|\xi - \eta\|^2}.$$

Adjusting η by a phase we may assume without loss of generality that $\langle T\xi\,, \eta\rangle$ is real, which establishes the result. □

2. Positive Operators

DEFINITION 2.1. We say that $T \in \mathcal{B}(H)$ is *positive* if $T^* = T$ and $\langle T\xi\,,\xi\rangle \geq 0$ for all $\xi \in H$. We denote the set of all positive operators by $\mathcal{B}(H)^+$.

For $T, S \in \mathcal{B}(H)$, both self-adjoint, we write $T \preceq S$ if $S - T$ is positive.

LEMMA 2.2. *The following are true:*

1. *For all $S \in \mathcal{B}(H)$, $S^*S \in \mathcal{B}(H)^+$.*
2. *$\mathcal{B}(H)^+$ is a cone and closed in the WOT.*
3. *$-\|T\|I \preceq T \preceq \|T\|I$ for all T self-adjoint.*
4. *If $0 \preceq T \preceq S$, then $\|T\| \leq \|S\|$.*
5. *If $-S \preceq T \preceq S$ with $0 \preceq S$, then $\|T\| \leq \|S\|$*

REMARK 2.3. By a set C in a vector space V being a *cone* we mean that $C + C \subseteq C$, $tC \subseteq C$ for all $t \geq 0$, and $C \cap (-C) = \{0\}$.

PROOF. For (1), we observe that $\langle S^*S\xi\,,\xi\rangle = \langle S\xi\,,S\xi\rangle \geq 0$.

For (2), it is clear that a convex combination of positive operators or a positive scalar multiple of positive operator is again positive. If T and $-T$ are both positive, then $\langle T\xi\,,\xi\rangle = 0$ for all ξ, which implies $T = 0$ by Proposition 1.22. This shows that $\mathcal{B}(H)^+$ is a cone. If $T_n \to T$ in the WOT with each T_n positive, we have that $\langle T_n\xi\,,\xi\rangle \to \langle T\xi\,,\xi\rangle$ so $\langle T\xi\,,\xi\rangle \geq 0$ as well.

For (3), we note that if $\|T\| \leq 1$, then $-\langle \xi\,,\xi\rangle \leq \langle T\xi\,,\xi\rangle \leq \langle \xi\,,\xi\rangle$.

For (4), we have that $(\xi, \eta) := \langle T\xi\,,\eta\rangle$ defines a pre-inner product, so by Cauchy–Schwarz we have that

$$|\langle T\xi\,,\eta\rangle| \leq \langle T\xi\,,\xi\rangle^{1/2} \langle T\eta\,,\eta\rangle^{1/2}.$$

This implies that $\|T\| = \sup_{\|\xi\|=1} \langle T\xi\,,\xi\rangle$ for all $T \succeq 0$. The conclusion follows directly.

For (5), we have that $-\langle S\xi\,,\xi\rangle \leq \langle T\xi\,,\xi\rangle \leq \langle S\xi\,,\xi\rangle$, so the result follows by the proof of the previous item and Proposition 1.22. □

PROPOSITION 2.4. *For all $T \in \mathcal{B}(H)^+$ there is $S \in \mathcal{B}(H)^+$ so that $T = S^2$.*

PROOF. Consider the following power series expansion of the square root

$$\sqrt{x} = \sum_{k=0}^{\infty} (-1)^k \binom{1/2}{k} (1-x)^k = \sum_{k=0}^{\infty} \frac{1}{2^{2k-2}} \frac{(2k-3)!}{k!(k-2)!} (1-x)^k.$$

Note that

$$O\left(\frac{1}{2^{2k-2}} \frac{(2k-3)!}{k!(k-2)!}\right) = O\left(\frac{1}{k} \cdot \frac{1}{4^k} \binom{2k}{k}\right) = O\left(\frac{1}{k^{3/2}}\right),$$

where the last equality follows by applying Stirling's approximation formula to $\binom{2k}{k}$. Therefore the series converges absolutely for all $|1-x| \leq 1$, in particular for all $0 \leq x \leq 1$.

Write $a_k := (-1)^k \binom{1/2}{k}$ for the coefficients above. Since $a_0 = 1$ and $a_k < 0$ for all $k \geq 1$, we may set $c_k := -a_k \geq 0$, so that

$$\sqrt{x} = 1 - \sum_{k=1}^{\infty} c_k (1-x)^k.$$

Evaluating at $x = 0$ gives $0 = 1 - \sum_{k=1}^{\infty} c_k$, hence $\sum_{k=1}^{\infty} c_k = 1$.

Now suppose $0 \preceq T \preceq I$, so that $0 \preceq I - T \preceq I$ and $\|I - T\| \leq 1$ by Lemma 2.2. Setting

$$S_N := I - \sum_{k=1}^{N} c_k (I-T)^k,$$

the bound $\|(I-T)^k\| \leq \|I-T\|^k \leq 1$ together with $\sum_k c_k < \infty$ shows that (S_N) is Cauchy, hence converges in norm to some self-adjoint $S \in \mathcal{B}(H)$.

We claim that $S \succeq 0$. First note that each $(I-T)^k$ is positive: for $k = 2m$ we have $(I-T)^{2m} = \big((I-T)^m\big)^*(I-T)^m \succeq 0$, while for $k = 2m+1$ we have $(I-T)^{2m+1} = (I-T)^m(I-T)(I-T)^m \succeq 0$ since $\langle (I-T)^m(I-T)(I-T)^m\xi \,, \xi\rangle = \langle (I-T)(I-T)^m\xi \,, (I-T)^m\xi\rangle \geq 0$. As $\|(I-T)^k\| \leq 1$, Lemma 2.2 gives $0 \preceq (I-T)^k \preceq I$, so that $\big\langle (I-T)^k\xi \,, \xi\big\rangle \in [0,1]$ for every unit vector ξ. Therefore

$$\langle S\xi \,, \xi\rangle = 1 - \sum_{k=1}^{\infty} c_k \left\langle (I-T)^k\xi \,, \xi\right\rangle \geq 1 - \sum_{k=1}^{\infty} c_k = 0,$$

which proves $S \in \mathcal{B}(H)^+$.

Finally, since S and T both lie in the commutative Banach algebra generated by $I - T$, and the series for S converges absolutely, we may form the Cauchy product. As $\big(\sum_k a_k y^k\big)^2 = 1 - y$ as an identity of formal power series, the Cauchy product yields

$$S^2 = I - (I - T) = T,$$

which completes the proof. □

Corollary 2.5. *For $T \in \mathcal{B}(H)$, the following are equivalent:*

(1) *T is positive.*
(2) *$T = \sum_i S_i^* S_i$.*
(3) *$T = S^* S$.*

Lemma 2.6. *Let $H_0 \subset H$ be a closed subspace, and let P be the orthogonal projection onto H_0. Then $P = P^* = P^2$. Conversely, any $P \in \mathcal{B}(H)$ with $P = P^* = P^2$ is the orthogonal projection onto its range.*

Notation 2.7. For $T \in \mathcal{B}(H)$ we write $|T| := (T^*T)^{1/2} \in \mathcal{B}(H)^+$, where the square root is defined as in Proposition 2.4.

Proposition 2.8. *Every self-adjoint operator $T \in \mathcal{B}(H)$ can be written as a difference of positive operators $T = S_1 - S_2$ with $\|S_1\|, \|S_2\| \leq \|T\|$*

PROOF. Let $S_1 = (|T|+T)/2$ and $S_2 = (|T|-T)/2$, and note that T, $|T|$, S_1, and S_2 are mutually commuting since T is self-adjoint which implies that $|T|$ is a power series in T. Let P be the orthogonal projection onto the kernel of S_1, which is a closed subspace of H. Then $S_1 P = 0$, which implies that $PS_1 = 0$, by taking adjoints. Since $S_1 S_2 = 0$ we have that $PS_2 = S_2$, so $S_2 P = S_2$ again by taking adjoints. This gives the following system of linear equations:

$$\begin{cases} PT & = -P|T| \\ (I-P)T & = (I-P)|T|. \end{cases}$$

Thus $T = (I-2P)|T| = |T|(I-2P)$. It therefore follows that

$$\begin{cases} S_1 = (I-P)|T| = |T|(I-P) \\ S_2 = P|T| = |T|P. \end{cases}$$

Since P is an orthogonal projection $P = P^* = P^2$ by Lemma 2.6, so $S_2 = P|T|P^*$ is a positive with $\|S_2\| \leq \||T|\| = \|T^*T\|^{1/2} = \|T\|$ by the C*-identity (Lemma 1.19). The same holds for S_1, which establishes the result. □

COROLLARY 2.9. *Every $T \in (\mathcal{B}(H))_1$ is a linear combination of four positive contractions.*

PROOF. We have that $T = (T+T^*)/2 + i[(T-T^*)/2i]$ is a decomposition of T into a linear combination of two self-adjoint contractions. □

3. States

NOTATION 3.1. For a (complex) normed space X, we write X^* for the normed space of all linear maps $\phi : X \to \mathbb{C}$ with $\|\phi\| := \sup_{\|x\|=1} |\phi(x)|$.

DEFINITION 3.2. A *state* $\phi \in \mathcal{B}(H)^*$ is linear functional satisfying $\phi(I) = 1$ and $\phi(T) \geq 0$ for all $T \in \mathcal{B}(H)^+$.

REMARK 3.3. Note that every state is $*$-*linear*, that is $\phi(T^*) = \overline{\phi(T)}$, since $\phi(T) \in \mathbb{R}$ for all T self-adjoint by Proposition 2.4.

REMARK 3.4. If ϕ is a state, then $T \preceq S$ implies that $\phi(T) \leq \phi(S)$.

PROPOSITION 3.5. *We have that $\phi \in \mathcal{B}(H)^*$ is a state if and only if $\phi(I) = 1$ and $\|\phi\| = 1$.*

PROOF. For the forward direction, we have that ϕ being a state implies that $(T,S) \mapsto \phi(S^*T)$ defines a pre-inner product on $\mathcal{B}(H)$, so by Cauchy–Schwarz we have that

$$|\phi(T)| \leq \phi(T^*T)^{1/2}\phi(I)^{1/2} \leq \|T^*T\|^{1/2} = \|T\|$$

by the C*-identity and the fact that $T^*T \preceq \|T^*T\|I$ (Lemma 2.2).

The reverse direction is slightly more involved. First, we claim that every functional with $\phi(I) = 1$ and $\|\phi\| = 1$ is $*$-linear. This is equivalent to showing that $\phi(T) \in \mathbb{R}$ if $T = T^*$. We see that for a parameter $s \in \mathbb{R}$

$$\begin{aligned}|\phi(T) + is| &= |\phi(T + isI)| \\ &\leq \|T + isI\| \\ &= \|(T + isI)^*(T + isI)\|^{1/2} \\ &= \|T^2 + s^2 I\|^{1/2} \leq (\|T^2\| + s^2)^{1/2}\end{aligned}$$

since $T^2 + s^2 I \leq (\|T^2\| + s^2)I$ by Lemma 2.2. Therefore, choosing s to have the same sign as $\operatorname{Im} \phi(T)$, we have that

$$|\operatorname{Im} \phi(T)| \leq (\|T^2\| + s^2)^{1/2} - |s| \to 0$$

as $|s| \to \infty$, which establishes the claim.

We next claim that every $*$-linear functional with $\phi(I) = 1$ and $\|\phi\| = 1$ is a state, which now suffices to establish the result. Taking $T \in \mathcal{B}(H)^+$, we may assume without loss of generality that $\|T\| \leq 1$. We have that $0 \preceq T \preceq I$; hence, $-I \preceq 2T - I \preceq I$ and $\|2T - I\| \leq 1$ by Lemma 2.2. It follows that $|2\phi(T) - 1| \leq 1$, which shows that $\phi(T) \geq 0$ since $\phi(T) \in \mathbb{R}$ by $*$-linearity. This establishes the claim, so the result. $\square$

Definition 3.6. Let X be a normed space. We define the *weak*-topology* on X^* by a net (ϕ_n) in X^* converges to $\phi \in X^*$ weak* if $\phi_n(x) \to \phi(x)$ for all $x \in X$.

Lemma 3.7 (Banach–Alaoglu). *We have that $(X^*)_1$ is compact in the weak*-topology.*

Recall, that for a convex subset $C \subset V$ of a vector space, a point $x \in C$ is *extreme* if $x = ty + (1-t)z$ for $y, z \in C$ implies that $x = y = z$.

Lemma 3.8 (Krein–Milman). *A closed, convex subset of $(X^*)_1$ is the weak* closure of the hull of its extreme points.*

Notation 3.9. We write $S(\mathcal{B}(H))$ for the set of all states on $\mathcal{B}(H)$.

Remark 3.10. By Proposition 3.5, we see that $S(\mathcal{B}(H))$ is a weak*-closed, convex subset of $(\mathcal{B}(H)^*)_1$. Thus $S(\mathcal{B}(H))$ is compact by Banach–Alaoglu and the weak*-closure of the convex hull of its extreme points by Krein–Milman.

Lemma 3.11. *If $\phi \in \mathcal{B}(H)^*$ is a state and $P \in \mathcal{B}(H)$ is a projection with $\phi(P) = 0$, then $\phi(PT) = 0 = \phi(TP)$ for all $T \in \mathcal{B}(H)$.*

Proof. By Cauchy–Schwarz,

$$|\phi(PT)| \leq \phi(PP^*)^{1/2}\phi(T^*T)^{1/2} = \phi(P)^{1/2}\phi(T^*T)^{1/2} = 0.$$

The other case follows by $*$-linearity. $\square$

Notation 3.12. We write a *rank-one operator* $T = \xi \otimes \bar{\eta}$, where $T(\zeta) = \langle \zeta , \eta \rangle \xi$; that is, ξ spans the range of T and η spans the range of T^*, where T^* can be computed to be $\eta \otimes \bar{\xi}$.

This notation is justified as follows. We have that $f_\zeta : H \times \bar{H} \to H$ given by

$$f_\zeta(\xi, \bar{\eta}) = \langle \zeta \, , \eta \rangle \, \xi$$

is bilinear and $f_\zeta(\lambda\xi, \bar{\eta}) = f_\zeta(\xi, \lambda\bar{\eta})$ for all $\lambda \in \mathbb{C}$, thus f_ζ extends to a canonical map on the (algebraic) tensor product $H \otimes \bar{H} \to H$ and so this induces a linear map $F : H \otimes \bar{H} \to \mathcal{B}(H)$ by

$$F(T)(\zeta) = f_\zeta(T) = \sum_i \langle \zeta \, , \eta_i \rangle \, \xi_i$$

for $T = \sum_i \xi_i \otimes \bar{\eta}_i$.

Lemma 3.13. *F is an embedding. In this way $H \otimes \bar{H}$ is identified with the* finite rank operators *on $\mathcal{B}(H)$, that is all $T \in \mathcal{B}(H)$ so that T and T^* both have finite-dimensional range.*

Proof. Writing $T = \sum \xi_i \otimes \bar{\eta}_i$ we may assume without loss of generality that the ξ_i's are linearly independent. Thus $F(T)(\zeta) = 0$ for all $\zeta \in H$ implies that $\langle \zeta \, , \eta_i \rangle = 0$ for all i and all ζ, so $\eta_i = 0$ for all i. $\square$

Notation 3.14. For $\xi \in H$ with $\|\xi\| = 1$ we define the *vector state* $\phi_\xi(T) := \langle T\xi \, , \xi \rangle$.

Lemma 3.15. *For each $\xi \in H$, we have that ϕ_ξ is an extreme state.*

Proof. Suppose that $\phi_\xi = t\phi_1 + (1-t)\phi_2$. We have that $\|\xi \otimes \bar{\xi}\| = 1$ (as an operator) and $\phi_\xi(\xi \otimes \bar{\xi}) = 1$, so $\phi_i(\xi \otimes \bar{\xi}) = 1$ for $i = 1, 2$. Let K be the orthogonal complement of ξ in H and let P be the orthogonal projection onto K. We have that $P + \xi \otimes \bar{\xi} = I$, so $\phi_i(P) = 0$ for $i = 1, 2$. By Lemma 3.11 we may conclude that $\phi_1 = \phi_\xi = \phi_2$. $\square$

Notation 3.16. We write $\mathcal{F}(H)$ for the algebra of all finite-rank operators on H.

Note that $\mathcal{F}(H)$ is an ideal in $\mathcal{B}(H)$, that is $xTy \in \mathcal{F}(H)$ for all $T \in \mathcal{F}(H)$ and $x, y \in \mathcal{B}(H)$.

Definition 3.17. We define a pairing between $\mathcal{F}(H)$ and $\mathcal{B}(H)$ as follows. For $T = \sum_i \xi \otimes \bar{\eta}_i \in \mathcal{F}(H)$ and $x \in \mathcal{B}(H)$, we write

$$\langle T \, , x \rangle := \sum_i \langle x\xi_i \, , \eta_i \rangle$$

We again note that $H \times \bar{H} \ni (\xi, \bar{\eta}) \mapsto \langle x\xi \, , \eta \rangle$ is bilinear and $(\lambda\xi, \bar{\eta})$ and $(\xi, \lambda\bar{\eta})$ map to the same scalar, so this pairing is well-defined.

Remark 3.18. When $x = I$, we write $\mathrm{Tr}(T) = \langle T \, , I \rangle = \sum_i \langle \xi_i \, , \eta_i \rangle$. It is clear that Tr is a positive linear functional on $\mathcal{F}(H)$ in the sense that $\mathrm{Tr}(T^*T) \geq 0$. Moreover, we have that for every unitary $u \in \mathcal{B}(H)$, i.e., $u^*u = uu^* = I$,

$$\mathrm{Tr}(uTu^*) = \sum_i \langle u\xi_i \, , u\eta_i \rangle = \sum_i \langle \xi_i \, , \eta_i \rangle = \mathrm{Tr}(T). \tag{5}$$

We refer to Tr as the *trace* on $\mathcal{F}(H)$.

PROBLEM 3.19. Check that $\mathrm{Tr}(TS) = \mathrm{Tr}(ST)$ for all $T, S \in \mathcal{F}(H)$.

DEFINITION 3.20. We define the *nuclear norm* on $\mathcal{F}(H)$ by

$$\|T\|_{\mathrm{nuc}} := \inf \left\{ \sum_i \|\xi_i\| \cdot \|\eta_i\| : T = \sum_i \xi_i \otimes \bar{\eta}_i \right\}$$

We write $\mathcal{L}^1(H)$ for the completion of $\mathcal{F}(H)$ in the nuclear norm.

LEMMA 3.21. *The following inequality is apparent from the definitions:*

$$|\langle T\,, x\rangle| \le \|x\| \cdot \|T\|_{\mathrm{nuc}}$$

for all $T \in \mathcal{F}(H)$ and $x \in \mathcal{B}(H)$.

LEMMA 3.22. *We have that $\|\xi \otimes \bar{\eta}\|_{\mathrm{nuc}} = \|\xi\| \cdot \|\eta\|$.*

PROOF. Suppose that $\xi \otimes \eta = \sum_i x_i \otimes \bar{y}_i$. Let P_ξ and P_η be the orthogonal projections onto $\mathbb{C}\xi$ and $\mathbb{C}\eta$, respectively. Then by linearity

$$\xi \otimes \eta = \sum_i P_\xi(x_i) \otimes \overline{P_\eta(y_i)},$$

and since orthogonal projections are contractions

$$\sum_i \|P_\xi(x_i)\| \cdot \|P_\eta(y_i)\| \le \sum_i \|x_i\| \cdot \|y_i\|.$$

It follows that we may assume that $x_i \otimes \bar{y}_i = c_i \xi \otimes \bar{\eta}$ with $\sum_i c_i = 1$. The result is then apparent by the triangle inequality. □

PROPOSITION 3.23. *We have that $T \mapsto \langle T\,, x\rangle$ isometrically identifies $\mathcal{B}(H)$ with $\mathcal{L}^1(H)^*$.*

PROOF. Let $\phi_x(T) := \langle T\,, x\rangle$. By Lemma 3.21 we have that ϕ_x is defined on $\mathcal{L}^1(H)$ with $\|\phi_x\| = \sup_{\|T\|_{\mathrm{nuc}}=1} |\langle T\,, x\rangle| \le \|x\|$. For vectors ξ and η of unit norm we have that by Lemma 3.22 that $\|\xi \otimes \bar{\eta}\|_{\mathrm{nuc}} = 1$, so

$$|\phi_x(\xi \otimes \bar{\eta})| = |\langle \xi \otimes \bar{\eta}\,, x\rangle| = |\langle x\xi\,, \eta\rangle|$$

verifies that $\|\phi_x\| = \|x\|$. □

REMARK 3.24. We observe that the pairing $x \mapsto \langle T\,, x\rangle$ also identifies $\mathcal{L}^1(H)$ isometrically with a subspace of $\mathcal{B}(H)^*$ by the theory of dual Banach spaces. In this way $\mathcal{L}^1(H)$ is weak*-dense in $\mathcal{B}(H)^*$ by Goldstine's Theorem. We refer to Proposition 3.26 below for the proof of a similar assertion.

LEMMA 3.25. *We have that $T \in \mathcal{F}(H)$, self-adjoint, defines a state on $\mathcal{B}(H)$ via $x \mapsto \langle T\,, x\rangle$ if and only if $T = \sum_i \xi_i \otimes \bar{\xi}_i$ with $\sum_i \|\xi_i\|^2 = 1$. Moreover, the ξ_i's may be taken to be an orthogonal family.*

Proof. In the reverse direction, we have that $\langle T\,, x\rangle = \sum_i \langle x\xi_i\,, \xi_i\rangle \geq 0$ for all $x \in \mathcal{B}(H)^+$ and that $\langle T\,, I\rangle = \sum_i \|\xi_i\|^2 = 1$, thus T defines a state.

In the forward direction, we have that $\langle T\,, \eta \otimes \bar{\eta}\rangle = \langle T\eta\,, \eta\rangle \geq 0$ for all $\eta \in H$, so T is positive, so $T = \sum_i \xi_i \otimes \bar{\xi}_i$ where the ξ_i's form an (orthogonal) eigenbasis by the Spectral Theorem for matrices. □

Proposition 3.26. *Viewing $\mathcal{F}(H) \subset \mathcal{B}(H)^*$ via the standard pairing, let $K := \mathcal{F}(H)^+ \cap S(\mathcal{B}(H))$. We have that K is weak*-dense in $S(\mathcal{B}(H))$.*

Proof. Assume that there is a state ϕ which does not belong to the weak*-closure of K. Since the weak*-closure of K is compact (Remark 3.10) we have that there is a basic weak* open neighborhood of the origin $V = \{\psi \in \mathcal{B}(H)^* : |\psi(x_i)| < \epsilon\}$ for some $x_1, \ldots, x_n \in \mathcal{B}(H)$ and $\epsilon > 0$ so that $(\phi + V) \cap (K + V) = \emptyset$. By the Geometric Hahn–Banach Theorem, there is $q \in \mathcal{B}(H)^{**}$ so that

$$\sup_{T \in K+V} \operatorname{Re} q(T) < \operatorname{Re} q(\phi). \tag{6}$$

Since $\operatorname{Re} q$ is bounded on V by the triangle inequality, we have that

$$\ker q \supseteq \cap_{i=1}^n \ker \{\mathcal{B}(H)^* \ni \psi \mapsto \psi(x_i)\}\,.$$

It follows that

$$q(\psi) = \psi(x)$$

for all ψ where x belongs to the span of $x_1, \ldots, x_n$. Setting $s = (x + x^*)/2$ we have that by Proposition 2.8 we have that $s = s_1 - s_2$, with s_1, s_2 positive with $s_1 = psp$ and $s_2 = -(1-p)s(1-p)$ for some orthogonal projection p. By Lemma 3.25 and Proposition 1.22 we therefore have that

$$\sup_{T \in K} \operatorname{Re} q(T) = \sup_{T \in K} \langle T\,, s\rangle = \sup_{\|\xi\|=1} \langle s\xi\,, \xi\rangle = \|s_1\|.$$

On the other hand, since ϕ is a state we have that

$$\operatorname{Re} q(\phi) = \phi(s) = \phi(s_1 - s_2) \leq \phi(s_1) \leq \|s_1\|.$$

Altogether this contradicts inequality (6). □

Remark 3.27. The content of the previous theorem could be stated as follows. Let $\phi \in S(\mathcal{B}(H))$. Then for any finite set of operators $x_1, \ldots, x_n \in \mathcal{B}(H)$ and $\epsilon > 0$ there is an orthonormal set $\xi_1, \ldots, \xi_k \in H$ with $\sum_i \|\xi_i\|^2 = 1$ so that

$$\left|\phi(x_i) - \sum_j \langle x_i\xi_j\,, \xi_j\rangle\right| < \epsilon.$$

4. Normal Operators, Spectra, and Abelian C*-Algebras

Recall that an operator $T \in \mathcal{B}(H)$ is *normal* if $T^*T = TT^*$. The following is the crucial fact about normal operators.

LEMMA 4.1. *$T \in \mathcal{B}(H)$ is normal if and only if $\|T\xi\| = \|T^*\xi\|$ for all $\xi \in H$.*

In what follows we will only make use of the "only if" direction.

DEFINITION 4.2. We say that $\lambda \in \mathbb{C}$ belongs to the *spectrum* of T if $T - \lambda I$ is not invertible in $\mathcal{B}(H)$. We write $\mathrm{sp}(T) \subset \mathbb{C}$ for the set of all spectral elements of T.

REMARK 4.3. We have that $\mathrm{sp}(T)$ is contained in the closed ball of radius $\|T\|$ about 0. This is because for any $|\lambda| > \|T\|$ we have that $\|\frac{1}{\lambda}T\| < 1$, hence

$$\left(I - \frac{1}{\lambda}T\right)^{-1} = I + \sum_{k=1}^{\infty}\left(\frac{1}{\lambda}T\right)^k$$

which converges in norm.

REMARK 4.4. We have that $\mathrm{sp}(T)$ is always nonempty. We have that

$$\frac{(T-\lambda I)^{-1} - (T-\omega I)^{-1}}{\lambda - \omega} = -(T-\lambda I)^{-1}(T-\omega I)^{-1}$$

which shows that for $\phi \in \mathcal{B}(H)^*$ that $f_\phi : \lambda \mapsto \phi\left((T-\lambda I)^{-1}\right)$ is analytic on any open region of $\mathbb{C} \setminus \mathrm{sp}(T)$. Thus if $\mathrm{sp}(T) = \emptyset$, we have that f_ϕ is entire and bounded since $\|(T-\lambda I)^{-1}\| \to 0$ as $|\lambda| \to \infty$, so by Liouville's theorem f_ϕ is constant 0 for all $\phi \in \mathcal{B}(H)^*$. By the Hahn–Banach Theorem, this implies that $(T-\lambda I)^{-1} = 0$, which is absurd.

LEMMA 4.5. *If $T \in \mathcal{B}(H)$ is not invertible, then one of the following must happen:*

(1) *there is a sequence $\xi_n \in H$ with $\|\xi_n\| = 1$ so that $\|T\xi_n\| = 0$;*
(2) *the closure of the range of T is not equal to H.*

PROOF. Suppose both do not occur. Then $\alpha := \inf_{\|\xi\|=1}\|T\xi\| > 0$ and $\overline{T(H)} = H$. Since the first condition implies that T is injective, we have that $T^{-1} : T(H) \to H$ exists and that $\|T^{-1}\| \leq \frac{1}{\alpha}$. Since T^{-1} is Lipschitz, this implies that T^{-1} extends to a bounded linear operator on $\overline{T(H)} = H$. Thus T is invertible in $\mathcal{B}(H)$. □

LEMMA 4.6. *If $T \in \mathcal{B}(H)$ is normal, then $\lambda \in \mathrm{sp}(T)$ if and only if there exists a (possibly constant) sequence (ξ_n) of unit vectors so that $\|T\xi_n - \lambda\xi_n\| \to 0$.*

PROOF. Since T is normal so is $T - \lambda I$. If the result did not hold, then by Lemma 4.5 we would have that there is nonzero η in the orthogonal complement of $(T-\lambda I)(H)$. Thus

$$0 = \langle (T-\lambda I)\xi \,, \eta\rangle = \langle \xi \,, (T-\lambda I)^*\eta\rangle$$

for all $\xi \in H$, that is, $\|(T-\lambda I)^*\eta\| = 0$. By Lemma 4.1 this means that $\|(T-\lambda I)\eta\| = 0$, so $T\eta = \lambda\eta$. □

Remark 4.7. For the case of normal operators the previous result gives a quick proof that $\mathrm{sp}(T)$ is closed. Indeed if $\lambda_i \to \lambda$ with $\lambda_i \in \mathrm{sp}(T)$, choose ξ_i a unit vector with $\|T\xi_i - \lambda_i \xi_i\| \leq 1/i$. Then $\|T\xi_i - \lambda\xi_i\| \to 0$.

Definition 4.8. A *C^*-algebra* $\mathcal{A}$ is an (operator) norm closed $*$-subalgebra of $\mathcal{B}(H)$ for some H.

Remark 4.9. In these notes we will usually insist that $I \in \mathcal{A}$.

Proposition 4.10. *The following facts are true about a C^*-algebra $\mathcal{A} \subset \mathcal{B}(H)$ (with $I \in \mathcal{A}$).*

(1) *If $x \succeq 0$ and $x \in \mathcal{A}$, then $x^{1/2} \in \mathcal{A}$. In particular for every $x \in \mathcal{A}$, $|x| = (x^*x)^{1/2} \in \mathcal{A}$.*
(2) *Every $x \in \mathcal{A}$ is a linear combination of four elements of $\mathcal{A}^+$.*
(3) *$\phi \in \mathcal{A}^*$ is a state if and only if $\phi(I) = 1$ and $\|\phi\| = 1$.*
(4) *The set of all states $S(\mathcal{A})$ is weak*-compact in $(\mathcal{A}^*)_1$.*
(5) *$S(\mathcal{A})$ is the weak*-closed convex hull of the extreme points.*

Lemma 4.11. *Let $\mathcal{B} \subset \mathcal{A} \subset \mathcal{B}(H)$ be C^*-algebras with $I \in \mathcal{B}$. Every extreme state on $\mathcal{B}$ has an extreme state extension to $\mathcal{A}$.*

Proof. Let $\phi \in S(\mathcal{B})$. By the Hahn–Banach theorem, there is a $\tilde{\phi} \in \mathcal{A}^*$ which extends ϕ so that $\|\tilde{\phi}\| = \|\phi\|$. So $\tilde{\phi}(I) = 1 = \|\tilde{\phi}\|$ and $\tilde{\phi}$ is a state by Proposition 4.10. Let $S_\phi \subseteq S(\mathcal{A})$ be the set of extensions of ϕ, which we can see is convex and weak*-compact, so has an extreme point by the Krein–Milman theorem. Let ϕ' be such an extreme point.

We claim that ϕ' is extreme in $S(\mathcal{A})$ as well. If $\phi' = t\psi_1 + (1-t)\psi_2$ for $0 < t < 1$, then

$$\phi = \phi'|_{\mathcal{B}} = t\psi_1|_{\mathcal{B}} + (1-t)\psi_2|_{\mathcal{B}},$$

so $\psi_i|_{\mathcal{B}} = \phi$ for $i = 1, 2$ by extremality of ϕ. Thus $\psi_1, \psi_2 \in S_\phi$ as well which establishes the result. $\square$

Definition 4.12. For $T \in \mathcal{B}(H)$ we let $C^*(T)$ be the C*-algebra generated by $\{I, T, T^*\}$.

Remark 4.13. We see that $C^*(T)$ is abelian if and only if T is normal.

Proposition 4.14. *Let $\mathcal{A} \subset \mathcal{B}(H)$ be an abelian C^*algebra with $I \in \mathcal{A}$. A state $\phi \in S(\mathcal{A})$ is extreme if and only if it is multiplicative.*

Proof. We begin by assuming that ϕ is multiplicative. For $x = x^* \in \mathcal{A}$ and a parameter $s \in \mathbb{R}$, define e^{isx} using the power series expansion, which is entire real anayltic, so $e^{isx} \in C^*(x) \subseteq \mathcal{A}$. Via the power series expansion we also see that

$$(e^{isx})^* = e^{-isx}$$

and that

$$e^{-isx}e^{isx} = e^{isx}e^{-isx} = I$$

since $C^*(x)$ is a commutative algebra. Thus e^{isx} is unitary and $\|e^{isx}\|^2 = \|(e^{isx})^* e^{isx}\| = 1$, so e^{isx} is a contraction.

Suppose that $\phi = t\psi_1 + (1-t)\psi_2$ for some $t \in (0,1)$ and $\psi_1, \psi_2 \in S(\mathcal{A})$. Since ϕ is multiplicative and $*$-linear since it is a state we have that

$$1 = \phi((e^{isx})^* e^{isx}) = \phi((e^{isx})^*)\phi(e^{isx}) = \overline{\phi(e^{isx})}\phi(e^{isx}),$$

so $\phi(e^{isx}) \in \mathbb{T}$, the complex units. Since $|\psi_j(e^{isx})| \leq 1$ for $j = 1, 2$, this implies that

$$\phi(e^{isx}) = \psi_1(e^{isx}) = \psi_2(e^{isx})$$

for all $x = x^*$, $s \in \mathbb{R}$. Using the power series expansion and the triangle inequality, we have that

$$\left\| \frac{e^{isx} - I}{s} - ix \right\| \to 0$$

as $s \to 0$, which implies that $\phi(x) = \psi_1(x) = \psi_2(x)$ for all $x \in \mathcal{A}$ self-adjoint. Hence $\phi = \psi_1 = \psi_2$ by $*$-linearity.

We next assume that ϕ is extreme. Let $0 \preceq x \preceq I$. We will first show that $\phi(xy) = \phi(x)\phi(y)$ for all $y \in \mathcal{A}$. We write

$$x_\epsilon := \frac{x + \epsilon I}{1 + 2\epsilon}$$

so $0 \preceq x_\epsilon \preceq I$ for all $\epsilon > 0$, $\|x_\epsilon - x\| \leq \epsilon$, and $\phi(x_\epsilon) \in (0,1)$. We then have that

$$\phi(x_\epsilon y) = \phi(x_\epsilon)\left[\frac{1}{\phi(x_\epsilon)}\phi(x_\epsilon y)\right] + (1 - \phi(x_\epsilon))\left[\frac{1}{\phi(I - x_\epsilon)}\phi((I - x_\epsilon)y)\right]$$

We claim that $\psi_1(y) = \frac{1}{\phi(x_\epsilon)}\phi(x_\epsilon y)$ and $\psi_2(y) = \frac{1}{\phi(I - x_\epsilon)}\phi((I - x_\epsilon)y)$ are states. It is clear that $\phi_i(I) = 1$ for $i = 1, 2$. If $y = (\sqrt{y})^2$, then $x_\epsilon y = (\sqrt{x_\epsilon}\sqrt{y})(\sqrt{x_\epsilon}\sqrt{y})^*$ since $\mathcal{A}$ is abelian. Thus $x_\epsilon y$ is positive and so is $\psi_1(y)$ since ϕ is a state. A similar argument holds for ψ_2.

By extremality, it therefore must be the case that $\phi(y) = \frac{1}{\phi(x_\epsilon)}\phi(x_\epsilon y)$ for all $\epsilon > 0$, so $\phi(x)\phi(y) = \phi(xy)$ for all $0 \preceq x \preceq 1$ and all $y \in \mathcal{A}$. By linearity, the same holds for all $x \in \mathcal{A}^+$. Since every element in $\mathcal{A}$ is a linear combination of four elements of $\mathcal{A}^+$, it follows that ϕ is multiplicative. □

COROLLARY 4.15. *Let $\mathcal{A}$ be an abelian C^*-algebra with $I \in \mathcal{A}$, and let $\phi \in S(\mathcal{A})$ be an extremal state. For $x \in \mathcal{A}$ we have that $\phi(x) \in \operatorname{sp}(x)$.*

PROOF. If $\phi(x) = \lambda$, then $\phi(x - \lambda I) = 0$, so $x - \lambda I$ cannot be invertible since ϕ is multiplicative. □

COROLLARY 4.16. *If $x \in \mathcal{B}(H)$ is normal, then there is a one-to-one correspondence between extremal states on $C^*(x)$ and $\operatorname{sp}(x)$.*

PROOF. Suppose $\lambda \in \operatorname{sp}(x)$. Then there is a sequence $\xi_n \in H$ of unit vectors so that $\|x\xi_n - \lambda\xi_n\| \to 0$. Let ϕ_λ be a weak*-cluster point of the sequence $\phi_{\xi_n}(x) = \langle x\xi_n, \xi_n\rangle$ of vector states which exists by weak*-compactness of $S(C^*(x))$. We have that $\phi_\lambda(x) = \lambda$ and that ϕ restricted to $C^*(x)$ is multiplicative, so extremal.

By Corollary 4.15, every extreme state in $S(C^*(x))$ must be equal to ϕ_λ for some $\lambda \in \mathrm{sp}(x)$ and this correspondence is one-to-one since for multiplicative states $\phi, \psi \in S(\mathcal{A})$ we have that $\phi = \psi$ if and only if $\phi(x) = \psi(x)$ by continuity. $\square$

DEFINITION 4.17. For $T \in \mathcal{B}(H)$ we define the *spectral radius* of T to be

$$|\operatorname{sp}|(T) := \sup\{|\lambda| : \lambda \in \mathrm{sp}(T)\}.$$

LEMMA 4.18. *If $T \in \mathcal{B}(H)$ is normal, then $|\operatorname{sp}|(T) = \|T\|$.*

PROOF. Since $S := T^*T$ is positive, we have that there is a sequence of unit vectors ξ_i so that $\langle S\xi_i\,, \xi_i\rangle \to \|S\| =: \sigma$. We compute using Lemma 2.2 that

$$\begin{aligned}\limsup_i \|S\xi_i - \sigma\xi_i\|^2 &= \limsup_i \left(\langle S\xi_i\,, S\xi_i\rangle - 2\sigma\,\langle S\xi_i\,, \xi_i\rangle + \sigma^2\right)\\ &= \left(\limsup_i \langle S^*S\xi_i\,, \xi_i\rangle\right) - \sigma^2 \le \|S^*S\| - \|S\|^2 = 0\end{aligned}$$

where the last equality holds by the C*-Identity.

Therefore, there is a multiplicative state ϕ on $C^*(T^*T)$ so that $\phi(T^*T) = \|T^*T\| = \|T\|^2$. By Lemma 4.11 and Proposition 4.14 we then have that there is a multiplicative state extension ϕ' of ϕ to $C^*(T)$. We have that

$$|\phi'(T)|^2 = \phi'(T^*)\phi'(T) = \phi(T^*T) = \|T\|^2, \tag{7}$$

which gives the result by Corollary 4.15. $\square$

PROPOSITION 4.19 (Spectral Mapping Theorem). *Let $T \in \mathcal{B}(H)$ be normal and $p(x)$ be a $*$-polynomial. Then $\mathrm{sp}(p(T)) = p(\mathrm{sp}(T))$.*

PROOF. If $\lambda \in \mathrm{sp}(T)$, then by Corollary 4.16 there is a multiplicative state ϕ on $C^*(T)$ so that $\phi(T) = \lambda$. Thus, $\phi(p(T)) = p(\lambda)$ and $p(\lambda) \in \mathrm{sp}(p(T))$ by Corollary 4.15. This shows that $p(\mathrm{sp}(T)) \subseteq \mathrm{sp}(p(T))$.

In the other direction, suppose that $\lambda \in \mathrm{sp}(p(T))$, so there is a multiplicative state ϕ on $C^*(p(T))$ so that $\phi(p(T)) = \lambda$. By Lemma 4.11 and Proposition 4.14 we have that there is a multiplicative state ϕ' on $C^*(T)$ which extends ϕ. We have again that $\phi'(T) := \theta \in \mathrm{sp}(T)$ and

$$p(\theta) = p(\phi'(T)) = \phi'(p(T)) = \phi(p(T)) = \lambda,$$

which shows that $\mathrm{sp}(p(T)) \subseteq p(\mathrm{sp}(T))$. $\square$

DEFINITION 4.20. Let X be compact hausdorff space. We let $C(X)$ be the algebra of continuous functions $f : X \to \mathbb{C}$. We have that $C(X)$ is an algebra under pointwise product with a involution given by $f^*(x) = \overline{f(x)}$ and a norm $\|f\| := \sup_{x\in X} |f(x)|$.

PROPOSITION 4.21 (Continuous Functional Calculus). *Let $T \in \mathcal{B}(H)$ be normal. There is an isometric $*$-isomorphism $C(\mathrm{sp}(T)) \to C^*(T)$ which is induced by*

$$p \mapsto p(T)$$

for every $*$-polynomial p.

Proof. We have that $\mathrm{sp}(T)$ is a closed and bounded subset of $\mathbb{C}$. Let $P(\mathrm{sp}(T))$ be the subalgebra of all continuous functions on X which are the restriction of some $*$-polynomial to X. By Lemma 4.18 and Proposition 4.19 we have that

$$\|p\| = \sup_{\lambda \in \mathrm{sp}(T)} |p(\lambda)| = \sup_{\theta \in \mathrm{sp}(p(T))} |\theta| = \|p(T)\|$$

so $p \mapsto p(T)$ induces a well-defined isometric $*$-homomorphim of $P(\mathrm{sp}(T))$ into $C^*(T)$. The Stone–Weierstrass Theorem then implies that $P(\mathrm{sp}(T))$ is dense in $C(\mathrm{sp}(T))$, which gives that this map extends to an isometric surjection from $C(\mathrm{sp}(T))$ onto $C^*(T)$. It is straightforward to check that the extension is a $*$-homomorphism as well. $\square$

5. Strong Operator Topology and Borel Functional Calculus

Definition 5.1. The *Strong Operator Topology* (SOT) is the topology on $\mathcal{B}(H)$ given by a net $T_n \to T$ if and only if $\|T_n\xi - T\xi\| \to 0$ for all $\xi \in H$.

The *Strong* Operator Topology* (S*OT) is the topology on $\mathcal{B}(H)$ given by a net $T_n \to T$ if and only if $\|T_n\xi - T\xi\| + \|T_n^*\xi - T^*\xi\| \to 0$ for all $\xi \in H$.

Remark 5.2. The operator norm, S*OT, SOT, and WOT are successively weaker topologies on $\mathcal{B}(H)$. Indeed, if $T_n \to T$ in the SOT, then $T_n \to T$ in the WOT. Indeed,

$$|\langle (T_n - T)\xi\,, \eta\rangle| \leq \|(T_n - T)\xi\| \cdot \|\eta\|$$

by Cauchy–Schwarz. It is clear that operator norm convergence implies S*OT convergence.

Remark 5.3. We have that $T_n \to T$ in the SOT if and only if $(T_n - T)^*(T_n - T) \to 0$ in the WOT. Moreover, we have that $T_n \to T$ in the SOT implies that $T_n^*T_n \to T^*T$ in the WOT. Indeed, by Cauchy–Schwarz we have that

$$\|(T_n - T)\xi\|^2 \geq \left(\langle T_n^*T_n\xi\,, \xi\rangle^{1/2} - \langle T^*T\xi\,, \xi\rangle^{1/2}\right)^2.$$

Both assertions now follow from that fact that if $\langle S_n\xi\,, \xi\rangle \to \langle S\xi\,, \xi\rangle$ for all $\xi \in H$, then $S_n \to S$ in the WOT since by the Polarization Identity

$$4\,\langle S\xi\,, \eta\rangle = \sum_{k=1}^{4} i^k \left\langle S(\xi + i^k\eta)\,, \xi + i^k\eta\right\rangle.$$

Example 5.4. Let ℓ^2 be the space of square-summable sequence, and let $S : \ell^2 \to \ell^2$ be the one-sided shift, that is, $Se_i = e_{i+1}$, where (e_i) is the standard orthonormal basis for ℓ^2. We have that (S^n) and (S^{*n}) both converge to 0 in the WOT; however, (S^{*n}) stills converges to 0 in the SOT but (S^n) has no cluster point in the SOT.

Lemma 5.5. *Let (P_n) be an net of projections in $\mathcal{B}(H)$ which is increasing in the operator order. Let $H_n = P_n(H)$ be the range of P_n, $K = \overline{\bigcup_n H_n}$, and P be the orthogonal projection onto K. Then $P_n \to P$ in the SOT.*

PROOF. Since $\langle P_n\xi\,,\xi\rangle = \|P_n\xi\|^2$, we have that (P_n) is increasing in the operator order if and only if $\|P_n\xi\| \le \|P_k\xi\|$ whenever $n < k$, which happens if and only if (H_n) is an increasing net of subspaces of H which happens if and only if $P_nP_k = P_n$ for all $n < k$.

For $\xi \in H$, we have that

$$\|(P_n - P)\xi\| = \|(P_n - P)(P\xi + (I - P)\xi)\| = \|(P_n - P)P\xi\|$$

since $P_nP = P_n$ and $P^2 = P$. Therefore it suffices to check SOT convergence only when $\xi \in P(H) = K$.

Since $\xi \in K$ for every $\epsilon > 0$ there is n and $\xi_n \in H_n$ so that $\|\xi_n - \xi\| < \epsilon$. However, it is a property of an orthogonal projection that

$$\|P\xi - \xi\| \le \|\eta - \xi\|$$

for all $\eta \in P(H)$. Therefore $\|P_n\xi - \xi\| < \epsilon$.

We finally note that that for $k > n$, we have that $I - P_k \preceq I - P_n$ and are orthogonal projections, so

$$\|P_k\xi - \xi\| \le \|P_n\xi - \xi\|,$$

and the result obtains. □

REMARK 5.6. Note that the range of P_n maybe be strictly increasing, in which case P_n has no cluster point in the operator norm topology.

DEFINITION 5.7. Let X be a compact Hausdorff space, and let $\mathrm{Bor}(X)$ be the σ-algebra of all Borel subsets of X. We say that a map

$$E : \mathrm{Bor}(X) \to \mathcal{B}(H)$$

is a *spectral measure* if the following hold:

(1) For all $A \in \mathrm{Bor}(X)$, $E(A)$ is an orthogonal projection;
(2) $E(\emptyset) = 0$ and $E(X) = I$;
(3) $E(A)E(B) = E(A \cap B)$; and
(4) $E(\bigcup_i A_i) = \sum_i E(A_i)$ for any sequence of pairwise disjoint Borel sets.

The third condition implies that E is increasing under set inclusion and that all $E(A), E(B)$ are mutually commuting. The infinite sum is understood in the sense that $\sum_{i=1}^N E(A_i) \to E(\bigcup_i A_i)$ in the strong operator topology, which is the natural topology for convergence, since $P_N = \sum_{i=1}^N E(A_i)$ can be seen under the assumptions to be an increasing sequence of projections.

NOTATION 5.8. We let $B(X)$ be the set of all bounded Borel functions $f : X \to \mathbb{C}$, which is a normed $*$-algebra under pointwise product, $f^*(x) = \overline{f(x)}$ and $\|f\| = \sup_{x\in X} |f(x)|$. Note that $C(X) \subset B(X)$ (isometrically) is a closed $*$-subalgebra.

PROPOSITION 5.9 (Borel Functional Calculus). *Let $\pi : C(X) \to \mathcal{B}(H)$ be an isometric $*$-isomorphism with $\pi(1) = I$. There is an extension $\tilde{\pi} : B(X) \to \mathcal{B}(H)$ so that*

(1) *$\tilde{\pi}$ is a contractive $*$-homomorphism;*

(2) $\tilde{\pi}(B(X)) \subseteq \overline{\pi(C(X))}^{WOT}$;
(3) $E(A) := \tilde{\pi}(1_A)$ *for all* $A \in \mathrm{Bor}(X)$ *is a spectral measure.*

Proof. The Riesz–Markov–Kakutani Theorem shows that $C(X)^*$ is isometrically isomorphic with the Banach space $M(X)$ of all complex Borel regular measures on X with total variational norm. Recall that $\mu : \mathrm{Bor}(X) \to \mathbb{C}$ is a complex measure if for every sequence of (A_i) of pairwise disjoint Borel sets we have that $\mu(\bigcup_i A_i) = \sum_i \mu(A_i)$. In particular, since the sum converges under rearrangements, the sum is converging absolutely. The total variational norm is then defined to be

$$\|\mu\| = \sup_{\sqcup A_i = X} \sum_i |\mu_i(A_i)|$$

Where the sup is taken over all countable Borel partitions of X.

Since $\int f d\mu$ is defined for all $f \in B(X)$ and $\mu \in M(X)$ and

$$\left| \int f d\mu \right| \leq \|f\| \|\mu\|$$

we have that there is an inclusion $B(X) \hookrightarrow M(X)^* = C(X)^{**}$. Since the point masses δ_x, where $\delta_x(A) = 1$ if and only if $x \in A$ and $\delta_x(A) = 0$ otherwise, belong to $M(X)$, we have this inclusion is isometric, hence we may regard $B(X) \subseteq C(X)^{**}$. In particular, by Goldstine's Theorem, $C(X) \subset B(X)$ is weak* dense. That is, for every $f \in B(X)$, there is a net f_n in $C(X)$ with $\|f_n\| \leq \|f\|$ so that $\int f_n d\mu \to \int f d\mu$ for all $\mu \in M(X)$.

For each $\xi, \eta \in H$, we have that $C(X) \ni f \mapsto \langle \pi(f)\xi\,, \eta\rangle$ defines a bounded linear functional on $C(X)$ with norm at most $\|\xi\|\|\eta\|$, hence there is a measure $\mu_{\xi,\eta} \in M(X)$ so that

$$\int f d\mu_{\xi,\eta} = \langle \pi(f)\xi\,, \eta\rangle$$

for all $f \in C(X)$. Note that $H \times H \ni (\xi, \eta) \mapsto \mu_{\xi,\eta} \in M(X)$ is contractive an sesquilinear, so for all $g \in B(X)$, we have that

$$H \times H \ni (\xi, \eta) \mapsto \int g d\mu_{\xi,\eta}$$

is a bounded sesquilinear form of norm at most $\|g\|$. Therefore, there is a unique operator $\tilde{\pi}(g) \in \mathcal{B}(H)$ with $\|\tilde{\pi}(g)\| \leq \|g\|$ so that

$$\langle \tilde{\pi}(g)\xi\,, \eta\rangle = \int g d\mu_{\xi,\eta}$$

for all $\xi, \eta \in H$. By uniqueness $\tilde{\pi}(f) = \pi(f)$ for all $f \in C(X)$ and $\tilde{\pi}$ is $*$-linear. Further we have that $\tilde{\pi}(g) \in \overline{C(X)}^{WOT}$ since for any bounded net (g_n) in $C(X)$ converging to g in the weak*-topology, we have that

$$\langle \pi(g_n)\xi\,, \eta\rangle = \int g_n d\mu_{\xi,\eta} \to \int g d\mu_{\xi,\eta} = \langle \tilde{\pi}(g)\xi\,, \eta\rangle\,.$$

We have so far that $\tilde{\pi} : B(X) \to \mathcal{B}(H)$ is a well-defined extension of π which is $*$-linear and contractive and so that $\tilde{\pi}(B(X)) \subseteq \overline{C(X)}^{WOT}$. We now show that $\tilde{\pi}$ is multiplicative. For $f \in C(X)$ and $g \in B(X)$, choose a bounded net (g_n) in $C(X)$ converging to g in the weak* topology. We have that $fg_n \to fg$ weak* as well. Thus

$$\pi(f)\pi(g_n) = \tilde{\pi}(fg_n) \to \tilde{\pi}(fg),$$

while $\pi(f)\pi(g_n) \to \pi(f)\tilde{\pi}(g)$ in the WOT as well. Thus $\tilde{\pi}(fg) = \pi(f)\tilde{\pi}(g)$. Now for $h \in B(X)$, we boundedly approximate through (h_n) in $C(X)$ in the weak* topology to get that

$$\pi(h_n)\tilde{\pi}(g) = \tilde{\pi}(h_n g) \to \tilde{\pi}(hg)$$

while $\pi(h_n)\tilde{\pi}(g) \to \tilde{\pi}(h)\tilde{\pi}(g)$ in the WOT. Thus $\tilde{\pi}$ is multiplicative.

Finally we show that $E(A) = \tilde{\pi}(1_A)$ is a spectral measure. It is clear that $E(\emptyset) = 0$ and $E(X) = I$. By multiplicativity, we have that

$$E(A \cap B) = \tilde{\pi}(1_{A\cap B}) = \tilde{\pi}(1_A 1_B) = \tilde{\pi}(1_A)\tilde{\pi}(1_B) = E(A)E(B).$$

Thus by $*$-linearity each $E(A)$ is a orthogonal projection. To establish countable additivity, let (A_i) be a pairwise disjoint sequence of Borel sets. Since E is clearly finitely additive we have that

$$E\left(\bigcup_i A_i\right) - \sum_{i=1}^{N} E(A_i) = E\left(\bigcup_{i=N+1} A_i\right),$$

therefore since the right hand side is an orthogonal projection

$$\begin{aligned} \left\| \left(E(\bigcup_i A_i) - \sum_{i=1}^{N} E(A_i) \right) \xi \right\|^2 &= \left\| E(\bigcup_{i=N+1} A_i)\xi \right\|^2 \\ &= \left\langle E(\bigcup_{i=N+1} A_i)\xi \, , \xi \right\rangle \\ &= \mu_{\xi,\xi}(\bigcup_{i=N+1} A_i) \to 0 \end{aligned}$$

with the last line following from countable additivity of $\mu_{\xi,\xi}$. □

Corollary 5.10. *Let $T \in \mathcal{B}(H)$ be a normal operator. Let $E : \mathrm{sp}(T) \to \mathcal{B}(H)$ be the spectral measure obtained from extending the Continuous Functional Calculus. We have that*

$$T = \int \lambda \, dE(\lambda)$$

in the sense that for any Borel partition $A_1, \ldots, A_n$ of the spectrum of T and $\lambda_i \in A_i$ satisfying $\| \mathrm{id}_{\mathrm{sp}(T)} - \sum_i \lambda_i 1_{A_i} \| \leq \epsilon$ we have that

$$\|T - \sum_i \lambda_i E(A_i)\| \leq \epsilon.$$

Indeed for any $f \in B(X)$ and we have that $\tilde{\pi}(f) = f(T) := \int f(\lambda)\, dE(\lambda)$ in the sense that for all Borel partitions $A_1, \ldots, A_n$ of $\mathrm{sp}(T)$ and $\lambda_i \in A_i$ so that $\|f - \sum_i f(\lambda_i) 1_{A_i}\| \leq \epsilon$ we have that

$$\|\tilde{\pi}(f) - \sum_i f(\lambda_i) E(A_i)\| \leq \epsilon.$$

We give a couple of applications of the Borel Functional Calculus before continuing.

LEMMA 5.11. *Suppose that P_n is an increasing sequence of projections converging strongly to the identity. If (T_k) is a bounded net of operators so that $\lim_k \|(T_k - T)P_n\| = 0$, then $T_k \to T$ in the SOT.*

PROPOSITION 5.12. *Suppose $P, Q \in \mathcal{B}(H)$ are orthogonal projections. Let $P \wedge Q$ be the orthogonal projection onto the intersection of their ranges. We have that $(PQP)^n \to P \wedge Q$ in the SOT.*

PROOF. We have that PQP is a positive contraction, so the spectrum belongs to $[0, 1]$. Let $A_n = \mathrm{sp}(PQP) \cap ([0, 1 - 1/n) \cup \{1\})$ and $E_n = E(A_n)$. We have that $\bigcup_n A_n = \mathrm{sp}(PQP)$, so $E_n \to I = E(\bigcup_n A_n)$ in the SOT by countable additivity of the spectral measure.

Further we have that (x^k) converges uniformly on A_n as $k \to \infty$ to the function δ_1 for which $\delta_1(1) = 1$ and $\delta_1(x) = 0$ otherwise. Therefore by Borel Functional Calculus we have that

$$((PQP)^k - E(\{1\}))E_n = \int_{A_n} (\lambda^k - \delta_1(\lambda)) dE(\lambda)$$

and

$$\|((PQP)^k - E(\{1\}))E_n\| \leq \sup_{\lambda \in A_n} |\lambda^k - \delta_1(\lambda)| \to 0$$

as $k \to \infty$.

By Lemma 5.11 we thus have that $(PQP)^k \to E(\{1\})$ in the SOT. But $E(\{1\})$ is by definition the set of vectors $\xi \in H$ so that $PQP\xi = \xi$. Clearly all ξ in the range of $P \wedge Q$ belong to the subspace. On the other hand, since P and Q are strict contraction off of their respective ranges $PQP\xi = \xi$ implies that $P\xi = \xi$ and $Q\xi = \xi$. In this way we see that $E(\{1\}) = P \wedge Q$. ☐

PROPOSITION 5.13 (Von Neumann's Mean Ergodic Theorem). *Let $U \in \mathcal{B}(H)$ be a unitary operator, that is $U^*U = I = UU^*$. Let F be the orthogonal projection onto the set of vectors fixed by U. We have that*

$$\frac{1}{N} \sum_{k=1}^{N} U^k \to F$$

in the SOT.

PROOF. Since U is a unitary, we have that $\mathrm{sp}(U) \subseteq \mathbb{T}$, the set of all complex units. Let $A_n = \{\lambda \in \mathrm{sp}(U) : |\lambda - 1| \geq 1/n\}$ and $B_n = A_n \cup \{1\}$. Let $E_n = E(A_n)$ and $F_n = E(B_n)$.

We first show that $(I-U)E_n = E_n(I-U)$ is bounded from below on the range of E_n. By partitioning A_n into small intervals, by Borel Functional Calculus for every $\epsilon > 0$ we have projections $P_1, \ldots, P_k$ and $\lambda_1, \ldots, \lambda_k \in A_n$ so that $\sum_i P_i = E_n$ and

$$\|(I-U)E_n - \sum_i (1-\lambda_i)P_i\| < \epsilon$$

For ξ in the range of E_n we compute

$$\|(\sum_i (1-\lambda_i)P_i)\xi\|^2 = \sum_i |1-\lambda_i|^2 \|P_i\xi\|^2 \geq \frac{1}{n^2}\|\xi\|^2$$

so $(I-U)E_n$ is bounded from below.

We see that

$$(I-U)E_n \left[\frac{1}{N}\sum_{k=1}^{N} U^k\right] E_n = \frac{1}{N}(I-U^{N+1})E_n \to 0$$

in operator norm as $N \to \infty$. Since $(I-U)E_n$ is bounded from below this shows that

$$\left\|\frac{1}{N}\sum_{k=1}^{N} U^k E_n\right\| \to 0$$

as $N \to \infty$. It follows that

$$\left\|\frac{1}{N}\sum_{k=1}^{N} U^k F_n - F\right\| \to 0$$

as $N \to \infty$. As in the proof of Proposition 5.12, we result obtains. $\square$

6. Von Neumann Algebras

Definition 6.1. A *von Neumann algebra* $\mathcal{M} \subset \mathcal{B}(H)$ is a $*$-subalgebra with $I \in \mathcal{M}$ which is closed in the WOT.

Lemma 6.2. *Let $\mathcal{C} \subset \mathcal{B}(H)$ be a convex subset. We have that*

$$\overline{\mathcal{C}}^{WOT} = \overline{\mathcal{C}}^{SOT}.$$

Proof. The Geometric Hahn–Banach theorem implies that for any convex set C in a locally convex topological vector space V, the closure is the set of points of V which cannot be separated from C by a continuous linear functional. We have that $\phi : \mathcal{B}(H) \to \mathbb{C}$ is WOT (respectively, SOT) continuous if and only if $\phi^{-1}(\mathbb{D})$ contains a WOT (resp. SOT)-basic open neighborhood of 0. Since any WOT-basic open neighborhood contains an SOT-basic open neighborhood, we have that any ϕ which is WOT continuous is also SOT continuous.

In order to prove the result it therefore suffices to check that any SOT-continuous linear functional on $\mathcal{B}(H)$ is WOT continuous. Let $\psi : \mathcal{B}(H) \to \mathbb{C}$ be SOT continuous. We have that $\psi^{-1}(\mathbb{D})$ contains some SOT-basic open neighborhood of 0, say

$$V = \{T \in \mathcal{B}(H) : \max_{1 \leq i \leq n} \|T\xi_i\| < \epsilon\}.$$

It follows that

$$|\psi(T)| \leq C(\sum_i \|T\xi_i\|^2)^{1/2} \tag{8}$$

for all $T \in \mathcal{B}(H)$ for some constant C depending only on n and ϵ.

Consider the subspace $K \subset H^{\oplus n}$ given by

$$K = \{(T\xi_1, \ldots, T\xi_n) : T \in \mathcal{B}(H)\},$$

and define $\psi' : K \to \mathbb{C}$ by

$$\psi'(T\xi_1, \ldots, T\xi_n) := \psi(T).$$

By the inequality (8) we have that ψ' is well-defined and continuous, so extends continuously to $\overline{K}$. Thus by the Riesz Representation Theorem we have that there exists $\eta = (\eta_1, \ldots, \eta_n) \in \overline{K}$ so that

$$\psi(T) = \psi'(T\xi_1, \ldots, T\xi_n) = \sum_i \langle T\xi_i\,, \eta_i\rangle$$

from which it is clear that ψ is WOT continuous. □

Definition 6.3. For a subset $\mathcal{S} \subset \mathcal{B}(H)$ we define the *commutant* of S by

$$\mathcal{S}' := \{T \in \mathcal{B}(H) : TS = ST,\ \forall S \in \mathcal{S}\}$$

Lemma 6.4. *The following statements are true.*

1. $I \in \mathcal{S}'$.
2. $\mathcal{S}'$ *is a subalgebra and a* $*$*-subalgebra if* $\mathcal{S}^* = \mathcal{S}$.
3. $\mathcal{S}'$ *is WOT closed.*
4. $\mathcal{S}'' \supset \mathcal{S}$.
5. $\mathcal{S}''' = \mathcal{S}'$.

Proof. The proofs of these facts are left as exercises. □

In this way $\mathcal{S} \mapsto \mathcal{S}''$ is a closure operation on $\mathcal{B}(H)$. Remarkably this algebraically defined closure operation is an analytic closure. The following cornerstone result of the subject is due to von Neumann.

Proposition 6.5 (Double Commutant Theorem). *The following statements are equivalent.*

1. $\mathcal{M} \subset \mathcal{B}(H)$ *is a von Neumann algebra.*
2. $\mathcal{M} \subset \mathcal{B}(H)$ *is a* $*$*-subalgebra so that* $\mathcal{M}'' = \mathcal{M}$.

Proof. For (2) implies (1), we have that $I \in \mathcal{M}''$ and $\mathcal{M} = \mathcal{M}''$ is WOT-closed since $\mathcal{M}'' = (\mathcal{M}')'$ by Lemma 6.4. Thus $\mathcal{M}$ is a von Neumann algebra.

For (1) implies (2), we have that $\mathcal{M}$ is SOT-closed. Let $H_N = H^{\bigoplus N}$. We define an embedding $\theta_N : \mathcal{B}(H) \to \mathcal{B}(H_N)$ by

$$\theta_N(T)(\xi_1, \ldots, \xi_N) := (T\xi_1, \ldots, T\xi_N).$$

It is apparent that $\theta_N(\mathcal{M})$ is SOT-closed in $\mathcal{B}(H_N)$ for all N.

There is a natural action of the symmetric group S_N by unitary operators on H_N by permuting coordinates. We may also consider the group $\Sigma_N = \{\epsilon = (\epsilon_1, \ldots, \epsilon_N) : \epsilon_i \in \{-1, 1\}\}$ which acts by unitary transformations on H_N by $\epsilon \cdot (\xi_1, \ldots, \xi_N) = (\epsilon_1 \xi_1, \ldots, \epsilon_N \xi_N)$. Together $G_N := \Sigma_N S_N$ forms a unitary action of the group of *signed permutations* on H_N. These operators belong to $\theta_N(\mathcal{B}(H))'$. Moreover, for $T \in \mathcal{B}(H_N)$ we wrte $T = [T_{ij}]$ where $T_{ij} = P_i T P_j$ with P_i being the projection onto the ith coordinate. We have that

$$\frac{1}{|G_N|} \sum_{g \in G_N} g^* T g = \theta_N(\overline{T})$$

where $\overline{T} = \frac{1}{N} \sum_i T_{ii}$. We conclude that

$$\theta_N(\mathcal{M})'' = \theta_N(\mathcal{M})'' \cap \theta_N(\mathcal{B}(H)) = \theta_N(\mathcal{M}'').$$

For $\xi_1, \ldots, \xi_N \in H$, let $\xi = (\xi_1, \ldots, \xi_N) \in H_N$. Consider the orthogonal projection P onto $K := \overline{\theta_N(\mathcal{M})\xi}$. Note that $\xi = I\xi \in K$ and that $\theta_N(\mathcal{M})K \subseteq K$. Therefore, we have that P belongs to $\theta_N(\mathcal{M})'$, so $TP = PT$ for all $T \in \theta_N(\mathcal{M})'' = \theta_N(\mathcal{M}'')$. It follows that $T(K) \subseteq K$ for all $T \in \theta_N(\mathcal{M}'')$ so

$$\overline{\theta_N(\mathcal{M}'')\xi} = \overline{\theta_N(\mathcal{M})\xi}$$

since $\xi \in K$. Thus given any $\epsilon > 0$ and $T \in \mathcal{M}''$ we have $S \in \mathcal{M}$ so that

$$\|\theta_N(T)\xi - \theta_N(S)\xi\|^2 = \sum_i \|(T - S)\xi_i\|^2 < \epsilon.$$

This shows that $\mathcal{M}'' \subseteq \overline{\mathcal{M}}^{SOT} = \mathcal{M}$, hence $\mathcal{M}'' = \mathcal{M}$ by Lemma 6.4. □

LEMMA 6.6. *For $T \in \mathcal{B}(H)$ we have that $\ker(T) = \ker(T^*T)$ and $\ker(T)^\perp = \overline{T^*(H)}$.*

PROOF. Clearly, $\ker(T) \subseteq \ker(T^*T)$. On the other hand, if $T^*T\xi = 0$, then $0 = \langle T^*T\xi, \xi\rangle = \|T\xi\|^2$, so $\ker(T^*T) = \ker(T)$.

Now, let $\xi \in \ker(T)$ so that $0 = \langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle$, thus $\overline{T^*(H)} \subseteq \ker(T)^\perp$. On the other hand, if $\xi \in T^*(H)^\perp$ then $0 = \langle \xi, T\eta\rangle = \langle T\xi, \eta\rangle$ for all $\eta \in H$ which implies that $\xi \in \ker(T)$, so $T^*(H)^\perp \subseteq \ker(T)$. Therefore by taking orthogonal complements $\ker(T)^\perp \subseteq \overline{T^*(H)}$. □

DEFINITION 6.7. A *partial isometry* $U \in \mathcal{B}(H)$ is a linear operator so that $\|U\xi\| = \|\xi\|$ for all $\xi \in \ker(U)^\perp$.

PROPOSITION 6.8 (Polar Decomposition). *For each $T \in \mathcal{B}(H)$ there is a unique partial isometry U with $\ker(U) = \ker(T)$ so that $T = U|T|$, where $|T| = (T^*T)^{1/2}$. Moreover $T = |T^*|U$.*

PROOF. If $\eta = |T|\xi$, define $U\eta = T\xi$. We have that U is well-defined since $\ker(T) = \ker(T^*T) = \ker(|T|)$ by Lemma 6.6. Since it is well-defined, it is clear that U is linear. Further, we have that

$$\|U\eta\|^2 = \|T\xi\|^2 = \langle T^*T\xi, \xi\rangle = \||T|\xi\|^2 = \|\eta\|^2$$

so U is an isometry on $|T|(H)$, thus extends to an isometry on $\overline{|T|(H)} = \ker(T)^\perp$ again using Lemma 6.6. By construction, it is clear that $U : \ker(T)^\perp \to H$ is the unique linear operator so that $U|T| = T$. We can extend U to H by setting $U\zeta = 0$ for all $\zeta \in \ker(T)$.

For the last part, we have that for $\eta = |T|\xi$ and $\eta' = |T|\xi'$

$$\langle U^*U\eta\,,\eta'\rangle = \langle U|T|\xi\,, U|T|\xi'\rangle = \langle T^*T\xi\,,\xi'\rangle = \langle |T|\xi\,, |T|\xi'\rangle = \langle \eta\,,\eta'\rangle\,,$$

Thus U^*U is the orthogonal projection onto $\overline{|T|(H)} = \overline{T^*T(H)} = \ker(T)^\perp$ by Lemma 6.6, so

$$U^*U|T| = |T| = |T|U^*U$$

using self-adjointness. Since $TT^* = U|T| \cdot |T|U^* = UT^*TU^*$ it follows that

$$(TT^*)^k = U(T^*T)^k U^*,$$

hence $|T^*| = (TT^*)^{1/2} = U(T^*T)^{1/2}U^* = U|T|U^*$ by using a power series expansion as in Proposition 2.4. Therefore $|T^*|U = U|T|U^*U = U|T| = T$, and the result obtains. □

Lemma 6.9. *Let $\mathcal{A} \subset \mathcal{B}(H)$ be a C*-algebra, and let $U(\mathcal{A})$ be the group of unitaries in $\mathcal{A}$. We have that $\mathcal{A}' = U(\mathcal{A})'$.*

Proof. Let $x \in \mathcal{A}$ be self-adjoint with $\|x\| \leq 1$. Consider the functions

$$f_\pm(t) = t \pm i\sqrt{1-t^2}$$

which send $[-1, 1]$ to $\mathbb{T}$. Since $\mathrm{sp}(x) \subseteq [-1, 1]$, we have that $f_\pm(x)$ are unitaries by the Continuous Functional Calculus. Further $2t = f_+(t) + f_-(t)$, so $f_+(x) + f_-(x) = 2x$. Thus every self-adjoint contraction is an average of two unitaries, which implies that $U(\mathcal{A})$ spans $\mathcal{A}$. The result now follows easily. □

Corollary 6.10. *If $\mathcal{M} \subset \mathcal{B}(H)$ is a von Neumann algebra, then for every $T \in \mathcal{M}$ with polar decomposition $T = U|T|$, $U \in \mathcal{M}$.*

Proof. Let $V \in \mathcal{M}'$ be a unitary. We have that

$$T = V^*TV = V^*UV \cdot V^*|T|V = V^*UV|T|$$

since $|T| \in \mathcal{M}$. Further we have that $\ker(T) = \ker(U)$ implies that

$$\ker(V^*UV) = \ker(V^*TV) = \ker(T) = \ker(U),$$

So $V^*UV = U$ for all $V \in \mathcal{M}'$ a unitary. Therefore by Lemma 6.9, we have that $U \in \mathcal{M}'' = \mathcal{M}$. □

Notation 6.11. For a C*-algebra $\mathcal{A}$, we write $\mathcal{A}^h$ for the set of all self-adjoint elements of $\mathcal{A}$.

Proposition 6.12 (Kaplansky's Density Theorem). *Let $\mathcal{A} \subset \mathcal{B}(H)$ be a C*-algebra with $I \in \mathcal{A}$. The following are true:*

(1) *$(\mathcal{A})_1$ is SOT-dense in $(\mathcal{A}'')_1$.*
(2) *$(\mathcal{A}^h)_1$ is SOT-dense in $((\mathcal{A}'')^h)_1$.*
(3) *$U(\mathcal{A})$ is SOT-dense in $U(\mathcal{A}'')$.*

PROOF. We first prove assertion statement (2). Consider the functions

$$f(t) = \frac{1}{1+t^2},\ g(t) = \frac{2t}{1+t^2},\ h(t) = \frac{t}{1+\sqrt{1-t^2}},$$

and note that g, h map $[-1, 1]$ to itself $g \circ h(t) = h \circ g(t)$ for all $t \in [-1, 1]$. Using the Continuous Functional Calculus (Proposition 4.21) we have that for any C*-algebra $\mathcal{B}$ that $f(\mathcal{B}^h),\ g(\mathcal{B}^h) \subseteq (\mathcal{B}^h)_1$. Moreover, we have that $g(h(x)) = x$ for all $x \in (\mathcal{B}^h)_1$. Thus we conclude that $g(\mathcal{A}^h) = (\mathcal{A}^h)_1$ and $g((\mathcal{A}'')^h) = ((\mathcal{A}'')^h)_1$.

Therefore it suffices to show that if $x_n \to x$, self-adjoint, in the SOT, then $g(x_n) \to g(x)$ in the SOT. We compute:

$$\begin{aligned}(g(x) - g(y))/2 &= (1 + x^2)^{-1}x - (1 + y^2)^{-1}y \\ &= (1 + x^2)^{-1}(x - y) + [(1 + x^2)^{-1} - (1 + y^2)^{-1}]y \\ &= (1 + x^2)^{-1}(x - y) + (1 + x^2)^{-1}(y^2 - x^2)(1 + y^2)^{-1}y \\ &= (1 + x^2)^{-1}(x - y) + (1 + x^2)^{-1}(y - x)(1 + y^2)^{-1}y^2 \\ &+ (1 + x^2)^{-1}x(y - x)(1 + y^2)^{-1}y.\end{aligned}$$

We thus have that

$$\|(g(x) - g(y))\xi\| \le 2\|(x - y)\xi\| + 2\|(x - y)\xi'\| + 2\|(x - y)\xi''\|$$

where $\xi' = g(y)y\xi$ and $\xi'' = g(y)\xi$ since $\|(1+x^2)^{-1}\| = \|f(x)\| \le 1$ and $\|(1+x^2)^{-1}x\| = \|g(x)\| \le 1$. It follows that if $x_n \to y$ in the SOT, then $g(x_n) \to g(y)$.

We now prove (1). Consider $M_2(\mathcal{A}) \subseteq \mathcal{B}(H \oplus H)$ and see that $M_2(\mathcal{A})'' = M_2(\mathcal{A}'')$. For $x \in \mathcal{A}''$ we have that

$$\tilde{x} = \begin{bmatrix} 0 & x \\ x^* & 0 \end{bmatrix} \in M_2(\mathcal{A}'')$$

and $\|\tilde{x}\| = \|x\|$. For $x \in (\mathcal{A}'')_1$ by the proof of (2), we have that there exists a net $\tilde{x}_n \in (M_2(\mathcal{A})^h)_1$ so that $\tilde{x}_n \to \tilde{x}$ in the SOT. Let p_1 and p_2 be the projections onto the first and second components in $H \oplus H$. We have that $y \mapsto p_1 y p_2$ is a contraction and is SOT-continuous, $p_1\tilde{x}_n p_2 \to x$ in the SOT.

We now turn to the proof of (3), which is more involved. We first claim that if x is normal and (x_n) is a net of normal operators so that $x_n \to x$ in the SOT, then $x_n^* \to x^*$ in the SOT. Since x_n and x are normal, we have that

$$\begin{aligned}\|(x_n^* - x^*)\xi\|^2 &= \|x_n^*\xi\|^2 + \|x^*\xi\|^2 - \langle x_n x^*\xi\,, \xi\rangle - \langle x x_n^*\xi\,, \xi\rangle \\ &= \|x_n\xi\|^2 - \|x\xi\|^2 + \langle (xx^* - x_n x^*)\xi\,, \xi\rangle + \langle (xx^* - x x_n^*)\xi\,, \xi\rangle \\ &= \|x_n\xi\|^2 - \|x\xi\|^2 + \langle (xx^* - x_n x^*)\xi\,, \xi\rangle + \langle \xi\,, (xx^* - x_n x^*\xi\rangle \\ &\le \|x_n\xi\|^2 - \|x\xi\|^2 + 2\|\xi\|\|(x_n - x)x^*\xi\| \to 0,\end{aligned}$$

which establishes the claim.

We next claim that if (x_n) and (y_n) are bounded nets so that $x_n \to x$ and $y_n \to y$ in the SOT, we have that $x_n y_n \to xy$ in the SOT. We leave the proof of this claim as an exercise.

Now we have that $\{f, g\}$ separate points in $\mathbb{R}$, so by the Stone–Weierstrass Theorem the $*$-algebra generated by $\{f, g\}$ is equal to $C_0(\mathbb{R})$. For any C*-subalgebra of $\mathcal{B}$ we have that f and g map $\mathcal{B}$ into $(\mathcal{B})_1$ it follows that every $h \in C_0(\mathbb{R})$ is SOT-continuous since by the previous claims for every $*$-polynomial p, we have that $p(x_n) \to p(x)$ in the SOT whenever $x_n \to x$ in the SOT and is uniformly bounded.

Finally, if $u \in \mathcal{A}''$ is a unitary, by choosing a branch of the logarithm, we have that $2\pi i h = \log(u)$ with $h \in (\mathcal{A}'')_1$ by the Borel Functional Calculus (Proposition 5.9). By (2), we may choose contractions h_n in $\mathcal{A}$ which converge to h in the SOT. Choosing any function $q \in C_0(\mathbb{R})$ so that $q(t) = \exp(2\pi i t)$ for all $t \in [-1, 1]$, we have that $q(h_n) \in U(\mathcal{A})$ by the Continuous Functional Calculus and by the previous paragraph that $q(h_n) \to q(h) = u$ in the SOT. □

Corollary 6.13. *If $I \in \mathcal{A} \subseteq \mathcal{B}(H)$ is a C*-algebra so that $(\mathcal{A})_1$ is SOT-closed, then $\mathcal{A}$ is a von Neumann algebra.*

Proof. Let $\mathcal{M} = \overline{\mathcal{A}}^{SOT}$. Suppose $x \in \mathcal{M}$. Without loss of generality we may assume that $x \in (\mathcal{M})_1$, which by Kaplansky's Density Theorem implies that $x \in (\mathcal{A})_1$ as $(\mathcal{A})_1$ is SOT-closed by assumption. Thus $\mathcal{A}$ is SOT-closed. □

In the last part of this section we return to the Banach space $\mathcal{L}^1(H)$ defined in Section 3.

Remark 6.14. Recall that in Proposition 3.23 we proved $\mathcal{L}^1(H)^*$ is isometrically identified with $\mathcal{B}(H)$ via the pairing $\langle T , x\rangle$, $T \in \mathcal{L}^1(H)$ and $x \in \mathcal{B}(H)$. It follows from the Hahn–Banach Theorem that $\mathcal{L}^1(H)$ embeds in $\mathcal{B}(H)^*$ isometrically via this pairing, thus

$$\|T\|_{\text{nuc}} = \sup_{\|x\| \leq 1} | \langle T , x\rangle |$$

Definition 6.15. We define the *σ-Weak Operator Topology* (σWOT) on $\mathcal{B}(H)$ by

$$\sum_{k=1}^{\infty} \langle T_n \xi_k , \eta_k \rangle \to \sum_{k=1}^{\infty} \langle T \xi_k , \eta_k \rangle$$

whenever $\sum_k \|\xi_k\|^2$, $\sum_k \|\eta_k\|^2 < \infty$.

Similarly, we define the *σ-Strong Operator Topology* (σSOT) by

$$\sum_{k=1}^{\infty} \|T_n \xi_k\|^2 \to \sum_{k=1}^{\infty} \|T \xi_k\|^2$$

whenever $\sum_k \|\xi_k\|^2 < \infty$.

Proposition 6.16. *Let $\mathcal{M} \subset \mathcal{B}(H)$ be a von Neumann algebra, and let $\phi \in \mathcal{M}^*$. The following statements are equivalent.*

1. *$\phi(x) = \langle T , x\rangle$ for some $T \in \mathcal{L}^1(H)$.*
2. *ϕ is WOT-continuous on $(\mathcal{M})_1$.*
3. *ϕ is SOT-continuous on $(\mathcal{M})_1$.*

(4) *ϕ is σSOT-continuous on $(\mathcal{M})_1$.*
(5) *$\phi(x) = \sum_k \langle x\xi_k , \eta_k\rangle$ where $\sum_k \|\xi_k\|^2$, $\sum_k \|\eta_k\|^2 < \infty$.*
(6) *ϕ is σWOT-continuous.*

Proof. (1) $\Rightarrow$ (2). We have that the finite-rank operators $\mathcal{F}(H)$ are dense in $\mathcal{L}^1(H)$ by construction, thus for any $\epsilon > 0$ and $T \in \mathcal{L}^1(H)$ by Remark 6.14 we can find a finite rank operator $T' = \sum_i \xi_i \otimes \bar{\eta}_i$ so that $|\langle T - T' , x\rangle| \leq \epsilon\|x\|$. Thus if $x_n \to x$ in the WOT, we have that $\langle T' , x_n\rangle = \sum_i \langle x_n\xi_i , \eta_i\rangle \to \sum_i \langle x\xi_i , \eta_i\rangle = \langle T' , x\rangle$, so if the net belongs to $(\mathcal{M})_1$, we have that $|\lim_n \langle T , x_n\rangle - \langle T , x\rangle| \leq 2\epsilon$. Since ϵ is chosen independently of n, this shows that $\lim_n \phi(x_n) = \phi(x)$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4). These follow trivially as each topology is successively stronger, that is, convergence in one implies convergence in any preceding.

(4) $\Rightarrow$ (5). We have that $\phi : (\mathcal{M})_1 \to \mathbb{C}$ is σSOT-continuous only if $\phi^{-1}(\mathbb{D})$ contains a relative basic σSOT-open neighborhood V around 0. That is,

$$V = \{x \in (\mathcal{M})_1 : (\sum_i \|x\xi_i\|^2)^{1/2} < \epsilon\}$$

for some $\epsilon > 0$ and some (ξ_i) with $C := \sum_i \|\xi_i\|^2 < \infty$. Thus we have that

$$|\phi(x)| \leq K(\sum_i \|x\xi_i\|^2)^{1/2}$$

for all $x \in \mathcal{M}$ where K depends only on C and ϵ. Setting $\xi := (\xi_i) \in \ell^2(H)$, we define $L := \overline{\mathcal{M}\xi}$ which is a closed subspace of the Hilbert space $\ell^2(H)$. We have that $\Phi(x\xi) := \phi(x)$ is well-defined and bounded on $\mathcal{M}\xi$, thus extends boundedly to L. By the Riesz Representation Theorem, we have that there is $\eta = (\eta_i) \in L$ so that

$$\phi(x) = \langle x\xi , \eta\rangle = \sum_i \langle x\xi_i , \eta_i\rangle .$$

Noting that $\|\xi\|^2 = \sum_i \|\xi_i\|^2$ with the same holding for η, the result obtains.

(5) $\Leftrightarrow$ (6). This is tautological.

(5) $\Rightarrow$ (1). We have by two applications of the Cauchy–Schwarz inequality that

$$|\sum_{i=N}^{\infty} \langle x\xi_i , \eta_i\rangle| \leq \sum_{i=N}^{\infty} \|x\xi_i\|\|\eta_i\| \leq \|x\| \left(\sum_{i=N}^{\infty} \|\xi_i\|^2\right)^{1/2} \left(\sum_{i=N}^{\infty} \|\eta_i\|^2\right)^{1/2} .$$

Thus setting $T_N = \sum_{i=1}^{N} \xi_i \otimes \bar{\eta}_i$ we have that $\| \langle T_N , x\rangle - \phi(x)\| \to 0$ in $\mathcal{M}^*$. By Remark 6.14 it follows that $T = \sum_{i=1}^{\infty} \xi_i \otimes \bar{\eta}_i \in \mathcal{L}^1(H)$ and $\langle T , x\rangle = \phi(x)$ for all $x \in \mathcal{M}$. $\square$

Proposition 6.17. *Let $\mathcal{M} \subset \mathcal{B}(H)$ be a von Neumann algebra. We have that a state $\phi \in S(\mathcal{M})$ is SOT-continuous on $(\mathcal{M})_1$ if and only if for every net (p_n) of projections in $\mathcal{M}$ so that $p_n \to 0$ in the SOT we have that $\phi(p_n) \to 0$.*

Proof. The "only if" direction is clear. Now suppose that for (x_n) in $(\mathcal{M})_1$ we have that $x_n \to 0$ in the SOT. Since $\|x_n^* x_n\xi\| \leq \|x_n^*\|\|x_n\xi\| \leq \|x_n\xi\|$, we have that

$y_n := x_n^* x_n \to 0$ in the SOT as well. We now fix $\epsilon > 0$. Since y_n is a positive contraction, we have that $\mathrm{sp}(y_n) \subseteq [0,1]$, so $p_n = \chi_{[\epsilon,1]}(y_n)$ defines a projection in $\mathcal{M}$ by the Borel Functional Calculus (Proposition 5.9). Moreover, since

$$\epsilon\chi_{[\epsilon,1]} \leq \mathrm{id} \leq \epsilon\chi_{[0,\epsilon)} + \chi_{[\epsilon,1]}$$

we have

$$\epsilon p_n \preceq y_n \preceq \epsilon(I - p_n) + p_n$$

by Borel Functional Calculus. The left inequality implies that $p_n \to 0$ in the SOT. Since ϕ is a state, we have by the second inequality and Cauchy–Schwarz that

$$|\phi(x_n)|^2 \leq \phi(y_n)\phi(I) \leq \epsilon\phi(I - p_n) + \phi(p_n) \leq \epsilon + \phi(p_n)$$

By assumption we therefore have that $\limsup_n |\phi(x_n)| \leq \sqrt{\epsilon}$ for $\epsilon > 0$ arbitrary which implies the result. □

Definition 6.18. For $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra, we define the *predual* of $\mathcal{M}$, denoted by $\mathcal{M}_*$ as the set of all linear functionals $\phi \in \mathcal{M}^*$ which are WOT-continuous on the unit ball $(\mathcal{M})_1$.

Proposition 6.19. *The following statements are true.*

(1) $\mathcal{M}_*$ *is norm-closed in* $\mathcal{M}^*$.
(2) $\mathcal{M}_*$ *is weak*-dense in* $\mathcal{M}^*$.
(3) $(\mathcal{M}_*)^* \cong \mathcal{M}$ *isometrically via the pairing* $\langle \phi\,, x\rangle = \phi(x)$.

Proof. For the first part, suppose $\phi_n \in \mathcal{M}_*$ and $\|\phi_n - \phi\| \to 0$. If (x_k) is a net in $(M)_1$ and $x_k \to x$ in the WOT, then for any $\epsilon > 0$, choose n so that $\|\phi_n - \phi\| < \epsilon$, then choose k so that $|\phi_n(x) - \phi_n(x_l)| < \epsilon$ for all $l > k$. We have that

$$|\phi(x) - \phi(x_l)| \leq |\phi(x) - \phi_n(x)| + |\phi_n(x_l) - \phi_n(x)| < 2\epsilon,$$

thus $\phi \in \mathcal{M}_*$.

For the second assertion, note that Proposition 6.16 applied to $\mathcal{B}(H)$ shows that $\mathcal{B}(H)_* \cong \mathcal{L}^1(H)$. In fact, Proposition 3.23 and the remark following it show that this identification is isometric. Every $\phi \in \mathcal{M}^*$ extends to some $\psi \in \mathcal{B}(H)^*$ by Hahn–Banach. We can find thus find a net (ψ_n) in $\mathcal{B}(H)_*$ so that $\psi_n \to \psi$ in the weak*-topology. If we denote by ϕ_n the restriction of ψ_n to $\mathcal{M}$ it is clear that $\phi_n \in \mathcal{M}_*$ and that $\phi_n \to \phi$ in the weak*-topology.

For the third and last assertion, we note that by Proposition 3.23 we have that $\mathcal{L}^1(H)^* \cong \mathcal{B}(H)$ isometrically via the pairing $\langle T\,, x\rangle$ defined in that section. By Proposition 6.16 we have that $\phi \in \mathcal{M}_*$ if and only if there is $T \in \mathcal{L}^1(H)$ so that $\langle T\,, x\rangle = \phi(x)$ for all $x \in \mathcal{M}$. It now follows easily that $\|x\| = \sup\{|\phi(x)| : \phi \in (\mathcal{M}_*)_1\}$. □

7. Unbounded Self-Adjoint Operators

Definition 7.1. A *linear operator* is a partially defined linear function $T : H \to H$ where H is a Hilbert space. We write $\mathcal{D}(T)$ for the domain of T which is a (not necessarily closed) linear subspace of H.

We say that the linear operator S *extends* T, written $T \subseteq S$, if $\mathcal{D}(T) \subseteq \mathcal{D}(S)$ and $S|_{\mathcal{D}(T)} = T$.

For a linear operator T we define the *graph* of T to be

$$\mathrm{gr}(T) := \{\xi \oplus T\xi : \xi \in \mathcal{D}(T)\} \subseteq H \oplus H$$

We note that $H \oplus H$ is a Hilbert space with inner product

$$\left\langle \xi \oplus \eta \,, \xi' \oplus \eta' \right\rangle = \left\langle \xi \,, \xi' \right\rangle + \left\langle \eta \,, \eta' \right\rangle.$$

LEMMA 7.2. *A subspace $K \subseteq H \oplus H$ is the graph of a linear operator if and only if $K \cap (0 \oplus H) = \{0 \oplus 0\}$. Moreover the linear operator is uniquely defined by K.*

PROOF. We note that if $\xi \oplus \eta, \xi \oplus \eta' \in K$, then so is $0 \oplus (\eta - \eta')$ which shows that $\eta = \eta'$. Therefore $T\xi = \eta$ gives a well-defined linear operator so that $\mathrm{gr}(T) = K$. Clearly T is the only such operator. The reverse reasoning shows that if $K = \mathrm{gr}(T)$ for some linear operator, then K satisfies the required properties. $\square$

COROLLARY 7.3. *For linear operators, we have that $T \subseteq S$ if and only if $\mathrm{gr}(T) \subseteq \mathrm{gr}(S)$.*

DEFINITION 7.4. We say that $T : H \to H$ is a *closed* linear operator if $\mathrm{gr}(T)$ is a closed subspace of $H \oplus H$. We say that T is *closeable* if it extends to a closed linear operator. We write $\mathcal{C}(H)$ for the set of all closed, densely-defined linear operators on H.

REMARK 7.5. If T is closed and $\mathcal{D}(T) = H$, then T is bounded by the Closed Graph Theorem.

REMARK 7.6. We have that T is closeable if and only if $\overline{\mathrm{gr}(T)}$ is a graph. Indeed, if $K = \overline{\mathrm{gr}(T)}$ is a graph, then by Lemma 7.2 there is a linear operator S with K as its graph. S is clearly a closed extension of T. Conversely, if $T \subseteq S$ is a closed extension, then $\overline{\mathrm{gr}(T)} \subseteq \overline{\mathrm{gr}(S)} = \mathrm{gr}(S)$ and any subspace of a graph is clearly a graph.

We now turn to the subject of adjoints for linear operators. Suppose that $T : H \to H$ is densely defined. We consider the set K of all $\eta \in H$ so that

$$\mathcal{D}(T) \ni \xi \mapsto \langle T\xi \,, \eta \rangle$$

extends to a bounded linear functional on H. Note that K is a subspace of H. By the Riesz Representation Theorem we have that there is $\zeta \in H$ so that

$$\langle T\xi \,, \eta \rangle = \langle \xi \,, \zeta \rangle$$

for all $\xi \in \mathcal{D}(T)$ and that such ζ is uniquely defined. Since ζ is unique we have that $T^*\eta = \zeta$ defines a linear operator with $\mathcal{D}(T^*) = K$. We refer to T^* as the *adjoint* of T.

NOTATION 7.7. We define an isometry $J : H \oplus H \to H \oplus H$ by

$$J(\xi \oplus \eta) = (-\eta) \oplus \xi$$

and note that $J^2 = -I$.

LEMMA 7.8. *Suppose T is densely defined. Then we have that $\mathrm{gr}(T^*) = [J\,\mathrm{gr}(T)]^{\perp}$.*

PROOF. Choosing $\eta \in \mathcal{D}(T^*)$, we have that for all $\xi \in \mathcal{D}(T)$

$$\langle \eta \oplus T^*\eta \,, J(\xi \oplus T\xi)\rangle = \langle \eta \oplus T^*\eta \,, (-T\xi) \oplus \xi\rangle = -\langle \eta \,, T\xi\rangle + \langle T^*\eta \,, \xi\rangle = 0$$

by definition of T^*, thus $\mathrm{gr}(T^*) \subseteq [J\,\mathrm{gr}(T)]^\perp$.

In the other direction, suppose $\xi \oplus \eta \in [J\,\mathrm{gr}(T)]^\perp$. We then have that

$$0 = \langle \xi \oplus \eta \,, (-T\zeta) \oplus \zeta\rangle = -\langle \xi \,, T\zeta\rangle + \langle \eta \,, \zeta\rangle$$

for all $\zeta \in \mathcal{D}(T)$, thus $\eta = T^*\xi$ by the definition of T^*. □

COROLLARY 7.9. *We have that T^* is closed if T is densely defined.*

PROOF. By the previous result we have that $\mathrm{gr}(T^*)$ is complemented, hence closed. □

COROLLARY 7.10. *Suppose that $T \in \mathcal{C}(H)$. We have that*

$$H \oplus H = \mathrm{gr}(T) \oplus J\,\mathrm{gr}(T^*) = (J\,\mathrm{gr}(T)) \oplus \mathrm{gr}(T^*)$$

where the "$\oplus$" symbol is interpreted to mean that the decomposition on the right hand side is orthogonal.

PROOF. Since J is an isometry $J\,\mathrm{gr}(T)$ is closed if and only if $\mathrm{gr}(T)$ is closed. We therefore have that

$$\mathrm{gr}(T^*)^\perp = [J\,\mathrm{gr}(T)]^{\perp\perp} = \overline{J\,\mathrm{gr}(T)} = J\,\mathrm{gr}(T).$$

The other equality follows by applying J, noting that $J^2 = -I$. □

COROLLARY 7.11. *If $T \in \mathcal{C}(H)$ and $\xi, \eta \in H$, then there exist $x, y \in H$, unique, with $x \in \mathcal{D}(T)$ and $y \in \mathcal{D}(T^*)$ so that*

$$-Tx + y = \xi$$
$$x + T^*y = \eta$$

PROOF. This is straightforward translation of Corollary 7.10, so is left as an exercise. □

PROPOSITION 7.12. *If $T \in \mathcal{C}(H)$, then $T^* \in \mathcal{C}(H)$ and $T^{**} = T$.*

PROOF. We first show that T^* is densely defined. Choose $\eta \in \mathcal{D}(T^*)^\perp$, and note that

$$\langle 0 \oplus \eta \,, (-T^*\xi) \oplus \xi\rangle = 0$$

for all $\xi \in \mathcal{D}(T)$. Thus by Corollary 7.10 it follows that $0 \oplus \eta \in \mathrm{gr}(T)$, which implies that $\eta = 0$ by the graph property (Lemma 7.2). Together with Corollary 7.9 this shows that $T^* \in \mathcal{C}(H)$.

Applying Corollary 7.10 to T and T^*, we have that

$$\mathrm{gr}(T) \oplus J\,\mathrm{gr}(T^*) = H \oplus H = \mathrm{gr}(T^{**}) \oplus J\,\mathrm{gr}(T^*)$$

from which it follows from uniqueness of orthogonal decompositions that $\mathrm{gr}(T) = \mathrm{gr}(T^{**})$. Again by Lemma 7.2 we have that $T^{**} = T$. □

PROPOSITION 7.13. *For $T \in \mathcal{C}(H)$ set $R = I + T^*T$ and $\mathcal{D}(R) = \{\xi \in H : \xi \in \mathcal{D}(T),\ T\xi \in \mathcal{D}(T^*)\}$. The following are true:*

(1) *$R : \mathcal{D}(R) \to H$ is a bijection.*
(2) *There exist contractions $B, C \in (\mathcal{B}(H))_1$ so that $B \succeq 0$, $C = TB$, and $BR \subseteq RB = I$.*
(3) *$\mathcal{D}(T^*T)$ is dense in $\mathcal{D}(T)$.*

PROOF. For $\xi \in \mathcal{D}(R)$ we have that

$$\|\xi\|^2 \leq \|\xi\|^2 + \|T\xi\|^2 = \langle \xi\,, \xi\rangle + \langle T^*T\xi\,, \xi\rangle = \langle R\xi\,, \xi\rangle \leq \|R\xi\|\|\xi\|$$

so $\|\xi\| \leq \|R\xi\|$ and R is injective. To prove that R is onto H, by Corollary 7.11 we have that for each $\xi \in H$ there are $x \in \mathcal{D}(T)$ and $y \in \mathcal{D}(T^*)$ uniquely determined by ξ so that

$$\begin{aligned} -Tx + y &= 0 \\ x + T^*y &= \xi, \end{aligned}$$

thus

$$\xi = x + T^*y = x + T^*Tx = Rx.$$

This concludes the proof of (1).

We set $B\xi := x$ and $C\xi := y = Tx = TB\xi$ and note that $\xi \mapsto B\xi$ is linear by uniqueness. Now since $0 \oplus \xi = [(-TB\xi) \oplus B\xi] + [C\xi \oplus T^*C\xi]$ is an orthogonal decomposition of $0 \oplus \xi$ by Corollary 7.10, we have that

$$\|\xi\|^2 = \|(-TB\xi) \oplus B\xi\|^2 + \|C\xi \oplus T^*C\xi\|^2 \geq \|B\xi\|^2 + \|C\xi\|^2$$

which shows that B and C are contractions. Moreover, by construction we have that $RB\xi = \xi$ for all $\xi \in H$ so it must be the case that $B = R^{-1} : H \to \mathcal{D}(R)$ bijectively, from which it follows that $BR \subseteq RB = I$.

Next, we have that for all $\xi \in H$ there exists $\eta \in \mathcal{D}(R)$ so that $\xi = R\eta$. It now follows that

$$\langle B\xi\,, \xi\rangle = \langle BR\eta\,, R\eta\rangle = \langle \eta\,, R\eta\rangle = \|\eta\|^2 + \|T\eta\|^2 > 0,$$

so $B \succeq 0$ since B is automatically self-adjoint by Remark 1.21. This concludes the proof of (2).

Finally, suppose that there is $\xi \oplus T\xi \in \mathrm{gr}(T)$ which is orthogonal to all $\eta \oplus T\eta$ with $\eta \in \mathcal{D}(T^*T) \subseteq \mathcal{D}(T)$. We would then have that

$$0 = \langle \xi \oplus T\xi\,, \eta \oplus T\eta\rangle = \langle \xi\,, \eta\rangle + \langle \xi\,, T^*T\eta\rangle = \langle \xi\,, R\eta\rangle \tag{9}$$

for all $\eta \in \mathcal{D}(R)$. But $R : \mathcal{D}(R) \to H$ is onto, so $\xi = 0$. Setting $T' = T|_{\mathcal{D}(T^*T)}$ and using that $\mathrm{gr}(T)$ is closed, this shows that $\mathrm{gr}(T')$ is dense in $\mathrm{gr}(T)$, hence $\mathcal{D}(T^*T) = \mathcal{D}(T')$ is dense in $\mathcal{D}(T)$. $\square$

DEFINITION 7.14. A densely defined operator T is said to be *self-adjoint* if $T^* = T$. Note this means that $\mathcal{D}(T^*) = \mathcal{D}(T)$, hence T is closed by Corollary 7.9.

LEMMA 7.15. *For $T \in \mathcal{C}(H)$ we have that T^*T is self-adjoint.*

PROOF. We keep all of the notation from the proof of Proposition 7.13, so $R = I + T^*T$ and $B = R^{-1}$ is a positive, bounded contraction. It clearly suffices to show that R is self-adjoint since $\mathcal{D}(R) = \mathcal{D}(T^*T)$.

We first claim that $\mathrm{gr}(R) = J\,\mathrm{gr}(-B)$. Indeed we have that $\xi \oplus \eta \in \mathrm{gr}(R)$ if and only if $\eta \oplus \xi \in \mathrm{gr}(B)$. But this is equivalent to saying that $\eta \oplus -\xi \in \mathrm{gr}(-B)$ which is again equivalent to $\xi \oplus \eta \in J\,\mathrm{gr}(B)$. This establishes the claim.

As a consequence, we observe that $\mathrm{gr}(R)$ is closed, so by Proposition 7.13, we have that $R \in \mathcal{C}(H)$. Since B is bounded, hence closed, and self-adjoint from the claim and Corollary 7.10 we have that

$$H \oplus H = J\,\mathrm{gr}(-B) \oplus \mathrm{gr}(-B) = \mathrm{gr}(R) \oplus J\,\mathrm{gr}(R).$$

Applying Corollary 7.10 to R, we have that

$$H \oplus H = \mathrm{gr}(R^*) \oplus J\,\mathrm{gr}(R),$$

hence comparing the two orthogonal decompositions, we have that $\mathrm{gr}(R) = \mathrm{gr}(R^*)$ from which it follows directly that $R^* = R$. □

LEMMA 7.16. *Suppose that $T \in \mathcal{C}(H)$ is self-ajoint. Writing $B = (I + T^2)^{-1}$ and $C = TB$, we have that $BT \subseteq C$ and $BC = CB$.*

PROOF. We first prove that $BT \subseteq TB = C$. Suppose that $\xi \in \mathcal{D}(BT) = \mathcal{D}(T)$, since $(I + T^2) : \mathcal{D}(T^2) \to H$ is onto by Proposition 7.13, we have that $\xi \in (I + T^2)\eta$ for some $\eta \in \mathcal{D}(T^2)$. Since $\xi \in \mathcal{D}(T)$, it is actually the case that $\eta \in \mathcal{D}(T^3)$. We thus have that

$$T\xi = T\eta + T^3\eta = (I + T^2)T\eta,$$

from which it follows by Proposition 7.13 that

$$BT\xi = B(I + T^2)T\eta = T\eta = TB\xi = C\xi$$

since $B\xi = \eta$. This establishes the claim.

We now show that $BC = CB$. We let $\xi \in H$ be arbitrary and choose $\eta \in \mathcal{D}(T^2)$ so that $\xi = (I + T^2)\eta$ as before by Proposition 7.13, noting again that $B\xi = \eta$. We compute that

$$BC\xi = BTB\xi = BT\eta = TB\eta = (TB)B\xi = CB\xi,$$

and the result obtains. □

We are now ready to state and prove the main result in this section on spectral decomposition of unbounded operators.

PROPOSITION 7.17. *Suppose that $T \in \mathcal{C}(H)$ is self-adjoint. There is an increasing sequence H_n of closed subspaces of H with $H = \overline{\bigcup_n H_n}$ so that letting P_n be the orthogonal projection onto H_n we have that*

(1) $H_n \subset \mathcal{D}(T)$;
(2) $P_nT \subseteq TP_n = P_nTP_n$;
(3) $\|TP_n\| \leq n$;

(4) *Setting* $H_0 = \bigcup_n H_n$, *we have that*

$$\mathrm{gr}(T) = \overline{\{\xi \oplus T\xi : \xi \in H_0\}}.$$

PROOF. Consider $B = (I + T^2)^{-1}$ which is a positive contraction by Proposition 7.13. Thus $\mathrm{sp}(B) \subseteq [0, 1]$. Let E_B be the spectral measure associated to B. By the Borel Functional Calculus define $P_n := E_B([1/n^2, 1]) = \chi_{[1/n^2,1]}(B)$, noting that by the same it holds that $P_n B = BP_n$ and $BP_n \succeq \frac{1}{n^2} P_n$. Thus $B_n := BP_n : H_n \to H_n$ is invertible.

Since B is injective, we have that $\ker(B) = E_B(\{0\}) = 0$, thus $E_B((0, 1]) = E_B(\bigcup_n [1/n^2, 1]) = I$. From the Borel Functional Calculus we conclude that $P_n \to I$ in the SOT, hence $\bigcup_n H_n$ is dense in H by Lemma 5.5.

It follows that for every $\xi \in H_n$ there is $\eta \in H_n$ so that $\xi = B\eta$. Therefore ξ belongs to the range of B which coincides with $\mathcal{D}(I + T^2) = \mathcal{D}(T^2) \subseteq \mathcal{D}(T)$ by Proposition 7.13. Hence $H_n \subseteq \mathcal{D}(T)$.

We now would like to show that $TP_n = P_n TP_n$. Again for $\xi \in H_n$ choose $\eta \in H_n$ so that $\xi - B\eta$. Since $BC = CB$ by Lemma 7.16 and B is self-adjoint, we have that $C \in C^*(B)'$. Thus C commutes with any operator in the WOT-closure of $C^*(B)$ as well, so $CP_n = P_n C$ by the Borel Functional Calculus. We now have that

$$T\xi = TB\eta = C\eta = CP_n\eta = P_n C\eta = P_n T\xi,$$

which establishes the claim. We have that $\mathcal{D}(T) \ni \xi \mapsto \langle P_n T\xi \, , \eta\rangle$ extends to a bounded linear functional for all $\eta \in H$ since $\langle P_n T\xi \, , \eta\rangle = \langle T\xi \, , P_n\eta\rangle$ and $P_n\eta \in H_n \subset \mathcal{D}(T) = \mathcal{D}(T^*)$. Thus

$$\langle P_n T\xi \, , \eta\rangle = \langle \xi \, , TP_n\eta\rangle = \langle \xi \, , P_n TP_n\eta\rangle = \langle P_n TP_n\xi \, , \eta\rangle$$

for all $\xi \in \mathcal{D}(T)$ and $\eta \in H$. Thus $P_n T \subseteq P_n TP_n$.

Next, for $\xi \in H_n$ we have that

$$\begin{aligned} \|T\xi\|^2 &= \left\langle T^2\xi \, , \xi \right\rangle \\ &= \left\langle [(I + T^2) - I]\xi \, , \xi \right\rangle \\ &= \left\langle \left[\int_{1/n^2}^{1} (t^{-1} - 1) dE_B(t) \right] \xi \, , \xi \right\rangle \leq (n^2 - 1)\|\xi\|^2 \end{aligned}$$

where E_B is the spectral measure associated to B. Therefore $\|TP_n\| \leq n$.

Lastly, for $\xi \oplus T\xi \in \mathrm{gr}(T)$, we have that $P_n\xi \oplus P_n T\xi \to \xi \oplus T\xi$ and

$$P_n\xi \oplus P_n T\xi = P_n\xi \oplus TP_n\xi$$

by the previous. □

PROPOSITION 7.18. *If* $T \in \mathcal{C}(H)$, *then there is a closeable operator* $|T|$ *with* $\mathcal{D}(|T|) = \mathcal{D}(T)$ *so that* $|T|^2 \supseteq T^*T$ *and* $T = U|T|$ *for a unique partial isometry so that* $\ker(U) = \ker(T)$.

PROOF. By Lemma 7.15 we have that $S = T^*T$ is self-adjoint. Let H_n be the subspaces for S as guaranteed by Proposition 7.17, copying all of the notation used therein. We have that $H_n \subseteq \mathcal{D}(S) \subseteq \mathcal{D}(T)$, so setting $S_n := SP_n = P_nSP_n$ we have that

$$\langle S_n\xi \,, \xi\rangle = \langle P_nT^*TP_n\xi \,, \xi\rangle = \langle TP_n\xi \,, TP_n\xi\rangle \geq 0$$

which means that $S_n \in \mathcal{B}(H_n)$ is positive for all n. Therefore using a power series expansion as in Proposition 2.4 $(S_n)^{1/2}$ exists. Since $P_kS_n = S_nP_k = S_k$ for all $k \leq n$, so $(S_n)^qP_k = P_k(S_n)^q = (S_k)^q$ for all $q \in \mathbb{N}$, using the power series expansion formula, we have that $(S_n)^{1/2}P_k = (S_k)^{1/2}$ as well. It follows that $S^{1/2}$ is a well-defined linear operator on $\mathcal{D}(S^{1/2}) := H_0 = \bigcup_n H_n$ and that $T^*T = (S^{1/2})^2$ on H_0.

By the proof of Proposition 7.13 we have that $\overline{\{\xi \oplus T\xi : \xi \in \mathcal{D}(T^*T)\}} = \mathrm{gr}(T)$, so $\mathcal{D}(T)$ is the completion of $\mathcal{D}(T^*T)$ under the norm $\xi \mapsto \|\xi\| + \|T\xi\|$. Similarly, from Proposition 7.17, part (4), we have that $\mathcal{D}(T^*T)$ is the completion of H_0 under the norm $\xi \mapsto \|\xi\| + \|T^*T\xi\|$. Since $\|S^{1/2}\xi\|^2 = \langle S\xi \,, \xi\rangle = \langle T^*T\xi \,, \xi\rangle = \|T\xi\|^2$ and $\|T\xi\|^2 = \langle T^*T\xi \,, \xi\rangle \leq \|T^*T\xi\|\|\xi\|$ for $\xi \in H_0$, we have that the closure of $S^{1/2}$ has domain $\mathcal{D}(T)$. We define $|T|$ to be the closure of $S^{1/2}$. Since $|T|^2\xi = S\xi$ for $\xi \in H_0$, another application of Proposition 7.17, part (4), shows that $|T|^2\xi = S\xi = T^*T\xi$ for $\xi \in \mathcal{D}(T^*T)$.

Note that by the proof of Proposition 7.17 we have that $\ker(T) = \ker(T^*T) \subset H_1$, so $\ker(T^*T) \subseteq \ker(|T|)$ as $|T|$ is derived from T^*T on H_1 via a power series. On the other hand $T^*T = |T|^2$ so $\ker(|T|) \subseteq \ker(T^*T)$. Therefore $\ker(|T|) = \ker(T^*T)$. It follows that the map

$$U(|T|\xi) = T\xi$$

is well-defined, hence linear, for $\xi \in \mathcal{D}(|T|) = \mathcal{D}(T)$. Note that

$$\|U(|T|\xi)\|^2 = \langle T\xi \,, T\xi\rangle = \langle T^*T\xi \,, \xi\rangle = \langle |T|\xi \,, |T|\xi\rangle = \||T|\xi\|^2$$

for all $\xi \in \mathcal{D}(T^*T)$ since $\xi \in H_n$ for some n, so U is isometric. Since $\ker(T) \subseteq H_1$ we have that $\overline{S_n^{1/2}(H_n)} = \ker(S_n^{1/2})^\perp = H_n \cap \ker(T)^\perp$, so U extends to an isometry from $\ker(T)^\perp$ to H and that $U^*U|T| = |T|$. $\square$

PROPOSITION 7.19. *If $A \in \mathcal{C}(H)$ is self-adjoint, then for all $t, s \in \mathbb{R}$, $\exp(itA)$ is a unitary, $\exp(itA)\exp(isA) = \exp(i(t+s)A)$, and $t \mapsto \exp(itA)$ is SOT-continuous.*

PROOF. Using the same notation as in Proposition 7.17 we have that $A_n := AP_n$ is self-adjoint, so $\exp(itA_n) \in B(H_n)$ is a unitary with $\exp(itA_n)\exp(isA_n) = \exp(i(t+s)A_n)$. For each $k \geq n$ we have that $\exp(itA_k)P_n = \exp(itA_n)$, so $\exp(itA)$ as an operator defined from $\bigcup_n H_n$ to itself is a bijective isometry. Thus this operator extends to a unitary on $\mathcal{B}(H)$ which we will also denote by $\exp(itA)$. Since $\exp(itA)\exp(isA) = \exp(i(t+s)A)$ on $\bigcup_n H_n$, the same holds everywhere on H.

By the multiplicative property and the fact that the image is uniformly bounded it suffice to check that $t \mapsto \exp(itA)$ is SOT-continuous at 0. For $\epsilon > 0$ and $\xi \in H$ there is n so that $\|P_n\xi - \xi\| < \epsilon$. Since A_n is bounded $\exp(itA_n) \to I$ in the operator norm as $t \to 0$. Thus there is $\delta > 0$ so that $\|P_n\xi - \exp(itA)P_n\xi\| = \|P_n\xi - \exp(itA_n)P_n\xi\| < \epsilon$

for all $|t| < \delta$. By the triangle inequality and the fact that $\exp(itA)$ is an isometry, we have that $\|\xi - \exp(itA)\xi\| \leq 3\epsilon$ for all $|t| < \delta$. $\square$

CHAPTER 2

Tomita–Takesaki Theory

1. σ-Finite von Neumann Algebras

DEFINITION 1.1. Let $\mathcal{A} \subset \mathcal{B}(H)$ be a $*$-subalgebra with $I \in \mathcal{A}$. We say that $\xi \in H$ is:

(1) *cyclic* if $\overline{\mathcal{A}\xi} = H$ and
(2) *separating* if for every $x \in \mathcal{A}$, $x\xi = 0$ implies that $x = 0$.

LEMMA 1.2. *Let $\mathcal{A} \subset \mathcal{B}(H)$ be a C^*-subalgebra with $I \in \mathcal{A}$. We have that $\xi \in H$ is $\mathcal{A}$-cyclic if and only if ξ is $\mathcal{A}'$-separating.*

PROOF. Suppose that ξ is cyclic for $\mathcal{A}$. Suppose that $x \in \mathcal{A}'$ is such that $x\xi = 0$. It follows that $x\mathcal{A}\xi = \mathcal{A}x\xi = \{0\}$, thus $\mathcal{A}\xi \subset \ker(x)$ and so $H = \overline{\mathcal{A}\xi} \subset \ker(x)$. Therefore $x = 0$.

Now suppose that ξ is separating for $\mathcal{A}'$. Let $K = (\mathcal{A}\xi)^{\perp}$. Since $\mathcal{A}$ is a $*$-subalgebra and $\mathcal{A}\xi$ is clearly $\mathcal{A}$-invariant, we claim that K is also $\mathcal{A}$ invariant. Indeed let $u \in \mathcal{A}$ be a unitary, so $u^*\mathcal{A}\xi = \mathcal{A}\xi$. For $\zeta \in K$ we have that

$$\sup_{\eta \in \mathcal{A}\xi} \langle \eta \,, u\zeta \rangle = \sup_{\eta \in \mathcal{A}\xi} \langle u^*\eta \,, \zeta \rangle = \sup_{\eta' \in \mathcal{A}\xi} \left\langle \eta' \,, \zeta \right\rangle = 0,$$

thus $uK \subseteq K$ for all $u \in \mathcal{A}$. Since $\mathcal{A}$ is spanned by its unitaries by the proof of Lemma 6.9 in Chapter 1 it follows that $\mathcal{A}K \subseteq K$ as well.

Let p be the orthogonal projection onto K. Since K is $\mathcal{A}$-invariant, we have that $p \in \mathcal{A}'$. As $I \in \mathcal{A}$, we have that $\xi \in \mathcal{A}\xi$ which implies that $p\xi = 0$. As ξ was assumed to be $\mathcal{A}'$-separating, this shows that $p = 0$ which shows that $\overline{\mathcal{A}\xi} = H$. $\square$

COROLLARY 1.3. *If $\mathcal{M} \subseteq \mathcal{B}(H)$ is a von Neumann algebra, we have that $\xi \in H$ is cyclic and separating for $\mathcal{M}$ if and only if it is cyclic and separating for $\mathcal{M}'$.*

DEFINITION 1.4. For $x \in \mathcal{B}(H)$ we define the *support* of x, denoted $\mathrm{supp}(x)$, to be the orthogonal projection onto $\ker(x)^{\perp} = \ker(x^*x)^{\perp} = \overline{x^*x(H)}$. Note that these are equivalent by Lemma 6.6 of Chapter 1.

REMARK 1.5. By the Borel Functional Calculus, we have that $\chi_{\{0\}}(x^*x)$ is the projection onto the kernel of x, so $\mathrm{supp}(x) = \chi_{(0,\|x*x\|]}(x^*x)$. Thus if $x \in \mathcal{M}$, a von Neumann algebra, we have that $\mathrm{supp}(x) \in \mathcal{M}$ as well. We also note that if $x\xi = 0$, then $\xi \in \ker(x)$, from which it follows by definition that $\mathrm{supp}(x)\xi = 0$.

LEMMA 1.6. *Let $\mathcal{M} \subset \mathcal{B}(H)$ be a von Neumann algebra. For $\xi \in H$, $\xi \neq 0$, there is a non-zero projection $p \in \mathcal{M}$ so that $\mathcal{M}p \ni x \mapsto x\xi$ is injective. In particular, ξ is pMp-separating.*

PROOF. Let Λ be the collection of all projections $p \in \mathcal{M}$ so that $p\xi = 0$. We have that $0 \in \Lambda$ and I is not, so Λ is non-empty and proper in the set of all projections in $\mathcal{M}$.

Suppose that $p, q \in \Lambda$. We can see that $\ker(p+q) = p \wedge q$, the projection onto the intersection of the ranges of p and q, hence $\operatorname{supp}(p+q) = (p \wedge q)^{\perp} = p \vee q$. Since $(p+q)\xi = 0$, we have that $(p \vee q)\xi = 0$ as well. It follows that for any finite collection $p_1, \ldots, p_k \in \Lambda$ that $\bigvee_i p_i \in \Lambda$ as well. Let $\mathcal{F}(\Lambda)$ be the directed set of all finite subsets of Λ and consider the net $\mathcal{F}(\Lambda) \ni F \mapsto \bigvee_{i \in F} p_i =: p_F$. This is an increasing net of projections, so by Lemma 5.5 of Chapter 1 we have that

$$\lim_{F \in \mathcal{F}(\Lambda)} p_F = \bigvee_{i \in \Lambda} p_i =: p$$

for convergence in the SOT. Since $p_F\xi = 0$, we have that $p\xi = 0$ as well. In this way we have shown that Λ contains a maximal element. Note by the previous that $p \neq I$.

Now let $q = I - p$ and suppose that there is $x \in \mathcal{M}q$, $xq \neq 0$, so that $x\xi = 0$. It follows that $\operatorname{supp}(x)(H) = \overline{qx^*xq(H)} \subseteq q(H)$, so that $\operatorname{supp}(x) \neq 0$ is orthogonal to p with the property that $\operatorname{supp}(x)\xi = 0$. It would then follow that $p + \operatorname{supp}(x) \in \Lambda$, contradicting the maximality of p. Therefore $x\xi \neq 0$ for all $x \in \mathcal{M}q$, $x \neq 0$. $\square$

REMARK 1.7 (GNS Construction). Let $I \in \mathcal{A} \subset \mathcal{B}(H)$ be a C*-algebra, and let ϕ be a state on $\mathcal{A}$. Let

$$\mathcal{I} = \{x \in \mathcal{A} : \phi(x^*x) = 0\}$$

and let $\widehat{\mathcal{A}} = \mathcal{A}/\mathcal{I}$. We define an inner product on $\widehat{\mathcal{A}}$ by $\langle \hat{x}, \hat{y} \rangle_{\phi} := \phi(y^*x)$. This is well-defined since $(x, y) \to \phi(y^*x)$ is a positive semidefinite sesquilinear form, so satisfies Cauchy–Schwarz, that is $|\phi(y^*x)|^2 \leq \phi(x^*x)\phi(y^*y)$. We define the completion of $\widehat{\mathcal{A}}$ under the norm $\|\hat{x}\|_{\phi} = \sqrt{\phi(x^*x)}$ to be the Hilbert space H_{ϕ}.

For $a, x \in \mathcal{A}$, we define $\pi_{\phi}(a)\hat{x} := \widehat{ax}$. Since $a^*a \preceq \|a\|^2 I$ we have that $x^*a^*ax \preceq \|a\|^2 x^*x$, which implies that

$$\|\pi_{\phi}(a)\hat{x}\|_{\phi}^2 = \phi(x^*a^*ax) \leq \|a\|^2\phi(x^*x) = \|a\|^2\|\hat{x}\|_{\phi}^2$$

as ϕ is a state. Thus we can see that $\mathcal{A} \ni a \mapsto \pi_{\phi}(a)$ is a well-defined $*$-linear contractive homomorphism from $\mathcal{A}$ in $\mathcal{B}(H_{\phi})$. Moreover, setting $\xi_{\phi} := \hat{I}$, we have that $\pi_{\phi}(\mathcal{A})\xi_{\phi} = \widehat{\mathcal{A}}$, so ξ_{ϕ} is a $\pi_{\phi}(\mathcal{A})$-cyclic vector. We refer to the triple $(H_{\phi}, \xi_{\phi}, \pi_{\phi})$ as the *GNS-construction* associated to $(\mathcal{A}, \phi)$.

DEFINITION 1.8. We say that a state ϕ on a C*-algebra $\mathcal{A}$ is *faithful* if $\phi(x^*x) = 0$ implies that $x = 0$.

REMARK 1.9. Let $I \in \mathcal{A} \subseteq \mathcal{B}(H)$ be a C*-algebra. The state ϕ is faithful if and only if the vector ξ_{ϕ} in the GNS-representation of $(\mathcal{A}, \phi)$ is separating. Indeed, we have that

ξ_ϕ is separating if and only if for all $x \in \mathcal{A}$, $\|x\xi_\phi\|^2 = 0$ implies $x = 0$. By construction $\|x\xi_\phi\|^2 = \phi(x^*x)$ which shows the equivalence.

LEMMA 1.10. *Let $I \in \mathcal{A} \subset \mathcal{B}(H)$ be a C^*-algebra. If $\pi : \mathcal{A} \to \mathcal{B}(K)$ is a $*$-homomorphism with $\pi(I) = I$ which is injective, then it is isometric.*

PROOF. Suppose that $x \in \mathcal{A}^h$ is self-adjoint. We have that $x - \lambda I$ is invertible in $\mathcal{B}(H)$ if and only if $\lambda \notin \mathrm{sp}(x)$ if and only if $t \mapsto (t-\lambda)^{-1}$ is continuous on $\mathrm{sp}(x)$ if and only if $(x - \lambda I)^{-1} \in \mathcal{A}$ by the Continuous Functional Calculus (Proposition 4.21 in Chapter 1). Since $\pi(I) = I$ and π is a homomorphism this implies that $x - \lambda I$ is invertible if and only if $\pi(x) - \lambda I$ is invertible in $\mathcal{B}(K)$, hence $\mathrm{sp}(\pi(x)) \subseteq \mathrm{sp}(x)$. It follows that $\|\pi(x)\| \leq \|x\|$ for all $x \in \mathcal{A}^h$, which taking $y \in \mathcal{A}$ arbitrary shows that $\|\pi(y)\|^2 = \|\pi(y)^*\pi(y)\| = \|\pi(y^*y)\| = \|y^*y\| = \|y\|^2$, that is, π is contractive.

If this inclusion is proper, pick $t_0 \in \mathrm{sp}(x) \setminus \mathrm{sp}(\pi(x))$ and choose continuous function f on $\mathrm{sp}(x)$ so that $f(t_0) = 1$ and $f \equiv 0$ on $\mathrm{sp}(\pi(x))$, since π is contractive, by the proof of the Continuous Functional Calculus we have that $\pi(f(x)) = f(\pi(x)) = 0$ while $f(x) \neq 0$, contradicting injectivity. Therefore $\mathrm{sp}(\pi(x)) = \mathrm{sp}(x)$, which implies that $\|\pi(x)\| = \|x\|$ by the Continuous Functional Calculus. Finally if $y \in \mathcal{A}$ is arbitrary as similar argument as in the previous paragraph shows that π is isometric. ☐

COROLLARY 1.11. *Let $I \in \mathcal{A} \subset \mathcal{B}(H)$ be a C^*-algebra. If ϕ is a faithful state, then $\pi_\phi : \mathcal{A} \to \mathcal{B}(H_\phi)$ is isometric. Hence $\pi_\phi(\mathcal{A})$ is closed, thus a C^*-subalgebra of $\mathcal{B}(H_\phi)$.*

DEFINITION 1.12. A von Neumann algebra $\mathcal{M} \subset \mathcal{B}(H)$ is said to be *σ-finite* if whenever $(p_i)_{i\in I}$ is a family of mutually orthogonal projections in $\mathcal{M}$ at most countably many are non-zero.

REMARK 1.13. If $\mathcal{M} \subseteq \mathcal{B}(H)$ admits a separating vector, then $\mathcal{M}$ is σ-finite. In order to see this, suppose $(p_i)_{i\in I}$ is a mutually orthogonal family of projections in $\mathcal{M}$. By Bessel's Inequality we have that for any $F \subset I$ finite that $\sum_{i\in F} \|p_i\xi\|^2 \leq \|\xi\|^2$. Therefore, letting $\mathcal{F}(I)$ be the directed set of all finite subsets of I, we have that the net $\mathcal{F}(I) \ni F \mapsto \sum_{i\in F} \|p_i\xi\|^2$ is convergent. It follows that for every $n \in \mathbb{N}$ there are at most finitely many $i \in I$ so that $\|p_i\xi\|^2 \geq 1/n$. Therefore, $\|p_i\xi\|^2 = 0$ for all but countably many indices. Since ξ is separating we have that $\|p_i\xi\| = 0$ implies that $p_i = 0$.

REMARK 1.14. If $\mathcal{M} \subset \mathcal{B}(H)$ is a σ-finite von Neumann algebra, then there is a vector $\xi \in H$ which is separating for $\mathcal{M}$.

The rest of this section will be devoted to proving the following result.

PROPOSITION 1.15. *Let $\mathcal{M} \subset \mathcal{B}(H)$ is a von Neumann algebra. The following are equivalent:*

1. *$\mathcal{M}$ is σ-finite;*
2. *$\mathcal{M}$ admits a faithful σWOT-continuous state;*
3. *$\mathcal{M}$ is isometrically $*$-isomorphic to a von Neumann algebra on some Hilbert space admitting a cyclic and separating vector.*

LEMMA 1.16. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra, and let $p \in \mathcal{M}$ be a projection. Then $pMp \subseteq \mathcal{B}(pH)$ is a von Neumann algebra.*

PROOF. We clearly have that $(pMp)_1 = p(M)_1 p$ and that $x \mapsto pxp$ is a WOT-continuous contraction. Thus $(pMp)_1$ is WOT-closed as $(\mathcal{M})_1$ is WOT-compact by the proof of Proposition 1.14 of Chapter 1. It follows that $(pMp)_1$ is SOT-closed, hence pMp is SOT-closed by Kaplansky's Density Theorem (see Corollary 6.13 in Chapter 1). $\square$

LEMMA 1.17. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra and let $\mathcal{I} \subset \mathcal{M}$ be an SOT-closed two-sided ideal. Then $\mathcal{I} = \mathcal{M}p$ for p a projection in the center of $\mathcal{M}$.*

PROOF. Suppose $x \in I$ is a contraction. There is a sequence of polynomials $(p_n(t))$ so that $p_n(0) = 0$ and converges uniformly to the constant function 1 on $(\epsilon, 1]$ for every $\epsilon > 0$. Since $p_n(0) = 0$, we have that $p_n(x^*x) \in \mathcal{I}$ and the convergence property implies that $p_n(x) \to \chi_{(0,1]}(x^*x) = \mathrm{supp}(x)$ in the SOT by the Borel Functional Calculus. Thus, if p and q are projections in $\mathcal{I}$, then $p \vee q = \mathrm{supp}(p + q) \in \mathcal{I}$ as well.

Since $\mathcal{I}$ is SOT-closed, we have similarly to the proof of Lemma 1.2 that $\mathcal{I}$ contains a maximal projection p. For each $u \in U(\mathcal{M})$ a unitary we have that $u^*pu \in \mathcal{I}$ is a projection, thus $u^*pu \preceq p$ by maximality. Since $upu^* \preceq p$, by conjugating both sides we have that $p \preceq u^*pu$, so $up = pu$ for all $u \in U(\mathcal{M})$. Since the unitaries span $\mathcal{M}$, we have that $xp = px$ for all $x \in \mathcal{M}$. Thus p is in the center of $\mathcal{M}$.

Finally, if $x \in \mathcal{I}$ but x does not belong to $\mathcal{M}p$, then $y = x(I-p) \neq 0$, so $\mathrm{supp}(y) \in \mathcal{I}$ and is non-zero and orthogonal to p, contradicting the maximality of p. Therefore $\mathcal{I} = \mathcal{M}p$. $\square$

PROPOSITION 1.18. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra, and let ϕ be a normal state on $\mathcal{M}$. For the GNS-construction associated to $(\mathcal{M}, \phi)$ we have that $\pi_\phi(\mathcal{M}) \subseteq \mathcal{B}(H_\phi)$ is a von Neumann algebra.*

PROOF. We first check that π_ϕ is WOT-continous on $(\mathcal{M})_1$. To see this, note that if $x_n \to x$ in the WOT is a uniformly bounded net, then

$$\langle \pi_\phi(x_n)\hat{y}\,, \hat{z}\rangle = \phi(z^*x_n y) \to \phi(z^*xy) = \langle \pi_\phi(x)\hat{y}\,, \hat{z}\rangle$$

for all $y, z \in \mathcal{M}$. As $\widehat{M}$ is dense in H_ϕ and (x_n) is uniformly bounded, this implies that $\pi_\phi(x_n) \to \pi_\phi(x)$ in the WOT.

Let $\mathcal{I} = \ker(\pi_\phi)$. Since π_ϕ is WOT-continuous on the unit ball of $\mathcal{M}$, we have that $(\mathcal{I})_1$ is WOT-closed, hence SOT-closed. Letting $\mathcal{A} = \mathbb{C}I + \mathcal{I}$, the unital C*-algebra generated by $\mathcal{I}$, we have that $(\mathcal{A})_1$ is SOT-closed as well, which shows by Kaplansky's Density Theorem via Corollary 6.13 of Chapter 1 that

$$\mathbb{C}I + \mathcal{I} = \mathcal{A} = \overline{\mathcal{A}}^{SOT} = \mathbb{C}I + \overline{\mathcal{I}}^{SOT},$$

thus $\mathcal{I}$ is SOT-closed.

From Lemmas 1.17 and 1.10 we have that there is a (non-zero) central projection $q \in \mathcal{M}$ so that $\pi_\phi(x) = \pi_\phi(xq)$ for all $x \in \mathcal{M}$ and $\pi_\phi : \mathcal{M}q \to \mathcal{B}(H_\phi)$ is isometric. This allows us to conclude that

$$(\pi_\phi(\mathcal{M}))_1 = (\pi_\phi(\mathcal{M}q))_1 = \pi_\phi((\mathcal{M}q)_1) = \pi_\phi((\mathcal{M})_1)$$

is WOT-closed as $(\mathcal{M})_1$ is WOT-compact by Proposition 1.14 of Chapter 1. Thus $(\pi_\phi(\mathcal{M}))_1$ is SOT-closed as well, and the same appeal to Kaplansky's Density Theorem as in the previous paragraph shows that $\pi_\phi(\mathcal{M})$ is SOT-closed. □

Proof of Proposition 1.15. We have that "(3) implies (1)" follows from Remark 1.13, while "(2) implies (3)" follows from Proposition 1.18. Thus it remains to show that every σ-finite von Neumann algebra admits a faithful, normal state.

To this end, suppose that (p_i) are a family of pairwise orthogonal, central projections in $\mathcal{M}$ so that ϕ_i is a faithful, normal state on $\mathcal{M}p_i$. By the assumption that $\mathcal{M}$ is σ-finite, we have that (p_i) is countable, so letting $p = \sum_i p_i$ under SOT-convergence, we see that p is central and we my also check that $\phi(x) := \sum_{i=1}^\infty \frac{1}{2^i}\phi_i(xp_i)$ is faithful on $\mathcal{M}p$ and WOT-continuous on $(\mathcal{M}p)_1$ as well.

We may assume that the (countable) family (p_i) is maximal by Zorn's Lemma, and we claim that $p = \sum_i p_i = I$. If not, let $q = I - p$ and $x \in \mathcal{M}q$, $x \neq 0$. We have that there is $\xi \in qH$, $\|\xi\| = 1$ so that $\|x\xi\| \neq 0$, therefore $\mathcal{M}q \ni y \mapsto \langle y\xi\,, \xi\rangle =: \phi_\xi(y)$ is a normal state so that $\phi_\xi(x^*x) \neq 0$. Let $p_0 \in \mathcal{M}q$ be the central projection corresponding to the kernel of the GNS-representation associated to $(\mathcal{M}q, \phi_\xi)$ which exists by Lemma 1.17 and the proof of Proposition 1.18, noting that $\mathcal{M}q \subseteq \mathcal{B}(qH)$ is a von Neumann algebra by Lemma 1.16. We have that ϕ_ξ is faithful and normal on $\mathcal{M}p_0$, which contradicts the maximality of p. Thus $p = I$ and $\phi(x) := \sum_{i=1}^\infty \frac{1}{2^i}\phi_i(xp_i)$ is faithful and normal on $\mathcal{M}$. □

2. The Tomita Operator in the Tracial Case

We now understand the class of objects to which Tomita–Takesaki Theory applies, namely triples $(\mathcal{M}, H, \xi)$ where $\mathcal{M} \subseteq \mathcal{B}(H)$ is a von Neumann algebra and $\xi \in H$ is a cyclic and separating vector for $\mathcal{M}$.

Remark 2.1. Note that since ξ is cyclic and separating for $\mathcal{M}$ it is also cyclic and separating for $\mathcal{M}'$ by Corollary 1.3. Therefore if $\mathcal{M}$ is represented in this way the commutant must be "sizable."

Definition 2.2. Given such a triple $(\mathcal{M}, H, \xi)$ we define two (unbounded) operators $S, F : H \to H$ by

$$S(x\xi) = x^*\xi, \quad F(x'\xi) = (x')^*\xi$$

for all $x \in \mathcal{M}$ and $x' \in \mathcal{M}'$. We have $\mathcal{D}(S) = \mathcal{M}\xi$ and $\mathcal{D}(F) = \mathcal{M}'\xi$. Both operators are well-defined as ξ is separating for both $\mathcal{M}$ and $\mathcal{M}'$. They are densely defined by cyclicity.

Remark 2.3. Upon inspection we see that S and F are not linear, *anti-linear* in the sense that $S(\lambda\eta) = \bar{\lambda}S(\eta)$ for all $\lambda \in \mathbb{C}$ and $\eta \in \mathcal{D}(S)$, with the corresponding statement holding for F as well. In this way we may construe S and F as linear operators from H to $\overline{H}$ or from $\overline{H}$ to H. Thus

$$\begin{pmatrix} 0 & S \\ S & 0 \end{pmatrix} : H \oplus \overline{H} \to H \oplus \overline{H}$$

is a linear operator, so all basic theory of linear operators will apply to anti-linear operators as well under this interpretation.

REMARK 2.4. We have that S^2 and F^2 are both linear and that $S^2(x\xi) = x\xi$ and $F^2(x'\xi) = x'\xi$ for all $x \in \mathcal{M}$ and $x' \in \mathcal{M}'$. Thus we have that $S^2, F^2 \subseteq I$.

Even though the operators S and F may appear to have little to do with each other, the remarkable fact is that they work in tandem to "intertwine" the actions of $\mathcal{M}$ and $\mathcal{M}'$. In this section we will focus on the simplest case, which occurs under the additional assumption of traciality.

DEFINITION 2.5. We will say that the triple $(\mathcal{M}, H, \xi)$ is *tracial* if

$$\langle x\xi\,, y\xi\rangle = \langle y^*\xi\,, x^*\xi\rangle = \overline{\langle x^*\xi\,, y^*\xi\rangle}$$

for all $x, y \in \mathcal{M}$.

REMARK 2.6. The term "tracial" is apt as the condition is equivalent to the state $\phi(x) = \langle x\xi\,, \xi\rangle$ having the trace property: $\phi(xy) = \phi(yx)$ for all $x, y \in \mathcal{M}$.

REMARK 2.7. If $(\mathcal{M}, H, \xi)$ is tracial, then $\|x\xi\| = \|x^*\xi\| = \|S(x\xi)\|$ for all $x \in \mathcal{M}$. Therefore S is an anti-linear isometry and so has a continuous anti-linear extension to H. We refer the extension by J and call it the *Tomita operator*. Since $S^2 \subseteq I$ with dense domain, we have that $J^2 = I$. Since J is the extension of an anti-linear isometry, it is an anti-linear isometry as well.

LEMMA 2.8. *If $(\mathcal{M}, H, \xi)$ is tracial, then $F \subseteq J$.*

PROOF. Using the Polarization Identity and $J^2 = I$ we have that

$$\begin{aligned} 4\left\langle J\eta\,, \eta'\right\rangle &= \sum_{k=1}^{4} i^k \|J(\eta) + i^k\eta'\|^2 \\ &= \sum_{k=1}^{4} i^k \|\eta + (-i)^k J\eta'\|^2 = 4\overline{\langle \eta\,, J\eta'\rangle} \end{aligned}$$

for all $\eta, \eta' \in H$.

For $x \in \mathcal{M}$ and $y' \in \mathcal{M}'$ we now calculate that

$$\left\langle Jx\xi\,, y'\xi\right\rangle = \left\langle x^*\xi\,, y'\xi\right\rangle = \left\langle (y')^*x^*\xi\,, \xi\right\rangle = \left\langle x^*(y')^*\xi\,, \xi\right\rangle = \left\langle (y')^*\xi\,, x\xi\right\rangle$$

from which it follows that

$$\overline{\langle x\xi\,, Jy'\xi\rangle} = \overline{\langle x\xi\,, (y')^*\xi\rangle} = \overline{\langle x\xi\,, Fy'\xi\rangle}.$$

Since J is bounded this suffice to show that $F \subseteq J$. $\square$

PROPOSITION 2.9. *If $(\mathcal{M}, H, \xi)$ is tracial, then $J\mathcal{M}J = \mathcal{M}'$.*

PROOF. For $x, y, z \in \mathcal{M}$ and $\eta \in H$ we have that

$$\begin{aligned}\langle JyJxz\xi\,,\eta\rangle &= \langle Jyz^*x^*\xi\,,\eta\rangle \\ &= \langle zy^*\xi\,,x^*\eta\rangle \\ &= \langle JyJz\xi\,,x^*\eta\rangle = \langle xJyJz\xi\,,\eta\rangle\,.\end{aligned}$$

Since this holds for all $z \in \mathcal{M}$ and $\overline{\mathcal{M}\xi} = H$ this shows that

$$xJyJ = JyJx,\ \forall x, y \in \mathcal{M}.$$

Therefore, $JMJ \subseteq \mathcal{M}'$. Since $F \subseteq J$, the same logic applies to $\mathcal{M}'$ and we have that $J\mathcal{M}'J \subseteq \mathcal{M}'' = \mathcal{M}$. Thus $\mathcal{M}' \subseteq J\mathcal{M}J$ as $J^2 = I$. $\square$

COROLLARY 2.10. *We have that* $(\mathcal{M}, H, \xi)$ *is tracial if and only if* $(\mathcal{M}', H, \xi)$ *is tracial.*

REMARK 2.11. Having a faithful trace τ implies that $\mathcal{M}$ is *finite* in the sense that every injective partial isometry $v \in \mathcal{M}$ is bijective. Indeed, by the trace property we have that $1 = \tau(I) = \tau(v^*v) = \tau(vv^*)$. Since vv^* is also a projection, so is $I - vv^*$. We then have that $\tau(I - vv^*) = 0$ implies that $vv^* = I$ since τ is faithful.

EXAMPLE 2.12. Let $M_n = M_n(\mathbb{C})$ be the $n \times n$ complex matrices. We have that M_n has a unique tracial state $\mathrm{tr}(x) = \frac{1}{n}\sum_i x_{ii}$. To see this, let Σ_n be the group of signed permutation matrices, and let τ be a tracial state. We have that

$$\frac{1}{|\Sigma_n|}\sum_{\sigma\in\Sigma_n} \sigma x \sigma^* = \mathrm{tr}(x) I,$$

which by the trace condition implies that

$$\tau(x) = \frac{1}{|\Sigma_n|}\sum_{\sigma\in\Sigma_n} \tau(\sigma x \sigma^*) = \mathrm{tr}(x).$$

Let $H = \widehat{M_n(\mathbb{C})}$ with $\langle \hat{x}\,, \hat{y}\rangle = \mathrm{tr}(y^*x) = \sum_{i,j} x_{ij}\overline{y_{ij}}$. We have the GNS-representation $\pi_{\mathrm{tr}} : M_n \to \mathcal{B}(H)$, that is, $\pi_{\mathrm{tr}}(x)\hat{y} = \widehat{xy}$. We also have a $*$-homomorphism $\pi'_{\mathrm{tr}} : M_n \to \mathcal{B}(H)$ given by $\pi'_{\mathrm{tr}}(x)\hat{y} = \widehat{yx^t}$ where x^t is the transpose of x. Note that π_{tr} and π'_{tr} having commuting images.

Let's show that π'_{tr} is a bounded representation of M_n. Using that $x^t(x^t)^* \preceq \|x^t\|^2 I$, so $yx^t(x^t)^*y^* \preceq \|x^t\|^2 yy^*$, we have that

$$\begin{aligned}\|\pi'_{\mathrm{tr}}(x)\hat{y}\|^2 &= \mathrm{tr}((x^t)^*y^*yx^t) \\ &= \mathrm{tr}(yx^t(xt)^*y^*) \\ &\le \|x^t\|^2\,\mathrm{tr}(yy^*)^2 \\ &= \|x\|^2\,\mathrm{tr}(y^*y) = \|x\|^2\|\hat{y}\|^2.\end{aligned}$$

Note that the trace condition was used crucially twice. Finally we have that $J\hat{x} = \widehat{x^*}$, from which it follows that $J\pi_{\mathrm{tr}}(x)J = \pi_{\mathrm{tr}'}(\bar{x})$ where $(\bar{x})^t = x^*$.

EXAMPLE 2.13. $\pi_k : M_{2^k} \to M_{2^{(k+1)}}$ be given by

$$\pi_k(A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$$

where we view $M_{2^{(k+1)}} \cong M_2(M_{2^k}) \cong M_2 \otimes M_{2^k}$. This is clearly an isometric $*$-embedding preserving the normalized traces. We write $\pi_{k,l} : M_{2^k} \to M_{2^{(k+l)}}$ as the composition of these homomorphisms. We define an equivalence relation on $\bigcup_k M_{2^k}$ by $[x,k] = [y,k+l]$ if $x \in M_{2^k}$ and $y = \pi_{k,l}(x)$ and denote the quotient by $\mathcal{R}_0$.

We may check that $\mathcal{R}_0$ is a $*$-algebra under the operations

$$[x,k] + \lambda[y,l] = [\pi_{k,l}(x) + \lambda\pi_{l,k}(y), k+l],$$
$$[x,k][y,l] = [\pi_{k,l}(x)\pi_{l,k}(y), k+l],$$

and

$$[x,k]^* = [x^*,k],\ [x,k]^t = [x^t,k].$$

Moreover, the class of $[I,k]$ is the unit in $\mathcal{R}_0$. $\mathcal{R}_0$ is normed by $\|[x,k]\| = \|x\|$ and this is a norm satisfies $\|x^*x\| = \|x\|^2$. We have that $\mathcal{R}_0$ admits a positive cone $\mathcal{R}_0^+$ consisting of all $[x,k]$ with $x \in M_{2^k}^+$ so $[x,k] \preceq [y,l]$ if and only if $\pi_{k,l}(x) \preceq \pi_{l,k}(y)$. Finally $\mathcal{R}_0$ admits a faithful, positive trace τ_0 given by $\tau_0([x,k]) = \mathrm{tr}(x)$. Note that $\mathcal{R}_0$ is not proven to be a C*-algebra, and in fact it is not complete in the given norm.

By copying the GNS construction for $(\mathcal{R}_0, \tau_0)$, we can complete $\widehat{\mathcal{R}_0}$ to a Hilbert space H and produce a $*$-homomorphic embedding $\pi : \mathcal{R}_0 \to \mathcal{B}(H)$ with a vector $\xi = \hat{I}$ so that $\tau_0(x) = \langle \pi(x)\xi\,, \xi\rangle$ for all $x \in \mathcal{R}_0$ and so that $\overline{\mathcal{R}_0\xi} = H$. Moreover π is isometric by Lemma 1.10 as it is an injective $*$-homomorphism on each $M_{2^k} \subset \mathcal{R}_0$. We define $\mathcal{R} = \pi(\mathcal{R}_0)''$ to be the *hyperfinite II_1 factor*. As τ_0 restricts to the normalized trace on each M_{2^k}, we have that this construction produces an $*$-homomorphic embedding $\pi' : \mathcal{R}_0 \to \mathcal{B}(H)$ with $\pi'(x)\hat{y} = \widehat{yx^t}$ for all $x, y \in \mathcal{R}_0$.

Note that as in the case of matrices we have the $J\pi(\mathcal{R}_0)J = \pi'(\mathcal{R}_0)$. Also note that since π and π' have commuting ranges, it follows that $\pi'(\mathcal{R}_0) \subseteq \mathcal{R}'$. If ξ was not separating for $\mathcal{R}$, then ξ would not be cyclic for $\mathcal{R}'$ by Lemma 1.2. However, we have that $\widehat{\mathcal{R}_0} = \pi'(\mathcal{R}_0)\xi \subseteq \mathcal{R}'\xi$ so $\mathcal{R}'\xi$ is dense. Therefore $(\mathcal{R}, H, \xi)$ is tracial, and it follows that $J\mathcal{R}J = \mathcal{R}'$.

3. Subspaces in General Position

REMARK 3.1. Throughout this section H will be a real Hilbert space. If H is a complex Hilbert space, we replace the inner product on H with its real part. In particular all complements are assumed to be taken with respect to the real part of the inner product.

DEFINITION 3.2. For a (real) Hilbert space H, we say two closed subspaces $K, L \subseteq H$ are in *general position* if $K \cap L = \{0\}$ and $\overline{K + L} = H$.

If (K, L) is a pair of subspaces in general position, we define $(K,L)' := (L^\perp, K^\perp)$, and note that by closedness $(K,L)'' = (K,L)$.

LEMMA 3.3. *If (K, L) is in general position, then so is $(K, L)'$.*

PROOF. We have that $L^\perp \cap K^\perp \subseteq (K+L)^\perp = \{0\}$ since $K+L$ in dense in H. We also see that $(L^\perp + K^\perp)^\perp \subseteq K \cap L$, since any vector orthogonal to $L^\perp$ lies in $L^{\perp\perp} = L$ and similarly for $K^\perp$. Since $K \cap L = \{0\}$ this shows that $L^\perp + K^\perp$ is dense in H. $\square$

DEFINITION 3.4. Suppose (K, L) is in general position in H. We define the operators, $S, F : H \to H$ for all $\xi \in K$, $\eta \in L$, $\xi' \in L^\perp$, and $\eta' \in K^\perp$ by

$$S(\xi + \eta) = \xi - \eta, \quad F(\xi' + \eta') = \xi' - \eta'.$$

Note that $\mathcal{D}(S) = K + L$ and $\mathcal{D}(F) = L^\perp + K^\perp$.

LEMMA 3.5. *The operators S and F are closed.*

PROOF. We will only prove the result for S as F corresponds to the S operator for $(K, L)'$. We need to show that $\mathrm{gr}(S) \subseteq H \oplus H$ is closed. Suppose that for sequences (ξ_k) in K and (η_k) in L, we have that

$$[(\xi_k + \eta_k) \oplus S(\xi_k + \eta_k)] = [(\xi_k + \eta_k) \oplus (\xi_k - \eta_k)] \to x \oplus y.$$

We thus have that $\xi_k + \eta_k \to x$ and $\xi_k - \eta_k \to y$ which implies that

$$\xi_k \to (x+y)/2, \quad \eta_k \to (x-y)/2.$$

This shows that $(x+y)/2 \in K$ and $(x-y)/2 \in L$. This means that $x = (x+y)/2 + (x-y)/2 \in \mathcal{D}(S)$ and

$$S(x) = S((x+y)/2 + (x-y)/2) = (x+y)/2 - (x-y)/2 = y.$$

We conclude that $x \oplus y = x \oplus S(x) \in \mathrm{gr}(S)$. $\square$

LEMMA 3.6. *If (K, L) is in general position in H, then $S^* = F$.*

REMARK 3.7. Note that this implies that $F^* = S$ as $F^* = S^{**} = S$ since S is closed.

PROOF. For $\xi \in K$, $\eta \in L$, $\xi' \in L^\perp$, and $\eta' \in K^\perp$ we have that

$$\begin{aligned} \left\langle S(\xi + \eta), \xi' + \eta' \right\rangle &= \left\langle \xi - \eta, \xi' + \eta' \right\rangle \\ &= \left\langle \xi, \xi' \right\rangle - \left\langle \eta, \eta' \right\rangle \\ &= \left\langle \xi + \eta, \xi' - \eta' \right\rangle = \left\langle \xi + \eta, F(\xi' + \eta') \right\rangle \end{aligned}$$

which shows that $S \subseteq F^*$ and $F \subseteq S^*$.

Suppose that $\langle \xi + \eta, y \rangle = \langle \xi - \eta, x \rangle$ for all $\xi \in K$, $\eta \in L$, that is, $y = S^*(x)$. By basic algebra, we would then have that $\langle \xi, y - x \rangle = -\langle \eta, x + y \rangle$ for all $\xi \in K$, $\eta \in K$. Setting ξ or η to 0 gives that

$$\langle \xi, y - x \rangle = 0 = \langle \eta, x + y \rangle,$$

thus $y - x \in K^\perp$ and $x + y \in L^\perp$. We then have that

$$y = \frac{x+y}{2} + \frac{y-x}{2} = F\left(\frac{x+y}{2} - \frac{y-x}{2}\right) = F(x)$$

We thus have that $S^* \subseteq F$, which by the first part gives that $S^* = F$. $\square$

We now turn to the case that H has a complex structure.

DEFINITION 3.8. Let H be a complex Hilbert space and let $K \subseteq H$ be a real subspace. We say that K is in *general position* if the pair (K, iK) is in general position.

REMARK 3.9. Note that if K is in general position, then $(K, iK)' = (iK^{\perp}, K^{\perp})$ where

$$K^{\perp} = \{y \in H : \operatorname{Re}\langle x\,, y\rangle = 0,\ \forall x \in K\}$$

is still the *real* orthogonal complement.

EXAMPLE 3.10. We now explore the connection between representations $(\mathcal{M}, H, \xi)$ of a von Neumann algebra on a Hilbert space with a cyclic and separating vector and real subspaces in general position. Specifically, given such a triple we define

$$K = \overline{\mathcal{M}^h \xi}, \quad K' = \overline{(\mathcal{M}')^h \xi}.$$

We would like to establish that both K is in general position and that $(K, iK)' = (K', iK')$.

LEMMA 3.11. *Keeping the same notation as in Example 3.10 we have that K and iK' are orthogonal to each other.*

PROOF. For $x \in \mathcal{M}^h$ and $y' \in (\mathcal{M}')^h$ we have that

$$\operatorname{Im}\left\langle x\xi\,, y'\xi\right\rangle = -\operatorname{Im}\left\langle y'\xi\,, x\xi\right\rangle = -\operatorname{Im}\left\langle xy'\xi\,, \xi\right\rangle = -\operatorname{Im}\left\langle y'x\xi\,, \xi\right\rangle = -\operatorname{Im}\left\langle x\xi\,, y\xi\right\rangle,$$

hence $\operatorname{Re}\langle x\xi\,, iy'\xi\rangle = -\operatorname{Im}\langle x\xi\,, y'\xi\rangle = 0$. The result follows by passing to the closures. □

COROLLARY 3.12. *Keeping the same notation as in Example 3.10 we have that*

$$K \cap iK = \{0\}.$$

PROOF. By Lemma 3.11 we have that

$$K \cap iK \subseteq (K' + iK')^{\perp} = \{0\},$$

where the last equality follows from $\mathcal{M}'\xi \subseteq K' + iK'$. □

PROPOSITION 3.13. *Keeping the same notation as in Example 3.10 we have that*

$$K^{\perp} = iK'.$$

PROOF. By Lemma 3.11 we have that K and iK' are orthogonal to each other. As both are closed subspaces, it suffices to show that $K + iK'$ is dense in H, that is, that $(K + iK')^{\perp} = 0$.

Define the representation $\pi : \mathcal{M} \to \mathcal{B}(H \oplus H)$ by $\pi(x)(\eta \oplus \zeta) := x\eta \oplus x\zeta$. We see that $\pi(\mathcal{M})$ is a von Neumann algebra and that $\pi(\mathcal{M})' = M_2(\mathcal{M}')$. Choose $\eta \in (K + iK')^{\perp}$. Consider the subspace

$$L = \overline{\pi(\mathcal{M})(\xi \oplus \eta)} \subseteq H \oplus H,$$

and let p be the orthogonal projection onto L. Since $\pi(\mathcal{M})L \subseteq L$ we have that $p \in \pi(\mathcal{M})'$ so that

$$p = \begin{pmatrix} q & s \\ s^* & r \end{pmatrix}$$

where $q, r, s \in \mathcal{M}'$. Since $p(\xi \oplus \eta) = \xi \oplus \eta$ we have that

$$q\xi + s\eta = \xi. \tag{10}$$

As $\eta \in K^{\perp}$, we have that $\operatorname{Re}\langle x\xi\,,\eta\rangle = 0$, hence

$$\langle x\xi\,,\eta\rangle - \langle \eta\,,x\xi\rangle = -\langle x\eta\,,\xi\rangle$$

for all $x \in \mathcal{M}^h$. This expression is complex linear in ξ or η, so it still holds for $x \in \mathcal{M}$ arbitrary. We conclude that $\eta \oplus \xi$ is orthogonal to L, that is, $p(\eta \oplus \xi) = 0$. This gives that

$$q\eta + s\xi = 0. \tag{11}$$

Since $\eta \in (iK')^{\perp}$, we have that $\operatorname{Im}\langle y'\xi\,,\eta\rangle = 0$, hence

$$\big\langle y'\xi\,,\eta\big\rangle = \big\langle \eta\,,y'\xi\big\rangle = \big\langle y'\eta\,,\xi\big\rangle \tag{12}$$

for all $y' \in (\mathcal{M}')^h$. This is expression is linear in y', so we conclude that the same holds for $y' \in \mathcal{M}'$ arbitrary.

Using the inequalities (11), (12), and (10) respectively we see that

$$0 \leq \langle q\eta\,,\eta\rangle = -\langle s\xi\,,\eta\rangle = -\langle s\eta\,,\xi\rangle = -\langle (I-q)\xi\,,\xi\rangle \leq 0$$

since q is a positive contraction as p is. This shows that $q\eta = 0 = (I-q)\xi$, by Cauchy–Schwarz. Since ξ is separating for $\mathcal{M}'$ and $I - q \in \mathcal{M}'$, we have that $q = I$, which implies that $\eta = 0$. □

Corollary 3.14. *If $(\mathcal{M}, H, \xi)$ is a von Neumann algebra $\mathcal{M}$ in $\mathcal{B}(H)$ with cyclic and separating vector ξ, setting $K = \overline{\mathcal{M}^h\xi}$, we have that the pair (K, iK) is in general position.*

4. The Tomita Operator

For the rest of this section the following notation will be fixed. Let (K, L) be a pair of closed subspaces in general position in a real Hilbert space H with $(K, L)' = (L^{\perp}, K^{\perp})$ the dual pair. We let P and Q be the orthogonal projections onto K and L, respectively, and write $P' = I - Q$ and $Q' = I = P$ for the corresponding projections for $(K, L)'$. We define the self-adjoint operators

$$\begin{aligned} \Sigma &:= I - P - Q, \qquad \Theta := P - Q, \\ \Sigma' &:= I - P' - Q', \quad \Theta' := P' - Q'. \end{aligned}$$

We note that

$$\Sigma' = -\Sigma,\ \Theta' = \Theta$$
$$-I \preceq \Sigma,\ \Theta \preceq I$$

and that

$$\begin{aligned}\Sigma^2 + \Theta^2 &= (I - P - Q)^2 + (P - Q)^2 \\ &= (I - P - Q + PQ + QP) + (P + Q - PQ - QP) = I.\end{aligned}$$

REMARK 4.1. If $L = K^{\perp}$, that is all angles between pairs of vectors in K and L are $\pi/2$, then $\Sigma = 0$ and $\Theta = I - 2P$, so that Σ acts as the "cosine" operator and Θ as the "sine" operator with respect to the relative positions of the subspaces K and L. If K and L are one-dimensional subspaces of $\mathbb{R}^2$, then Σ and Θ can be seen to depend only on $\sin(\theta)$ and $\cos(\theta)$, where θ is the angle between K and L.

LEMMA 4.2. *We have that Θ is injective. Considering the polar decomposition $\Theta = JT$, the following are true.*

(1) *T is injective.*
(2) *J is a self-adjoint orthogonal operator.*
(3) *$JT = TJ$*
(4) *$T = (I - \Sigma)^{1/2}(I + \Sigma)^{1/2}$.*

PROOF. To see that Θ is injective, suppose $\Theta\xi = P\xi - Q\xi = 0$. Thus $P\xi = Q\xi$ so $\xi \in K \cap L = \{0\}$. This implies that T is injective as well.

Since Θ is self-adjoint, we have that the $T = |\Theta| = |\Theta^*|$, hence $JT = TJ$, and $JT = (JT)^* = TJ^*$. Since $TJ = TJ^*$ and T is injective, it follows that $J = J^*$. Now $T^2 = \Theta^2 = (TJ)^2 = T^2J^2$. Since T is self-adjoint $\ker(T^2) = \ker(T)$, so $J^2 = I$ by injectivity of T^2.

Finally, a standard computation verifies that $T^2 = \Theta^2 = (I - \Sigma)(I + \Sigma)$. $\square$

LEMMA 4.3. *The following are true.*

(1) *T commutes with P and Q, hence with Σ.*
(2) *$JP = (I - Q)J$, $JQ = (I - P)J$, and $J\Sigma = -\Sigma J$.*

PROOF. For the first assertion, we have that

$$PT^2 = P(P - Q)^2 = P(P - PQ - QP + Q) = P - PQP,$$
$$T^2P = (P - Q)^2P = (P - PQ - QP + Q)P = P - PQP.$$

Thus P and T^2 commute. It follows that P and T commute by the power series expansion of $(T^2)^{1/2}$. A similar calculation shows that T and Q commute.

For the second assertion, we have that

$$TJP = JTP = (P - Q)P = (I - Q)P = (I - Q)(P - Q) = (I - Q)JT = T(I - Q)J,$$

where we have used that T commutes with Q and J. It follows that $JP = (I - Q)J$ since T is injective. A similar argument shows that $JQ = (I - P)J$. $\square$

The following shows that J is completely characterized by the properties we have established.

PROPOSITION 4.4. *J is the unique self-adjoint orthogonal operator satisfying:*

(1) *$JK = L^\perp$, $JL = K^\perp$;*
(2) *$J \succeq 0$ on K and $J \preceq 0$ on L.*

Proof. That $JK = L^\perp$ and $JL = K^\perp$ follow directly from Lemma 4.3 and that J is self-adjoint orthogonal. Using Lemma 4.2 we now calculate

$$PJP = P(I-Q)J = P(P-Q)J = PJTJ = PT,$$

which is positive since P and T are commuting and positive. Similarly,

$$QJQ = Q(I-P)J = Q(Q-P)J = -QJTJ = -QT,$$

which is negative in the operator order.

If J' is a self-adjoint orthogonal operator which satisfies these properties, then JJ' commutes with P and Q, hence Σ, Θ, and $T = |\Theta|$ as well. From uniqueness of the polar decomposition, we then have that JJ' commutes with J.

From Lemma 4.2 we have that $I + \Sigma = P + Q$ is injective, so

$$(P+Q)TJJ' = (P+Q)(P-Q)J' = [P(I-Q) - Q(I-P)]J' = PJ'P - QJ'Q \succeq 0.$$

Since $(P+Q)T = T(P+Q)$ is positive and injective, we have that $JJ' = I$ by uniqueness of the polar decomposition. □

Recall the operators S and F introduced in Definition 3.4.

Proposition 4.5. *We have that $JSJ = F$ and that JS and JF are self-adjoint.*

Proof. Since $\mathcal{D}(S) = K + L$ and $\mathcal{D}(F) = L^\perp + K^\perp$ we have that

$$J\mathcal{D}(S) = \mathcal{D}(F), \quad J\mathcal{D}(F) = \mathcal{D}(S)$$

by Proposition 4.4. Thus $\mathcal{D}(JSJ) = \mathcal{D}(F)$. For $\xi' \in L^\perp$ and $\eta' \in K^\perp$ we have that $J\xi' \in K$ and $J\eta' \in L$, hence

$$JSJ(\xi' + \eta') = JS(J\xi' + J\eta') = J(J\xi' - J\eta') = \xi' - \eta' = F(\xi' + \eta').$$

This shows that $JSJ = F$. Reasoning similarly about the domains, we see that

$$JS = (JSJ)J = FJ.$$

Now for $\zeta \in H$, we have that

$$\mathcal{D}(S) \ni \xi \mapsto \langle JS\xi \, , \zeta \rangle = \langle S\xi \, , J\zeta \rangle$$

extends to a bounded linear functional on H if and only $J\zeta \in \mathcal{D}(S^*)$. But $S^* = F$ by Lemma 3.6, which shows that $(JS)^* = FJ = JS$. □

5. The Modular Operator

In this section H will be a *complex* Hilbert space and K will be a real subspace so that (K, iK) is in general position. As complex operators, we observe that Σ and Θ as introduced in the last section are linear and anti-linear respectively. It follows that Θ^2 is linear, hence T is linear and J therefore anti-linear.

DEFINITION 5.1. We define the *modular operator* associated to $K \subseteq H$ as

$$\Delta = (I - \Sigma)^{-1}(I + \Sigma).$$

The following lemmas will be key to understanding the properties of Δ.

LEMMA 5.2. *Let $B \in \mathcal{B}(H)$ be positive and injective with (dense) range $H_0 \subseteq H$. We have that B^{-1} is self-adjoint, $\mathcal{D}(B^{-1}) = H_0$, and $B^{-1} : H_0 \to H$ is a bijection.*

PROOF. Note that $(H_0)^{\perp} = \ker(B^*) = \ker(B) = \{0\}$, so H_0 is in fact dense. By definition, $B^{-1}\eta = \xi$ if and only if $\eta = B\xi$, so the latter two properties are clear. The remaining part is a straightforward adaptation of Lemma 7.15 from Chapter 1. □

LEMMA 5.3. *Suppose that for $B \in \mathcal{B}(H)$ we have that both B and $I - B$ are positive and injective, then $(I - B)^{-1}B = B(I - B)^{-1}$ is self-adjoint with domain $\mathcal{D}((I - B)^{-1})$.*

PROOF. We first check that $(I - B)^{-1}B$ and $B(I - B)^{-1}$ have the same domain, namely $\mathcal{D}((I - B)^{-1})$. If $\xi \in \mathcal{D}(B(I - B)^{-1})$, then $\xi = (I - B)\eta$. Thus

$$B\xi = B(I - B)\eta = (I - B)B\eta \in \mathcal{D}((I - B)^{-1}),$$

which shows that $\xi \in \mathcal{D}((I - B)^{-1}B)$. On the other hand if $\xi' \in \mathcal{D}((I - B)^{-1}B)$, then $B\xi' = (I - B)\eta'$, which implies that $B(\xi' + \eta') = \eta'$. It follows that

$$(I - B)(\xi' + \eta') = (\xi' + \eta') - \eta' = \xi',$$

thus $\xi' \in \mathcal{D}(B(I - B)^{-1})$. We leave it as an exercise to use these identities to check that $(I - B)^{-1}B$ and $B(I - B)^{-1}$ agree on $\mathcal{D}((I - B)^{-1})$.

Now for $\zeta \in H$ we have that $\mathcal{D}((I-B)^{-1} \ni \xi \mapsto \left\langle B(I - B)^{-1}\xi \,, \zeta \right\rangle = \left\langle (I - B)^{-1}\xi \,, B\zeta \right\rangle$ extends to a bounded linear functional on H if and only if $B\zeta \in \mathcal{D}((I - B)^{-1})$, which by the previous paragraph occurs if and only if $\zeta \in \mathcal{D}((I - B)^{-1})$. Since $(I - B)^{-1}$ is self-adjoint, we now see that

$$\left\langle B(I - B)^{-1}\xi \,, \zeta \right\rangle = \left\langle \xi \,, (I - B)^{-1}B\zeta \right\rangle = \left\langle \xi \,, B(I - B)^{-1}\zeta \right\rangle,$$

which establishes the result. □

COROLLARY 5.4. *We have that Δ is self-adjoint (thus densely defined).*

PROOF. We see that $I - \Sigma = P + Q \preceq 2I$ and $I + \Sigma = (I - Q) + (I - P) \preceq 2I$ are both positive and injective since (K, iK) are in general position, similarly to the proof of Lemma 4.2. Since $I - \Sigma = 2I - (I + \Sigma)$, the result follows from Lemma 5.3. □

COROLLARY 5.5. *We have that $\Delta^{1/2} := (I - \Sigma)^{-1/2}(I + \Sigma)^{1/2}$ is self-adjoint and densely defined and that $(\Delta^{1/2})^2 = \Delta$.*

PROOF. The same reasoning as for Δ shows that $\Delta^{1/2}$ is self-adjoint and densely-defined. By Lemma 5.3 we have that

$$\begin{aligned}(\Delta^{1/2})^2 &= [(I - \Sigma)^{-1/2}(I + \Sigma)^{1/2}][(I - \Sigma)^{-1/2}(I + \Sigma)^{1/2}] \\ &= (I - \Sigma)^{-1/2}(I - \Sigma)^{-1/2}(I + \Sigma)^{1/2}(I + \Sigma)^{1/2} = \Delta\end{aligned}$$

on the range of $(I-\Sigma)$, that is $\mathcal{D}(\Delta)$, which is contained in the range of $(I-\Sigma)^{1/2}$, the domain of $\Delta^{1/2}$. Thus $(\Delta^{1/2})^2 \supseteq \Delta$. Since $(\Delta^{1/2})^2 = (\Delta^{1/2})^*\Delta^{1/2}$ is again self-adjoint, it follows that $(\Delta^{1/2})^2 = ((\Delta^{1/2})^2)^* \subseteq \Delta^* = \Delta$, thus $(\Delta^{1/2})^2 = \Delta$. □

COROLLARY 5.6. *We have that* $\Delta^{-1} = (I+\Sigma)^{-1}(I-\Sigma)$ *and* $\Delta^{-1/2} = (I+\Sigma)^{-1/2}(I-\Sigma)^{1/2}$.

PROOF. Indeed, we have that $\mathcal{D}(\Delta) = (I-\Sigma)(H)$ which is the range of Δ^{-1} as $(I-\Sigma)^{-1} : (I-\Sigma)(H) \to H$ is onto and $\mathcal{D}(\Delta^{-1}) = (I+\Sigma)(H)$ which is the range of Δ. From the previous two lemmas we have that $\Delta\Delta^{-1}, \Delta^{-1}\Delta \subseteq I$.

A similar argument works for $\Delta^{1/2}$. □

We now explore the relation between Δ and the operators S and F.

PROPOSITION 5.7. *We have that* $S = J\Delta^{1/2} = \Delta^{-1/2}J$ *and* $F = J\Delta^{-1/2} = \Delta^{1/2}J$.

PROOF. We begin by showing that $S = J\Delta^{1/2}$ and $F = J\Delta^{-1/2}$. To do so we first note that

$$(I+\Sigma)S \subseteq JT, \quad (I-\Sigma)F \subseteq JT. \tag{13}$$

Indeed, for $\xi, \eta \in K$

$$\begin{aligned}
(I+\Sigma)S(\xi+i\eta) &= (I+\Sigma)(\xi-i\eta) \\
&= [(I-P)+(I-Q)](\xi-i\eta) \\
&= (I-Q)\xi-(I-P)i\eta \\
&= (P-Q)\xi-(Q-P)i\eta \\
&= (P-Q)(\xi+i\eta) = JT(\xi+i\eta),
\end{aligned}$$

where we have used that $P\xi = \xi$ and $Qi\eta = i\eta$. A similar computation works for F.

Applying J on the left to the identities (13) and using Lemma 4.3, we see that

$$(I-\Sigma)JS \subseteq T = (I-\Sigma)^{1/2}(I+\Sigma)^{1/2}, \quad (I+\Sigma)JF \subseteq T = (I+\Sigma)^{1/2}(I-\Sigma)^{1/2}.$$

Since $I+\Sigma$ and $I-\Sigma$ are injective, we have that

$$(I-\Sigma)^{1/2}JS \subseteq (I+\Sigma)^{1/2}, \quad (I+\Sigma)^{1/2}JF \subseteq (I-\Sigma)^{1/2},$$

hence,

$$JS \subseteq \Delta^{1/2}, \quad JF \subseteq \Delta^{-1/2}.$$

Since JS and JF are self-adjoint by Proposition 4.5 and that $\Delta^{1/2}$ and $\Delta^{-1/2}$ are self-adjoint by Lemma 5.3 and its corollaries, we have that

$$JS = (JS)^* \supseteq (\Delta^{1/2})^* = \Delta^{1/2}, \quad JF = (JF)^* \supseteq (\Delta^{-1/2})^* = \Delta^{-1/2}$$

which completes the demonstration of equality in this case.

Since $JT = TJ$ by Lemma 4.2, we can repeat the argument by multiplying by J on the right of the identities (13) to derive the other set of equations. □

While the following is not strictly necessary for our approach, it is good to point out the connection between the last result and polar decompositions as they were defined for unbounded operators in Proposition 6.8 of Chapter 1.

Corollary 5.8. *We have that* $FS = S^*S = \Delta$ *and* $SF = F^*F = \Delta^{-1}$*, hence* $S = J\Delta^{1/2} = J|S|$ *and* $F = J\Delta^{-1/2} = J|F|$ *are the polar decompositions of* S *and* F *respectively.*

Before continuing, we pause to collect some of the algebraic relations we have implicitly proven.

Corollary 5.9. *The following identities hold among the operators* S, F, J, Δ*, and* T*.*

(1) $FS = \Delta$ *and* $SF = \Delta^{-1}$
(2) $J\Delta J = \Delta^{-1}$
(3) $S + F = (JT)^{-1} = 2\,\Theta^{-1}$

6. The KMS Condition

As in the previous section, H will be a *complex* Hilbert space and K will be a real subspace so that (K, iK) is in general position.

Definition 6.1. An SOT-continuous one-parameter group of unitaries (U_t) on H is said to satisfy the *KMS condition* with respect to K if $U_t(K) = K$ for all $t \in \mathbb{R}$ and if for all $\xi, \eta \in K$ there is a function $f : \{-1 \le \mathrm{Im}(z) \le 0\} \to \mathbb{C}$ so that f is continuous and analytic on the interior and so that

$$f(t) = \langle U_t\xi\,, \eta\rangle\,, \quad f(t-i) = \langle \eta\,, U_t\xi\rangle = \overline{f(t)}$$

for all $t \in \mathbb{R}$.

The goal of this section is to create a SOT-continuous one-parameter group of unitaries $\mathbb{R} \ni t \mapsto \Delta^{it}$ from the modular operator Δ constructed in the previous section and show that it *uniquely* satisfies the KMS condition for the inclusion $K \subset H$ to which it is associated. Before continuing further, it will serve to recall a few basic results from complex analysis.

Proposition 6.2 (Morera's Theorem). *Let* Ω *be a region in the complex plane. A function* $f : \Omega \to \mathbb{C}$ *is analytic if* $\oint_\gamma f(z)dz = 0$ *for all closed, rectifiable paths in* γ *in* Ω*.*

Proof. We will provide a sketch only. Fixing $z_0 \in \Omega$ for all $z \in \Omega$ the function $F(z) = \int_{z_0}^{z} f(w)dw$ is independent of the path from z_0 to z. It follows that $F'(z) = f(z)$ is complex differentiable, thus holomorphic. $\square$

Proposition 6.3 (Schwarz Reflection Principle). *Suppose that* Ω *is a simply connected region in* $\{\mathrm{Im}(z) \ge 0\}$ *so that* $\Omega \cap \mathbb{R}$ *is connected. If* $f : \Omega \to \mathbb{C}$ *is continuous, analytic in the interior, and real-valued on* $\Omega \cap \mathbb{R}$*, then* f *my be extended to an analytic function* $g : \Omega \cup \overline{\Omega} \to \mathbb{C}$ *so that* $g(z) = \overline{f(z)}$ *for all* $z \in \overline{\Omega}$*.*

PROOF. We will provide sketch only. The hypotheses show that any integral $\oint_\gamma g(z)dz$ for γ a path in $\Omega \cup \overline{\Omega}$ can be split over $\Omega \cap \mathbb{R}$. The two resulting integrals can be seen to be zero by the assumptions of simple connectedness, continuity, and analyticity on the interior, in which case Morera's Theorem applies. □

LEMMA 6.4. *Let $f, g : \{-1 \leq \mathrm{Im}(z) \leq 0\} \to \mathbb{C}$ be continuous functions which are analytic on the interior. If $f(t) = g(t)$ for all $t \in \mathbb{R}$, then $f \equiv g$.*

PROOF. Applying the Schwarz Reflection Principle, we may extend $f - g$ to an analytic function $h : \{-1 < \mathrm{Im}(z) < 1\} \to \mathbb{C}$. Since $h(t) = 0$ for all $t \in \mathbb{R}$, we must have that h is identically zero. □

COROLLARY 6.5. *For $\xi, \eta \in K$ fixed, there is at most one function $f : \{-1 \leq \mathrm{Im}(z) \leq 0\} \to \mathbb{C}$ which is continuous and analytic on the interior so that $f(t) = \langle U_t\xi\,, \eta\rangle$ for all $t \in \mathbb{R}$.*

The following result gives a reformulation of the KMS condition which is more convenient to use in practice.

LEMMA 6.6. *An SOT-continuous one-parameter group of unitaries (U_t) on H satisfies the KMS condition with respect to $K \subset H$ if and only if for all $\xi, \eta \in K$ there is a function $f : \{-1/2 \leq \mathrm{Im}(z) \leq 0\} \to \mathbb{C}$ which is continuous and analytic in the interior so that $f(t) = \langle U_t\xi\,, \eta\rangle$ and so that $f(t - i/2) \in \mathbb{R}$ for all $t \in \mathbb{R}$.*

PROOF. If $f : \{-1/2 \leq \mathrm{Im}(z) \leq 0\} \to \mathbb{C}$ is such a function, we may apply the Schwarz Reflection Principle over the line $\mathbb{R} - i/2$ to obtain a function $g : \{-1 \leq \mathrm{Im}(z) \leq 0\} \to \mathbb{C}$ which extends f continuously, is analytic on the interior, and satisfies $g(t - i) = \overline{f(t)}$. The existence of such g verifies the KMS condition.

In the other direction, if $f : \{-1 \leq \mathrm{Im}(z) \leq 0\} \to \mathbb{C}$ satisfies the KMS condition for (U_t) for some $\xi, \eta \in K$, we have that $g(z) := \overline{f(\bar{z} - i)}$ can also be seen to satisfy the KMS condition for the same. Thus by Lemma 6.4 we have that $f(z) = g(z)$, so

$$f(t - i/2) = g(t - i/2) = \overline{f(t - i/2)}$$

for all $t \in \mathbb{R}$. □

LEMMA 6.7. *Suppose that $R \in \mathcal{B}(H)$ is positive and injective. There is a SOT-continuous map $z \mapsto R^{iz}$ which is defined on $\{\mathrm{Im}(z) \leq 0\}$ with the following properties:*

(1) *R^{iz} is bounded on horizontal strips of finite width;*
(2) *$z \mapsto \langle R^{iz}\xi\,, \eta\rangle$ analytic on $\{\mathrm{Im}(z) < 0\}$ for all $\xi, \eta \in H$; and*
(3) *$\mathbb{R} \ni t \mapsto R^{it}$ defines a one-parameter group of unitaries.*

PROOF. The SOT-continuity in the parameter z is somewhat tricky to prove via direct application of the Borel Functional Calculus as stated above, so while we can define R^{iz} by the Borel Functional Caluculus applied to the function $(0, \infty) \ni \lambda \mapsto \exp(iz \log(\lambda))$, we will work instead directly with the spectral measure. Let E be the spectral measure associated to R. For each $n \in \mathbb{N}$ let $E_n = E([1/n, \infty))$, $H_n = E_n(H)$, and $R_n = RE_n$. Since R is injective, $E(\{0\}) = 0$, so $\bigcup_n H_n$ is dense in H and $E_n \to I$ in the SOT. We have that $\log(R_n) \in \mathcal{B}(H_n)$ is bounded and hermitian and that $|\lambda^{iz}| \leq \|R\|^{-\operatorname{Im}(z)}$ for all $0 < \lambda \leq \|R\|$, so

$$z \mapsto \exp(iz\log(R_n)) =: R_n^{iz}$$

is operator norm-continuous on $\{\operatorname{Im}(z) \leq 0\}$, analytic on $\{\operatorname{Im}(z) < 0\}$, and uniformly bounded on horizontal strips with the bound being independent of n. Since $E_k R_n^{iz} = R_n^{iz} E_k = R_k^{iz}$ for all $k \leq n$, we have that for fixed z the sequence (R_n^{iz}) defines a bounded operator on $\bigcup H_n$. Thus by density we may extend this to a bounded operator on H of norm at most $\|R\|^{-\operatorname{Im}(z)}$. which we denote by R^{iz}. We note that for z real, R_n^{iz} is a unitary, thus so is R^{iz} as being a unitary is the same as being a bijective isometry. Since $R_n^{it}R_n^{is} = R_n^{i(t+s)}$ by the Continuous Functional Calculus, it follows that $R^{it}R^{is} = R^{i(t+s)}$ as well for all $t, s \in \mathbb{R}$.

Now, fix z with $\operatorname{Im}(z) \leq 0$ and $\epsilon > 0$. For $\xi \in H$, we may choose $\xi_n \in H_n$ with $\|\xi_n\| \leq \|\xi\|$ for n sufficiently large so that $\|\xi - \xi_n\| < \epsilon$. For this value of n we can find $\delta >$ so that $\|R_n^{iz} - R_n^{iw}\| < \epsilon$ when $|z - w| < \delta$ and $\operatorname{Im}(w) \leq 0$. We then have that

$$\begin{aligned}
\|(R^{iz} - R^{iw})\xi\| &\leq \|R^{iz}(\xi - \xi_n)\| + \|(R^{iz} - R^{iw})\xi_n\| + \|R^{iw}(\xi - \xi_n)\| \\
&= \|R^{iz}(\xi - \xi_n)\| + \|(R_n^{iz} - R_n^{iw})\xi_n\| + \|R^{iw}(\xi - \xi_n)\| \\
&\leq \|R\|^{-\operatorname{Im}(z)}\|\xi - \xi_n\| + \|R_n^{iz} - R_n^{iw}\|\|\xi_n\| + \|R\|^{-\operatorname{Im}(w)}\|\xi - \xi_n\| \\
&\leq (\|R\|^{-\operatorname{Im}(z)} + \|\xi\| + \|R\|^{-\operatorname{Im}(w)})\epsilon.
\end{aligned}$$

This suffices to show that $z \mapsto R^{iz}$ is SOT-continuous.

Let γ be a closed, rectifiable path in $\{\operatorname{Im}(z) < 0\}$. Since R_n^{iz} is analytic in this simply connected region, it is clear that $\oint_\gamma \left\langle R_n^{iz}\xi\,, \eta\right\rangle dz = 0$ for all $\xi, \eta \in H$. Since all R_n^{iz} and R^{iz} are uniformly bounded on compact sets, and $R_n^{iz}\xi \to R_n^{iz}\xi$ we have by the Dominated Convergence Theorem that

$$\oint_\gamma \left\langle R_n^{iz}\xi\,, \eta\right\rangle dz \to \oint_\gamma \left\langle R^{iz}\xi\,, \eta\right\rangle dz.$$

Thus $\oint_\gamma \left\langle R^{iz}\xi\,, \eta\right\rangle dz = 0$, so $\left\langle R^{iz}\xi\,, \eta\right\rangle$ is analytic on $\{\operatorname{Im}(z) < 0\}$ by Morera's Theorem. □

LEMMA 6.8. *Suppose that $R, S \in \mathcal{B}(H)$ are commuting positive operators which are both injective. For a fixed $\alpha > 0$ we have that for all $\xi, \eta \in H$ that $z \mapsto \left\langle R^{iz}S^{-iz+\alpha}\xi\,, \eta\right\rangle$ is continuous on $\{-\alpha \leq \operatorname{Im}(z) \leq 0\}$ and analytic on the interior.*

PROOF. By Lemma 6.7 we have that $z \mapsto R^{iz}$ and $z \mapsto S^{-iz+\alpha}$ are uniformly bounded and SOT-continuous on the specified domain. This, along with the fact that the ranges are commuting by the Borel Functional Calculus allows us to use the standard proof of the product rule for differentiation to obtain the result. The details are left as an exercise. □

We are now ready to introduce the one-parameter unitary group associated to Δ for a real subspace $K \subset H$ in general position.

Lemma 6.9. *We have that (Δ^{it}) is an SOT-continuous one-parameter group of unitaries on H and that $\Delta^{it} = (I+\Sigma)^{it}(I-\Sigma)^{-it}$ for all $t \in \mathbb{R}$. Moreover, we have that*

$$J\Delta^{it} = \Delta^{it}J, \quad \Delta^{it}K = K$$

for all $t \in \mathbb{R}$.

Proof. We have that Δ is self-adjoint by Corollary 5.4, injective by Corollary 5.6, and is equal to $(\Delta^{1/2})^2$ with $\Delta^{1/2}$ self-adjoint by Corollary 5.5. Therefore by the proof of Proposition 7.17 of Chapter 1, we have that there is an increasing sequence of closed subspaces (H_n) of H so that $\bigcup_n H_n$ is dense in H, each H_n lies in $\mathcal{D}(\Delta)$, $\Delta(H_n) \subseteq H_n$, and the restriction of Δ_n of Δ to H_n is bounded, positive, and injective. By Lemma 6.7, we have that Δ_n^{it} is an SOT-continuous one-parameter group of unitaries for H_n and that $\Delta_k^{it}\xi$ and $\Delta_n^{it}\xi$ agree for all $\xi \in H_{\min(k,n)}$. Arguing similarly to the proof of that lemma allows us to extend $(\Delta_n^{it})_{n\in\mathbb{N}}$ for t fixed to a unitary Δ^{it} on H and show that these unitaries are SOT-continuous in t.

By Lemma 5.3 we have that

$$(I-\Sigma)\Delta \subseteq \Delta(I-\Sigma) = I+\Sigma.$$

Since $\Delta_n : H_n \to H_n$ is bounded, positive, and injective, $\overline{\Delta_n(H_n)} = H_n$ (see Lemma 6.6 in chapter 1), thus each H_n is an invariant subspace for $I-\Sigma$, hence $I+\Sigma$ as well. It follows from the Continuous Functional Calculus that

$$\Delta_n^{it}(I-\Sigma)^{it}\xi = (I-\Sigma)^{it}\Delta_n^{it}\xi = (I+\Sigma)^{it}\xi$$

for all $\xi \in H_n$. Consequently, $(I-\Sigma)^{it}\Delta^{it} = (I+\Sigma)^{it}$.

Since $(I \pm \Sigma)K \subseteq K$, we have that $\Delta^{it}K \subseteq K$ by the previous result and the Borel Functional Calculus. Since $\Delta^{-it}K \subseteq K$ by the same, we have equality. Since $(I\pm\Sigma)K^\perp = (I\mp\Sigma')K^\perp \subseteq K^\perp$, we have that $\Delta^{it}K^\perp = K^\perp$ as well. This shows that $J' := \Delta^{it}J\Delta^{-it}$ satisfies the hypotheses of Proposition 4.4, thus $\Delta^{it}J\Delta^{-it} = J$. ☐

This final Corollary will be key to our analysis of the KMS condition for Δ^{it}.

Corollary 6.10. *For all $\xi, \eta \in H$, we have that $z \mapsto \langle \Delta^{iz}(I-\Sigma)^{1/2}\xi\,,\eta\rangle$ is continuous on the strip $\{-1/2 \le \mathrm{Im}(z) \le 0\}$ and analytic in the interior.*

Proof. This is a straightforward application of Lemma 6.8 and Lemma 6.9 noting that both $I-\Sigma$ and $I+\Sigma$ are positive and injective. ☐

We are now ready to prove the main results of this section.

Proposition 6.11. *We have that (Δ^{it}) satisfies the KMS condition with respect to $K \subset H$.*

PROOF. For $\xi \in K$ note by Lemmas 4.2 and 4.3 that

$$\xi = P\xi = \frac{1}{2}((I-\Sigma) + JT)\xi = \frac{1}{2}(I-\Sigma)^{1/2}(I+J)(I-\Sigma)^{1/2}\xi$$

Therefore setting

$$\zeta := \frac{1}{2}(I+J)(I-\Sigma)^{1/2}\xi \tag{14}$$

we have that $\xi = (I-\Sigma)^{1/2}\zeta$. For $\xi, \eta \in K$ we therefore define the function $f : \{-1/2 \leq \mathrm{Im}(z) \leq 0\} \to \mathbb{C}$ by

$$f(z) = \left\langle \Delta^{iz}(I-\Sigma)^{1/2}\zeta \,, \eta \right\rangle = \left\langle (I+\Sigma)^{iz}(I-\Sigma)^{-iz+1/2}\zeta \,, \eta \right\rangle$$

which by Corollary 6.10 is continuous and analytic on the interior. We also note that for $t \in \mathbb{R}$ we have that

$$f(t) = \left\langle \Delta^{it}(I-\Sigma)^{1/2}\zeta \,, \eta \right\rangle = \left\langle \Delta^{it}\xi \,, \eta \right\rangle.$$

By Lemma 6.6 it will suffice to show that $f(t - i/2)$ is real for all $t \in \mathbb{R}$.

To this end, we see that

$$f(t-i/2) = \left\langle (I+\Sigma)^{iz+1/2}(I-\Sigma)^{-iz}\zeta \,, \eta \right\rangle = \left\langle \Delta^{it}(I+\Sigma)^{1/2}\zeta \,, \eta \right\rangle.$$

We have that

$$\begin{aligned}
(I+\Sigma)^{1/2}\zeta &= \frac{1}{2}(I+\Sigma)^{1/2}(I+J)(I-\Sigma)^{1/2}\xi \\
&= \frac{1}{2}[(I+\Sigma)^{1/2}(I-\Sigma)^{1/2} + J(I-\Sigma)]\xi \\
&= \frac{1}{2}J[JT + (I-\Sigma)]\xi \\
&= JP\xi = J\xi,
\end{aligned} \tag{15}$$

thus

$$f(t-i/2) = \left\langle J\xi \,, \Delta^{-it}\eta \right\rangle.$$

This quantity is real since $JK = iK^{\perp}$ by Proposition 4.4 and $\Delta^{-it}\eta \in K$ by Lemma 6.9. ☐

PROPOSITION 6.12. *We have that Δ^{it} is the unique SOT-continuous one-parameter group of unitaries which satisfies the KMS condition with respect to $K \subset H$.*

PROOF. Let (U_t) be SOT-continuous one-parameter group of unitaries on H. We say that $\xi \in H$ is *entire* if $h(t) = U_t\xi$ extends to an entire function $h : \mathbb{C} \to H$. We have that the entire vectors are dense since for each n and $\xi \in H$ we can define

$$\xi_n := \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} \exp(-nt^2)U_t\xi dt,$$

noting that $\xi_n \to \xi$ by SOT-continuity of U_t and that

$$h_n(z) := \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} \exp(-n(t-z)^2) U_t \xi dt$$

is entire.

Suppose that $\xi \in H$ is entire with function $h(z)$. We have that $h(t+z) = U_t(h(z))$ for $z \in \mathbb{R}$ and both sides are analytic in z, so $h(t+z) = U_t(h(z))$ for all $z \in \mathbb{C}$. In particular, $h(t+is) = U_t(h(is))$ for all $s \in \mathbb{R}$.

Now suppose (U_t) satisfies the KMS condition with respect to K. Since $U_t(K) = K$, we have that the set of entire vectors for (U_t) is dense in K by the argument in the first paragraph. Let K_0 be the entire vectors. Given $\xi, \eta \in K_0$, we would like to show that

$$\left\langle U_t \xi \,, \Delta^{it} J \eta \right\rangle = \langle \xi \,, J\eta \rangle$$

as the complex spans of K_0 and JK_0 are dense in H, this would show that $U_t = \Delta^{it}$.

To this end, let $h(z)$ be entire for ξ and define

$$g(z) = \left\langle h(z) \,, J \Delta^{iz} (I - \Sigma)^{1/2} \zeta \right\rangle = \left\langle h(z) \,, (I + \Sigma)^{iz} (I - \Sigma)^{-iz + 1/2} \zeta \right\rangle$$

where $\eta = (I - \Sigma)^{1/2} \zeta$ as derived in line (14) above. Note that $g(z)$ is bounded, continuous on $\{-1/2 \leq \mathrm{Im}(z) \leq 0\}$, and analytic on the interior. We have that $g(t) = \left\langle U_t \xi \,, \Delta^{it} J\eta \right\rangle \in \mathbb{R}$ since $U_t \xi \in K$ and $\Delta^{it} J \eta = J \Delta^{it} \eta \in iK^{\perp}$. If we can show that $g(t - i/2)$ is real for all $t \in \mathbb{R}$, then since g would assume only real values on its boundary, we could repeatedly apply the Schwarz Reflection Principle to extend g to a bounded entire function, implying that g is constant. Consequently we would see that $g(z) = g(0) = \langle \xi \,, J\eta \rangle$, and the proof would be concluded.

We will now show that $g(t - i/2)$ is real. Similarly to the proof of Proposition 6.11, we have that

$$\begin{aligned} g(t - i/2) &= \left\langle h(t - i/2) \,, J \Delta^{it} (I + \Sigma)^{1/2} \zeta \right\rangle \\ &= \left\langle h(t - i/2) \,, J \Delta^{it} J \eta \right\rangle \\ &= \left\langle h(t - i/2) \,, \Delta^{it} \eta \right\rangle \end{aligned}$$

where we have used line (15) above. Let $s \in \mathbb{R}$ be fixed, and let $f(z)$ be the KMS function for (U_t) associated to ξ, $\Delta^{is}\eta$. We have that

$$f(t) = \left\langle U_t \xi \,, \Delta^{is} \eta \right\rangle = \left\langle h(t) \,, \Delta^{is} \eta \right\rangle$$

for all $t \in \mathbb{R}$, which implies that $f(z) = \left\langle h(z) \,, \Delta^{is} \eta \right\rangle$ whenever both are defined by Lemma 6.4. By Lemma 6.6, this shows that $\left\langle h(t - i/2) \,, \Delta^{is} \eta \right\rangle = f(t - i/2) \in \mathbb{R}$. Setting $s = t$ shows that $g(t - i/2)$ is real. This completes the proof of the claim, hence the result obtains. $\square$

We now give an application of these results to the study of von Neumann algebras.

DEFINITION 6.13. For a von Neumann algebra $\mathcal{M}$ we will say that a homomorphism $\alpha : \mathbb{R} \to \operatorname{Aut}(\mathcal{M})$ is a *strongly continuous one-parameter group of automorphisms* if $\alpha_t(x) \to x$ in the SOT as $t \to 0$ for all $x \in \mathcal{M}$.

DEFINITION 6.14. For a von Neumann algebra $\mathcal{M}$ and a faithful, normal state ϕ we say that a strongly continuous one-parameter group of automorphisms satisfies the *KMS-condition* with respect to ϕ if $\phi(\alpha_t(x)) = \phi(x)$ for all $t \in \mathbb{R}$ and $x \in \mathcal{A}$ and if for all $x, y \in \mathcal{M}$ there is a function $f : \{-1 \leq \operatorname{Im}(z) \leq 0\} \to \mathbb{C}$ which is continuous and analytic in the interior so that

$$f(t) = \phi(\alpha_t(x)y), \quad f(t-i) = \phi(y\alpha_t(x))$$

for all $t \in \mathbb{R}$.

LEMMA 6.15. *Suppose that α is a strongly continuous one-parameter group of automorphisms of a von Neumann algebra $\mathcal{M}$, and let ϕ be a faithful normal state on $\mathcal{M}$ so that $\phi \circ \alpha_t = \phi$ for all $t \in \mathbb{R}$. Let $(\mathcal{M}, H_\phi, \xi_\phi)$ be the representation of $\mathcal{M}$ on the Hilbert space H_ϕ with cyclic, separating vector ξ_ϕ obtained from the GNS-construction associated to $(\mathcal{M}, \phi)$ as described in section 1. (Note that we are implicitly identifying $\mathcal{M}$ with $\pi_\phi(\mathcal{M})$.) We have that*

$$U_t(x\xi_\phi) := \alpha_t(x)\xi_\phi$$

for all $x \in \mathcal{M}$ defines a SOT-continuous one-parameter unitary group on H_ϕ. Moreover, for all $t \in \mathbb{R}$, $x \in \mathcal{M}$, and $\eta \in H_\phi$, we have that

$$\alpha_t(x)\eta = U_t x U_t^* \eta.$$

PROOF. As ξ_ϕ is separating, U_t is well-defined, thus linear, on $\mathcal{M}\xi_\phi$. We have that

$$\|U_t(x\xi_\phi)\|^2 = \langle \alpha_t(x)^*\alpha_t(x)\xi_\phi\,, \xi_\phi\rangle = \phi(\alpha_t(x^*)\alpha_t(x)) = \phi(\alpha_t(x^*x)) = \phi(x^*x) = \|x\xi_\phi\|^2,$$

so U_t is isometric. Hence U_t extends to a unitary on H_ϕ for all $t \in \mathbb{R}$. Further, we have by similar computations that

$$\begin{aligned}
\|(U_t - I)x\xi_\phi\|^2 &= \langle \alpha_t(x)^*\alpha_t(x)\xi_\phi\,, \xi_\phi\rangle - \langle \alpha_t(x)^*x\xi_\phi\,, \xi_\phi\rangle - \langle x^*\alpha_t(x)\xi_\phi\,, \xi_\phi\rangle + \langle x^*x\xi_\phi\,, \xi_\phi\rangle \\
&= 2\phi(x^*x) - \phi(\alpha_t(x^*)x) - \phi(x^*\alpha_t(x)) \\
&= 2\,\phi(x^*x) - 2\operatorname{Re}\phi(\alpha_t(x^*)x).
\end{aligned}$$

Since ϕ is normal, $(\alpha_t(x^*))$ is uniformly bounded, and $\alpha_t(x^*) \to x^*$ in the SOT, we have that that $\phi(\alpha_t(x^*)x) \to 0$, hence $\|(U_t - I)x\xi_\phi\| \to 0$ as $t \to 0$. Since $\mathcal{M}\xi_\phi$ is dense in H_ϕ and (U_t) is uniformly bounded, this suffices to show that $U_t \to I$ in the SOT as $t \to 0$.

For the second assertion, we note that

$$U_t x U_t^* y\xi_\phi = U_t x U_{-t} y\xi_\phi = U_t x\alpha_{-t}(y)\xi_\phi = \alpha_t(x\alpha_{-t}(y))\xi_\phi = \alpha_t(x)y\xi_\phi$$

for all $x, y \in \mathcal{M}$. The assertion now follows as it holds on the dense subspace $\mathcal{M}\xi_\phi$ of H_ϕ. $\square$

PROPOSITION 6.16. *Let $\mathcal{M}$ be a von Neumann algebra, and let ϕ be a faithful, normal state on $\mathcal{M}$. If α is a strong continuous one-parameter group of automorphisms of $\mathcal{M}$ which satisfies the KMS-condition with respect to ϕ, then the one-parameter unitary group (U_t) defined in Lemma 6.15 satisfies the KMS-condition with respect to $K = \overline{\mathcal{M}^h \xi_\phi}$.*

PROOF. We begin by noting that by definition $U_t\xi_\phi = \xi_\phi$. Let $x, y \in \mathcal{M}^h$. We have that

$$\phi(\alpha_t(x)y) = \langle U_t x U_{-t} y\xi_\phi \,, \xi_\phi\rangle = \langle U_t x U_{-t} y\xi_\phi \,, U_t\xi_\phi\rangle = \langle xU_{-t}y\xi_\phi \,, \xi_\phi\rangle = \langle y\xi_\phi \,, U_t x\xi_\phi\rangle$$

and

$$\phi(y\alpha_t(x)) = \phi(\alpha_t(\alpha_{-t}(y)x)) = \phi(\alpha_{-t}(y)x) = \langle U_{-t}yU_t x\xi_\phi \,, \xi_\phi\rangle = \langle U_t x\xi_\phi \,, y\xi_\phi\rangle \,,$$

hence if $f : \{-1 \leq \mathrm{Im}(z) \leq 0\} \to \mathbb{C}$ witnesses the KMS-condition for α with respect to x and y, we have that

$$\overline{f(t)} = \langle U_t x\xi_\phi \,, y\xi_\phi\rangle = f(t - i)$$

which verifies the KMS-condition with respect to (U_t) for all $x\xi_\phi$ and $y\xi_\phi$. Now if $\eta, \zeta \in K$, we can find sequences (x_n) and (y_n) in $\mathcal{M}^h$ so that $\|x_n\xi_\phi - \eta\|$, $\|y_n\xi_\phi - \zeta\| \to 0$.

Let f_n be the function witnessing the KMS condition with respect to $x_n\xi_\phi$ and $y_n\xi_\phi$. Using Cauchy–Scwarz and the fact that U_t is a unitary, we have that

$$\begin{aligned}
&|\langle U_t x_n\xi_\phi \,, y_n\xi_\phi\rangle - \langle U_t\eta \,, \zeta\rangle| \\
&= |\langle U_t x_n\xi_\phi \,, y_n\xi_\phi\rangle - \langle U_t x_n\xi_\phi \,, \zeta\rangle + \langle U_t x_n\xi_\phi \,, \zeta\rangle - \langle U_t\eta \,, \zeta\rangle| \\
&\leq |\langle U_t x_n\xi_\phi \,, y_n\xi_\phi\rangle - \langle U_t x_n\xi_\phi \,, \zeta\rangle| + |\langle U_t x_n\xi_\phi \,, \zeta\rangle - \langle U_t\eta \,, \zeta\rangle| \\
&\leq \|U_t x_n\xi_\phi\|\|y_n\xi_\phi - \zeta\| + \|U_t(x_n\xi_\phi - \eta)\|\|\zeta\| \\
&= \|x_n\xi_\phi\|\|y_n\xi_\phi - \zeta\| + \|x_n\xi_\phi - \eta\|\|\zeta\|,
\end{aligned}$$

thus $(f_n(t))$ and $(f_n(t - i))$ are uniformly Cauchy sequences of functions with $t \in \mathbb{R}$. By the Maximal Principle for analytic functions, we therefore have that (f_n) converges uniformly on $\{-1 \leq \mathrm{Im}(z) \leq 0\}$ to a continuous function f satisfying $\overline{f(t)} = \langle U_t\eta \,, \zeta\rangle = f(t - i)$. That f is analytic in the interior is a consequence of Morera's theorem and is similar to the argment given in the proof of Lemma 6.7. $\square$

COROLLARY 6.17. *Let $\mathcal{M}$ be a von Neumann algebra with a normal, faithful state ϕ if (α_t) is a strongly continuous one-parameter group of automorphisms which satisfied the KMS condition with respect to ϕ, then (α_t) is the unique such strongly continuous one-parameter automorphism group.*

PROOF. By the previous result, this follows directly from Proposition 6.12. $\square$

7. The Main Theorem of Tomita–Takesaki Theory

We are now ready to prove the main theorem of Tomita–Takeski Theory. In this section we will fix $(\mathcal{M}, H, \xi)$ to be a von Neumann algebra $\mathcal{M}$ (faithfully) represented on a Hilbert space H with cyclic and separating vector ξ with $\|\xi\| = 1$. We will let

$K = \overline{\mathcal{M}^h \xi}$, and note that $K \subset H$ is in general position by Corollary 3.14. The operators J and Δ are defined with respect to K.

PROPOSITION 7.1. *We have that* $\Delta^{it} \mathcal{M} \Delta^{-it} = \mathcal{M}$ *and* $J\mathcal{M}J = \mathcal{M}'$.

The proof will occupy the rest of this section, and we will need several lemmas. The first is a variant of Sakai's Radon–Nikodym Theorem.

LEMMA 7.2. *Suppose that* $x' \in (\mathcal{M}')^h$. *For any* $\lambda \in \mathbb{C}$ *with* $\lambda \neq 0$ *there is a unique* $x \in \mathcal{M}^h$ *so that*

$$\langle y\xi \,, x'\xi \rangle = \frac{\lambda}{2} \langle y\xi \,, x\xi \rangle + \frac{\bar{\lambda}}{2} \langle x\xi \,, y^*\xi \rangle$$

for all $y \in \mathcal{M}$.

PROOF. Since both expressions are real linear in x', x, and λ, we may assume without loss of generality that x' is a positive contraction and that $\operatorname{Re} \lambda = 1$. We define

$$\phi(y) := \langle y\xi \,, x'\xi \rangle \,, \quad \psi_x(y) := \frac{\lambda}{2} \langle y\xi \,, x\xi \rangle + \frac{\bar{\lambda}}{2} \langle x\xi \,, y^*\xi \rangle \,.$$

Note that ϕ and ψ_x are linear and $\phi(y^*) = \overline{\phi(y)}$ and $\psi_x(y^*) = \overline{\psi_x(y)}$. We recall the predual $\mathcal{M}_*$ of $\mathcal{M}$ as introduced in Definition 6.18 of Chapter 1. It is easy to see that $\phi, \psi_x \in \mathcal{M}_*$ for all $x \in \mathcal{M}$.

Consider the set $C := \{\psi_x : x \in (\mathcal{M}^h)_1\} \subset \mathcal{M}_*$ which is convex as the map $x \mapsto \psi_x$ is real linear. If (x_n) is a net in $(\mathcal{M}^h)_1$ so that $x_n \to x$ in the WOT, we clearly have that $\psi_{x_n}(y) \to \psi_x(y)$ for any $y \in \mathcal{M}$. Since $\mathcal{M} = (\mathcal{M}_*)^*$ and $(\mathcal{M}^h)_1$ is compact in the WOT, we have that C is compact in the weak topology on $\mathcal{M}_*$. The result would follow from establishing $\phi \in C$.

If it was the case that $\phi \notin C$, then by the Geometric Hahn–Banch Theorem, we would have that there is $z \in \mathcal{M}$ so that $\operatorname{Re} \psi_x(z) < \operatorname{Re} \phi(z)$ for all $x \in (\mathcal{M}^h)_1$. Since $2 \operatorname{Re} \psi_x(z) = \psi_x(z + z^*)$ and $2 \operatorname{Re} \phi(x) = \phi(z + z^*)$, we may have that there is $w \in \mathcal{M}^h$ so that $\psi_x(w) < \phi(w)$ for all $x \in (\mathcal{M}^h)_1$. Since w is hermitian, we have that the polar decomposition $w = j|w|$ is such that $j = j^*$ and $jw = |w|$. Notice that if $y \in \mathcal{M}$ is positive, then since y, x' are positive and commuting and $\|x'\| \leq 1$, we have that $0 \preceq x'y = y^{1/2} x' y^{1/2} \preceq \|x'\| y \preceq y$. It follows that ϕ is a positive linear functional and that

$$\phi(|w|) = \langle |w|\xi \,, x'\xi \rangle = \langle x'|w|\xi \,, \xi \rangle \leq \langle |w|\xi \,, \xi \rangle = \langle w\xi \,, j\xi \rangle \,.$$

However, since $\operatorname{Re} \lambda = 1$ and $w \preceq |w|$ this shows that

$$\psi_j(w) < \phi(w) \leq \phi(|w|) \leq \langle w\xi \,, j\xi \rangle = \operatorname{Re}(\lambda \langle w\xi \,, j\xi \rangle) = \psi_j(w),$$

a contradiction. ☐

Recall the anti-linear operator $\Theta = P - Q = J|\Theta| = JT$ defined in section 3. The next lemma is crucial to our approach as it shows that the function Θ maps $\mathcal{M}'\xi$ into $\mathcal{M}\xi$ without taking closures.

LEMMA 7.3. *For each $x' \in M'$ there is $x \in M$ so that*

$$\Theta x'\xi = x\xi, \quad \Theta(x')^*\xi = x^*\xi.$$

PROOF. We first assume that $x' \in (\mathcal{M}')^h$. Setting $\lambda = 1$, by Lemma 7.2 we have that there is $x \in \mathcal{M}^h$ so that for all $y \in \mathcal{M}^h$

$$\mathrm{Re}\left\langle y\xi\,, x'\xi\right\rangle = \left\langle y\xi\,, x'\xi\right\rangle = \mathrm{Re}\left\langle y\xi\,, x\xi\right\rangle$$

where we have used that x' and y commute to show the first equality. This shows that $Px'\xi = Px\xi = x\xi$. On the other hand, $x'\xi \in iK^\perp$ by Proposition 3.13, so $Qx'\xi = 0$. Therefore

$$\Theta x'\xi = (P - Q)x'\xi = x\xi.$$

The general result follows from anti-linearity of Θ. □

REMARK 7.4. We note that by considering the triple $(\mathcal{M}', H, \xi)$ instead of $(\mathcal{M}, H, \xi)$, we have that the Θ function for this "dual" triple is equal to $\Theta' = \Theta$ for $(\mathcal{M}, H, \xi)$, hence the result holds verbatim exchanging the roles of $\mathcal{M}'$ and $\mathcal{M}$.

REMARK 7.5. Note that as $\Theta^{-1} = S+F = J(\Delta^{1/2}+\Delta^{-1/2})$, the above lemma shows that for every $x' \in \mathcal{M}'$, there is $x \in \mathcal{M}$ so that $x'\xi = (S+F)x\xi$ and $(x')^*\xi = (S+F)x^*\xi$.

LEMMA 7.6. *For $x \in \mathcal{M}$ and $x' \in \mathcal{M}'$, we have that*

$$\Theta x\xi = (I + \Sigma)x^*\xi, \quad \Theta x'\xi = (I - \Sigma)(x')^*\xi.$$

PROOF. We begin by assuming that $x \in \mathcal{M}^h$ and $x' \in (\mathcal{M}')^h$. We have that $Px\xi = x\xi$ and $Qx'\xi = 0$, thus

$$\begin{aligned}\Theta x\xi &= (P - Q)x\xi = (2I - P - Q)x\xi = (I + \Sigma)x\xi,\\ \Theta x'\xi &= (P - Q)x'\xi = (P + Q)x'\xi = (I - \Sigma)x'\xi.\end{aligned}$$

The result follows since Θ is anti-linear and $I \pm \Sigma$ are (complex) linear. □

LEMMA 7.7. *For each $x' \in M'$ and $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ there is $x \in \mathcal{M}$ so that*

$$\Theta x'\Theta = \lambda(I + \Sigma)x(I - \Sigma) + \bar{\lambda}(I - \Sigma)x(I + \Sigma).$$

PROOF. Since the expressions are linear on both sides of the equation, we may assume that $x' \in (\mathcal{M}')^h$. By Lemma 7.2 there is $x \in \mathcal{M}^h$ with

$$\left\langle y\xi\,, x'\xi\right\rangle = \lambda\left\langle y\xi\,, x\xi\right\rangle + \bar{\lambda}\left\langle x\xi\,, y^*\xi\right\rangle, \tag{16}$$

where the x in that lemma has been rescaled by a factor of $1/2$. For $y, z \in \mathcal{M}$, we may substitute y by z^*y in (16) to obtain

$$\left\langle y\xi\,, x'z\xi\right\rangle = \lambda\left\langle y\xi\,, zx\xi\right\rangle + \bar{\lambda}\left\langle yx\xi\,, z\xi\right\rangle.$$

Given $y', z' \in \mathcal{M}'$, we can find $y, z \in \mathcal{M}$ so that $\Theta y'\xi = y\xi$ and $\Theta z'\xi = z\xi$, hence we obtain

$$\left\langle \Theta y\xi\,, x'\Theta z'\xi\right\rangle = \lambda\left\langle \Theta y'\,, zx\xi\right\rangle + \bar{\lambda}\left\langle yx\xi\,, \Theta z'\xi\right\rangle.$$

Since Θ is anti-linear and $\Theta^* = (JT)^* = TJ = JT = \Theta$ by Lemma 4.2, we derive from this that

$$\left\langle \Theta x' \Theta z' \xi \,, y' \xi \right\rangle = \lambda \left\langle \Theta z x \xi \,, y' \xi \right\rangle + \bar{\lambda} \left\langle z' \xi \,, \Theta y x \xi \right\rangle . \tag{17}$$

From Lemmas 7.3 and 7.6, we have that

$$\Theta z x \xi = (I + \Sigma) x^* z^* \xi = (I + \Sigma) x^* \Theta (z')^* \xi = (I + \Sigma) x^* (I - \Sigma) z' \xi$$

and similarly

$$\Theta y x \xi = (I + \Sigma) x^* (I - \Sigma) y' \xi.$$

Substituting these expressions into (17) we obtain

$$\left\langle \Theta x' \Theta z' \xi \,, y' \xi \right\rangle = \lambda \left\langle (I + \Sigma) x^* (I - \Sigma) z' \xi \,, y' \xi \right\rangle + \bar{\lambda} \left\langle z' \xi \,, (I + \Sigma) x^* (I - \Sigma) y' \xi \right\rangle .$$

Using the assumption that x' and x are hermitian, this shows that

$$\left\langle \Theta x' \Theta z' \xi \,, y' \xi \right\rangle = \lambda \left\langle (I + \Sigma) x (I - \Sigma) z' \xi \,, y' \xi \right\rangle + \bar{\lambda} \left\langle (I + \Sigma) x (I - \Sigma) z' \xi \,, y' \xi \right\rangle ,$$

hence the result holds by density of $\mathcal{M}'\xi$ in H. □

REMARK 7.8. One may consider Lemma 7.7 as a "quantized" version of Lemma 7.3. In the unbounded operator picture it asserts that for every $x' \in \mathcal{M}'$ and $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$ we have that there is $x \in \mathcal{M}$ so that

$$J x' J = \lambda \Delta^{1/2} x \Delta^{-1/2} + \bar{\lambda} \Delta^{-1/2} x \Delta^{1/2}.$$

We would like to solve this equation for x, which is somewhat akin to inverting $\cosh(x) = e^{-x}/2 + e^{x}/2$. The following technical lemma may be viewed in this context.

LEMMA 7.9. *Let $\theta \in (-\pi, \pi)$ and set $\lambda = e^{i\theta/2}$. Suppose that $f : \{|\mathrm{Re}(z)| \leq 1/2\} \to \mathbb{C}$ is bounded, continuous, and analytic on the interior. Then*

$$f(0) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-\theta t} \cosh(\pi t)^{-1} \left(\lambda f(it + 1/2) + \bar{\lambda} f(it - 1/2) \right) dt$$

PROOF. Set $g(z) := \pi e^{i\theta z} \sin(\pi z)^{-1} f(z)$, which is defined and meromorphic on the interior of the strip $\{|\mathrm{Re}(z)| \leq 1/2\}$ with a simple pole at $z = 0$ ($\pi \sin(\pi z)^{-1} \sim z$) with residue $f(0)$. We have that $g(z) \to 0$ as $|z| \to 0$, thus by Cauchy's Integral Formula,

$$f(0) = \mathrm{Res}(g)(0) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} [g(it + 1/2) - g(it - 1/2)] i dt.$$

The result follows by noting that $\sin(\pi i t \pm 1/2) = \cosh(\pi t)$. □

LEMMA 7.10. *For $x' \in \mathcal{M}'$ and $\theta \in (-\pi, \pi)$, set $\lambda = e^{i\theta/2}$ and let $x \in \mathcal{M}$ so that*

$$\Theta x' \Theta = \lambda (I + \Sigma) x (I - \Sigma) + \bar{\lambda} (I - \Sigma) x (I + \Sigma). \tag{18}$$

as guaranteed by Lemma 7.7. We have that

$$x = \frac{1}{2} \int_{-\infty}^{\infty} e^{-\theta t} \cosh(\pi t)^{-1} \Delta^{it} J x' J \Delta^{-it} dt$$

PROOF. For $\zeta, \eta \in H$, we define

$$f(z) := \left\langle (I+\Sigma)^{z+1/2}(I-\Sigma)^{-z+1/2}x(I-\Sigma)^{z+1/2}(I+\Sigma)^{-z+1/2}\zeta\,,\eta \right\rangle.$$

From Lemma 6.9 we have that $\Delta^{it} = (I+\Sigma)^{it}(I-\Sigma)^{-it}$, hence by Lemma 6.8 we have that $f(z)$ is continuous and bounded on and analytic in the interior of $\{|\operatorname{Re}(z)| \leq 1/2\}$ with

$$\begin{aligned}\lambda f(it+1/2) &= \lambda \left\langle \Delta^{it}(I+\Sigma)x(I-\Sigma)\Delta^{-it}\zeta\,,\eta\right\rangle \\ \bar{\lambda} f(it-1/2) &= \bar{\lambda} \left\langle \Delta^{it}(I-\Sigma)x(I+\Sigma)\Delta^{-it}\zeta\,,\eta\right\rangle \\ f(0) &= \langle TxT\zeta\,,\eta\rangle\,,\end{aligned}$$

where $\Theta = JT = J(I+\Sigma)^{1/2}(I-\Sigma)^{1/2}$. Since J, $T = T^*$, and Δ^{it} mutually commute by Lemma 4.3 and Lemma 6.9, we have by (18), Lemma 7.9, and the identities above that

$$\langle xT\zeta\,,T\eta\rangle = \frac{1}{2}\int_{-\infty}^{\infty} e^{-\theta t}\cosh(\pi t)^{-1}\left\langle \Delta^{it}Jx'J\Delta^{-it}T\zeta\,,T\eta\right\rangle dt.$$

Since T is positive and injective by Lemma 4.2, we have that the range of T is dense in H (see Lemma 6.6 in chapter 1), hence the result holds. □

LEMMA 7.11. *For all $x' \in \mathcal{M}'$ and $t \in \mathbb{R}$, we have that $\Delta^{it}Jx'J\Delta^{-it} \in \mathcal{M}$.*

PROOF. We fix $\zeta, \eta \in H$, $x', y' \in \mathcal{M}'$, and define

$$g(t) := \left\langle [y', \Delta^{it}Jx'J\Delta^{-it}]\zeta\,,\eta\right\rangle,$$

where $[A, B] = AB - BA$ is the commutator. Note that $g(t)$ is uniformly bounded. Setting

$$h(z) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-zt}\cosh(\pi t)g(t)dt,$$

we see that h is analytic on $\{|\operatorname{Re}(z)| < \pi\}$. By Lemma 7.10, it follows that

$$h(\theta) = \left[y', \int_{-\infty}^{\infty} e^{-\theta t}\cosh(\pi t)^{-1}\left\langle \Delta^{it}Jx'J\Delta^{-it}\zeta\,,\eta\right\rangle dt\right] = 0$$

for all $\theta \in (-\pi, \pi)$ which implies that $h(z) = 0$ identically. However, for $s \in \mathbb{R}$ we have that $h(is)$ is the Fourier transform of $\cosh(\pi t)^{-1}g(t)$. Since the Fourier transform is injective, it follows that $g(t) = 0$ identically, establishing the result. □

We are now ready to prove the main result of the section.

PROOF OF PROPOSITION 7.1. Lemma 7.11 shows that for every $x' \in \mathcal{M}'$, we have that $\Delta^{-it}Jx'J\Delta^{-it} \in \mathcal{M}$. Setting $t = 0$, this shows that $J\mathcal{M}'J \subseteq \mathcal{M}$. Considering the triple $(\mathcal{M}', H, \xi)$, we have that by the beginning of section 4 that the Tomita and modular operators, respectively J' and Δ', for this triple satisfy $J' = J$ and $\Delta' = \Delta^{-1}$. It now follows by the same reasoning that $J\mathcal{M}J \subseteq \mathcal{M}'$, thus $J\mathcal{M}J = \mathcal{M}'$. Having established this, it is now straightforward to see from Lemma 7.11 that $\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}$ and $\Delta^{it}\mathcal{M}'\Delta^{-it} = \mathcal{M}'$ for all $t \in \mathbb{R}$. □

8. The Modular Automorphism Group: First Properties

As a ready consequence of Proposition 7.1 we have the following result.

PROPOSITION 8.1. *Let $\mathcal{M}$ be a von Neumann algebra, and let ϕ be a faithful, normal state. We have there is a (unique) strongly-continuous one-parameter group of automorphisms (σ_t^ϕ) of $\mathcal{M}$ which satisfies the KMS condition with respect to ϕ.*

REMARK 8.2. We refer to σ^ϕ as the *modular automorphism group* associated to ϕ.

Before giving the proof, we pause to present a key lemma.

LEMMA 8.3. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra. Given a faithful, normal state on $\mathcal{M}$, we define $\|x\|_\phi := \phi(x^*x)^{1/2}$. We have that $\|x-y\|_\phi$ metrizes the SOT on $(\mathcal{M})_1$.*

PROOF. Let $\pi_\phi : \mathcal{M} \to \mathcal{B}(H_\phi)$ be the GNS representation associated to $(\mathcal{M}, \phi)$ as described in 1 with cyclic vector $\xi_\phi \in H_\phi$ so that $\phi(x) = \langle \pi_\phi(x)\xi_\phi , \xi_\phi\rangle$. Since ϕ is faithful, by the results in that section π_ϕ is an isometric $*$-isomorphism and ξ_ϕ is separating. Further, by Proposition 1.18 and its proof we have that $\pi_\phi(\mathcal{M}) \subseteq \mathcal{B}(H_\phi)$ is a von Neumann algebra and π_ϕ is continuous from $(\mathcal{M})_1$ equipped with the WOT to $\pi_\phi((\mathcal{M})_1) = (\pi_\phi(\mathcal{M}))_1$ with the WOT. Since $(\mathcal{M})_1$ and $(\pi_\phi(\mathcal{M}))_1$ are both WOT-compact, π_ϕ is a continuous bijection between them, and the WOT is Hausdorff, we have that π_ϕ is a homeomorphism, that is $\pi_\phi^{-1} : (\pi_\phi(\mathcal{M}))_1 \to (\mathcal{M})_1$ is WOT–WOT continuous as well. Since $x_n \to x$ in the SOT if and only if $(x_n - x)^*(x_n - x) \to 0$ in the WOT topology, we conclude that a net (x_n) in $(\mathcal{M}_1)$ converges in the SOT if and only if $(\pi_\phi(x_n))$ converges in the SOT.

Now suppose that (x_n) is a net in $(\mathcal{M})_1$ so that $\|x_n - x\|_\phi \to 0$ for some $x \in (\mathcal{M})_1$. Since ξ_ϕ is separating for $\pi_\phi(\mathcal{M})$, it is cyclic for $\pi_\phi(\mathcal{M})'$ by Lemma 1.2, thus $\pi_\phi(\mathcal{M})'\xi_\phi$ is dense in H_ϕ. By abuse of notation we will identify x_n and $\pi_\phi(x_n)$. We note that

$$\|x_n - x\|_\phi^2 = \phi((x_n - x)^*(x_n - x)) = \langle (x_n - x)\xi_\phi , (x_n - x)\xi_\phi\rangle .$$

For $a \in \pi_\phi(\mathcal{M})'$ we have that

$$\begin{aligned}
\|(x_n - x)a\xi_\phi\|^2 &= \langle (x_n - x)a\xi_\phi , (x_n - x)a\xi_\phi\rangle \\
&= \langle a^*(x_n - x)^*(x_n - x)a\xi_\phi , \xi_\phi\rangle \\
&= \langle (x_n - x)^*a^*a(x_n - x)\xi_\phi , \xi_\phi\rangle \\
&\leq \|a\|^2 \langle (x_n - x)^*(x_n - x)\xi_\phi , \xi_\phi\rangle \\
&= \|a\|^2\|x_n - x\|_\phi^2 \to 0.
\end{aligned}$$

For $\epsilon > 0$ and $\eta \in H_\phi$, there is some $a \in \pi_\phi(\mathcal{M})'$ so that $\|\eta - a\xi_\phi\| < \epsilon$. We then have that

$$|\|(x_n - x)a\xi_\phi\| - \|(x_n - x)\eta\|| \leq \|x_n - x\|\|a\xi_\phi - \eta\| \leq 2\epsilon.$$

It follows that $\|(x_n - x)\eta\| \to 0$ for all $\eta \in H_\phi$, that is, $\pi_\phi(x_n) \to \pi_\phi(x)$ in the SOT. Hence $x_n \to x$ in the SOT. The converse is easily seen by the considerations above. ☐

PROOF OF PROPOSITION 8.1. Let $(\mathcal{M}, H_\phi, \xi_\phi)$ be the GNS construction associated to $(\mathcal{M}, \phi)$, and let Δ be the modular operator associated to this triple. By Proposition 7.1 we have that $\sigma_t^\phi(x) := \Delta^{it} x \Delta^{-it}$ is a $*$-automorphism of $\mathcal{M}$. Clearly $\mathbb{R} \ni t \mapsto \sigma_t^\phi \in \mathrm{Aut}(\mathcal{M})$ is a homomorphism, so in order to show that σ_t^ϕ is a strongly continuous one-parameter group, we need to check that $\sigma_t^\phi(x) \to x$ as $t \to 0$ in the SOT for all $x \in \mathcal{M}$. We compute

$$\begin{aligned}\|\sigma_t^\phi(x) - x\|_\phi^2 &= \left\langle (\sigma_t^\phi(x) - x)^*(\sigma_t^\phi(x) - x)\xi_\phi \,, \xi_\phi \right\rangle \\ &= \left\langle (\sigma_t^\phi(x^*x) - x^*\sigma_t^\phi(x) - \sigma_t^\phi(x^*)x - x^*x)\xi_\phi \,, \xi_\phi \right\rangle \\ &= \left\langle \Delta^{it} x^* x \Delta^{-it}\xi_\phi \,, \xi_\phi \right\rangle - 2\,\mathrm{Re}\left\langle x^*\Delta^{it} x \Delta^{-it}\xi_\phi \,, \xi_\phi \right\rangle + \langle x^* x\xi_\phi \,, \xi_\phi \rangle \\ &= 2\,\langle x\xi_\phi \,, x\xi_\phi \rangle - 2\,\mathrm{Re}\left\langle \Delta^{it} x\xi_\phi \,, x\xi_\phi \right\rangle\end{aligned}$$

as $\Delta^{it}\xi_\phi = \xi_\phi$. Since (Δ^{it}) is SOT-continuous, the claim holds by Lemma 8.3. $\square$

REMARK 8.4. Considering the GNS representation $(\mathcal{M}, H_\phi, \xi_\phi)$ associated to $(\mathcal{M}, \phi)$, we notice that $\|x\|_\phi^2 = \phi(x^*x) = \|x\xi_\phi\|^2$ while

$$\|x^*\|_\phi^2 = \phi(xx^*) = \langle x^*\xi_\phi \,, x^*\xi_\phi \rangle = \langle Sx\xi_\phi \,, Sx\xi_\phi \rangle = \left\langle J\Delta^{1/2} x\xi_\phi \,, J\Delta^{1/2} x\xi_\phi \right\rangle = \|\Delta^{1/2} x\xi_\phi\|^2.$$

This shows that Δ can be viewed as the "Radon–Nikodym derivative" for $\phi(xx^*)$ with respect to $\phi(x^*x)$. The following result uses the modular automorphism group to make this precise.

DEFINITION 8.5. Let ϕ be a faithful, normal state on $\mathcal{M}$. We define the *centralizer* of ϕ to be

$$\mathcal{M}^\phi = \{a \in \mathcal{M} : \phi(ax) = \phi(xa),\ \forall x \in \mathcal{M}\}.$$

We notice that $\mathcal{M}^\phi$ is a von Neumann subalgebra of $\mathcal{M}$ and that ϕ is a trace on $\mathcal{M}^\phi$.

LEMMA 8.6. *We have that $a \in \mathcal{M}^\phi$ if and only if $\sigma_t^\phi(a) = a$ for all $t \in \mathbb{R}$.*

PROOF. For $a \in \mathcal{M}^\phi$ we have that

$$\phi(ax) = \langle x\xi_\phi \,, a^*\xi_\phi \rangle = \langle x\xi_\phi \,, Sa\xi_\phi \rangle\,,$$

so $x\xi_\phi \mapsto \phi(ax) = \phi(xa)$ extends to a bounded linear function on H_ϕ. On the other hand, $\phi(xa) = \langle a\xi_\phi \,, Sx\xi_\phi \rangle$, so altogether $a\xi_\phi$ lies in the domain of $S^* = F$. Thus,

$$\langle x\xi_\phi \,, Sa\xi_\phi \rangle = \phi(ax) = \phi(xa) = \langle a\xi_\phi \,, Sx\xi_\phi \rangle = \langle x\xi_\phi \,, Fa\xi_\phi \rangle\,, \tag{19}$$

where in the last equality we have accounted for the fact that F is anti-linear. It follows that $Sa\xi_\phi = Fa\xi_\phi$. As F as range of F is the same as its domain and $F^2 \subseteq I$, we thus have that

$$\Delta a\xi_\phi = SFa\xi_\phi = F^2 a\xi_\phi = a\xi_\phi.$$

Therefore

$$\sigma_t^\phi(a)\xi_\phi = \Delta^{it} a\Delta^{-it}\xi_\phi = \Delta^{it} a\xi_\phi = a\xi_\phi,$$

and we can conclude that $\sigma_t^\phi(a) = a$ as ξ_ϕ is separating.

In the other direction, we have that $\sigma_t^\phi(a) = a$ for all $t \in \mathbb{R}$ implies, as above, that $\Delta^{it} a\xi_\phi = a\xi_\phi$ for all $t \in \mathbb{R}$. By Lemma 6.9 it follows that

$$(I - \Sigma)^{it} a\xi_\phi = (I + \Sigma)^{it} a\xi_\phi$$

for all $t \in \mathbb{R}$.

By Lemma 6.7 we have that for all $\eta \in H$ that

$$f(z) = \left\langle (I - \Sigma)^{iz} a\xi_\phi\,, \eta \right\rangle, \quad g(z) = \left\langle (I + \Sigma)^{iz} a\xi_\phi\,, \eta \right\rangle$$

are defined and continuous on $\{\mathrm{Im}(z) \leq 0\}$ and analytic on the interior. Since $f(t) = g(t)$, by the Schwarz Reflection Principle it must be the case that $f(z) = g(z)$ identically, from which we conclude using $z = -i$ that

$$(I - \Sigma) a\xi_\phi = a\xi_\phi = (I + \Sigma) a\xi_\phi,$$

which implies (for instance, by the powers series expansion of the square root)

$$(I - \Sigma)^{1/2} a\xi_\phi = a\xi_\phi = (I + \Sigma)^{1/2} a\xi_\phi.$$

Since $S = J(I + \Sigma)^{1/2}(I - \Sigma)^{-1/2}$ and $F = J(I - \Sigma)^{1/2}(I + \Sigma)^{-1/2}$ by Proposition 5.7, we have that $Sa\xi_\phi = Fa\xi_\phi$. Thus by the same reasoning as used in line (19), it follows that $\phi(ax) = \phi(xa)$ for all $x \in \mathcal{M}$. □

Remark 8.7. This lemma, though rather simple looking, has profound consequences. Indeed, let $\mathcal{N} = M_2(\mathcal{M})$. If ϕ_1 and ϕ_2 are faithful normal traces on $\mathcal{M}$, then

$$\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) := \phi_1(a) + \phi_2(d)$$

is a faithful, normal state on $\mathcal{N}$. Let $\{e_{ij}\}$ be the matrix units for M_2 viewed as elements of $\mathcal{N}$. It is straightforward to check that $e_{ii} \in \mathcal{N}^\phi$ for $i = 1, 2$, thus

$$\sigma_t^\phi(e_{ii} x e_{jj}) = e_{ii}\sigma_t^\phi(x) e_{jj}$$

for all $x \in \mathcal{N}$ and $i, j \in 1, 2$. In other words,

$$\sigma_t^\phi(x_{ij}) = [\sigma_t^\phi(x)]_{ij}$$

where x_{ij} denotes the operator $e_{ii} x e_{jj} \in \mathcal{N}$ for $x \in \mathcal{M}$. As $\phi_i(x) = \phi(x_{ii})$, we see that $\alpha_t^i(x) = \sigma_t^\phi(x_{ii})$ is a strongly continuous one-parameter group of automorphisms for $\mathcal{M}$ satisfying the KMS condition with respect to ϕ_i. By Corollary 6.17 it follows that $\sigma_t^\phi(x_{ii}) = \sigma_t^{\phi_i}(x)$ for all $x \in \mathcal{M}$.

Setting $u_t := \sigma_t^\phi(e_{12}) = [\sigma_t^\phi(e_{12})]_{12}$ as an element of $\mathcal{M}$ we have that

$$u_t^* u_t = \sigma_t^\phi(e_{21} e_{12}) = \sigma_t^\phi(e_{11}) = \sigma_t^\phi(I_{11})$$

and

$$u_t u_t^* = \sigma_t^\phi(e_{12} e_{21}) = \sigma_t^\phi(e_{22}) = \sigma_t^\phi(I_{22}).$$

Thus $u_t \in \mathcal{M}$ is a unitary. Moreover,

$$u_t \sigma_t^{\phi_2}(x) u_t^* = \sigma_t^\phi(e_{12} x_{22} e_{21}) = \sigma_t^\phi(x_{11}) = \sigma_t^{\phi_1}(x)$$

which shows that $\sigma_t^{\phi_1}$ and $\sigma_t^{\phi_2}$ are related by an inner automorphism of $\mathcal{M}$.

Further, we see that

$$[u_{t+s}]_{12} = \sigma_t^{\phi}([u_s]_{12}) = \sigma_t^{\phi}(e_{12}[u_s]_{22}) = [u_t\sigma^{\phi_2}(u_s)]_{12},$$

so we have that (u_t) satisfies the cocycle relation $u_{t+s} = u_t\sigma^{\phi_2}(u_s)$ for all $t, s \in \mathbb{R}$.

We denote by $\mathrm{Inn}(\mathcal{M})$ the normal subgroup of $\mathrm{Aut}(\mathcal{M})$ consisting of all inner automorphisms, that is, all automorphisms of the form $x \mapsto uxu^*$ for some $u \in \mathcal{M}$ a unitary. Thus we have shown the following.

LEMMA 8.8. *The homomorphism* $\mathbb{R} \ni t \mapsto [\sigma_t^{\phi}] \in \mathrm{Aut}(\mathcal{M})/\mathrm{Inn}(\mathcal{M})$ *is independent of the choice of faithful, normal state* ϕ.

COROLLARY 8.9. *The set* $T(\mathcal{M}) := \{t \in \mathbb{R} : \sigma_t^{\phi} \in \mathrm{Inn}(\mathcal{M})\}$ *is independent of the choice of faithful, normal state. Thus* $T(\mathcal{M})$ *is a* $*$*-isomorphism invariant of* $\mathcal{M}$.

EXAMPLE 8.10. It follows that if $\mathcal{M}$ admits a faithful, normal tracial state τ, then $T(\mathcal{M}) = \mathbb{R}$ as $\mathcal{M}^{\tau} = \mathcal{M}$.

EXAMPLE 8.11. We have that $T(\mathcal{B}(H)) = \mathbb{R}$ for H separable. Indeed, by Proposition 6.16 of chapter 1 a normal linear functional ϕ on $\mathcal{B}(H)$ is of the form $\phi(x) = \sum_{i=1}^{\infty} \langle T\xi_i\,, \eta_i\rangle$ with $\sum_{i=1}^{\infty} \|\xi_i\|^2$, $\sum_{i=1}^{\infty} \|\eta_i\|^2 < \infty$. These inequalities show that $\rho := \sum_{i=1}^{\infty} \xi_i \otimes \bar{\eta}_i$ is a bounded operator on H, and by Remark 3.24 of chapter 1 we may compute that $\phi(x) = \mathrm{Tr}(x\rho) = \mathrm{Tr}(\rho x)$ for all $x \in \mathcal{B}(H)$. Indeed, if $x, y \in \mathcal{B}(H)^h$, we have that

$$\mathrm{Tr}(x\rho y) = \sum_i \langle x\xi_i\,, y\eta_i\rangle = \sum_i \langle yx\xi_i\,, x\eta_i\rangle = \mathrm{Tr}(yx\rho), \tag{20}$$

and the result holds since the relation is clearly linear in x and y. Since $\langle \rho\zeta\,, \zeta\rangle = \mathrm{Tr}(\rho(\zeta \otimes \bar{\zeta})) = \phi(\zeta \otimes \bar{\zeta}) > 0$ if ϕ is a faithful state, we have that ρ is positive and injective. (It also follows that H must necessarily be separable.)

By Lemma 6.7 we have that ρ^{it} is a unitary in $\mathcal{B}(H)$ and ρ^{iz} is analytic in z when $\mathrm{Im}(z) < 0$. Using line (20) we may verify directly that $\alpha_t : x \mapsto \rho^{it}x\rho^{-it}$ satisfies the KMS condition with respect to ϕ. Indeed, for $x, y \in \mathcal{B}(H)$ and $z \in \mathbb{C}$ with $-1 \leq \mathrm{Im}(z) \leq 0$, we let

$$f(z) = \mathrm{Tr}(\rho^{iz+1}x\rho^{-iz}y)$$

which is continuous on $\{-1 \leq \mathrm{Im}(z) \leq 0\}$ and analytic on the interior. We see that for $t \in \mathbb{R}$,

$$f(t) = \mathrm{Tr}(\rho\rho^{it}x\rho^{-it}y) = \mathrm{Tr}(\alpha_t(x)y\rho) = \phi(\alpha_t(x)y)$$

while

$$f(t-i) = \mathrm{Tr}(\rho^{it}x\rho^{-it+1}y) = \mathrm{Tr}(\rho^{it}x\rho^{-it}\rho y) = \mathrm{Tr}(y\alpha_t(x)\rho) = \phi(y\alpha_t(x)).$$

In the next section we will calculate some non-trivial examples of the $T(\mathcal{M})$.

9. Powers Factors

Definition 9.1. By a *pointed Hilbert space* (H, ξ) we will mean a complex Hilbert space H with a distinguished unit vector ξ.

For a sequence (H_n, ξ_n) of pointed Hilbert spaces, we can define the tensor product $\bigotimes_{n\in\mathbb{N}}(H_n, \xi_n)$ as follows. Notice that $(\bigotimes_{n=1}^K H_n, \xi_1 \otimes \cdots \otimes \xi_K)$ is a pointed Hilbert space. For each K there is an isometric embedding

$$\iota_K : \bigotimes_{n=1}^{K} H_n \hookrightarrow \bigotimes_{n=1}^{K+1} H_n \tag{21}$$

given by

$$\iota_K : \eta_1 \otimes \cdots \otimes \eta_K \mapsto \eta_1 \otimes \cdots \otimes \eta_K \otimes \xi_{K+1}.$$

These connecting maps define an inductive limit of pointed Hilbert spaces, which can be see to have the structure of a pointed inner product space, the completion of which is a pointed Hilbert space. We denote this pointed Hilbert space by $\bigotimes_{n\in\mathbb{N}}(H_n, \xi_n)$.

Remark 9.2. Given a sequence of pairs $(\mathcal{M}_i, \phi_i)$ of von Neumann algebras equipped with faithful, normal states, we have that $\mathcal{M} := \bigotimes_{n=1}^K \mathcal{M}_i$ is a von Neumann algebra and $\phi := \phi_1 \otimes \cdots \otimes \phi_K$ is a faithful, normal state on $\mathcal{M}$. To see this, let $(\mathcal{M}_i, H_i, \xi_i)$ be the GNS construction associated to (M_i, ϕ_i), and set $\mathcal{M}_i'$ be the commutant of $\mathcal{M}_i$ in $\mathcal{B}(H_i)$. Since each ϕ_i is faithful, we have that ξ_i is cyclic for both M_i and $\mathcal{M}_i'$, thus $\xi := \xi_1 \otimes \cdots \xi_K$ is cyclic for both $\mathcal{M}$ and $\mathcal{M}' \supseteq \bigotimes_{i=1}^K \mathcal{M}_i'$ acting on $\bigotimes_{n=1}^K H_i$. It follows by Lemma 1.2 that ξ is separating for $\mathcal{M}$ and $\langle x\xi\,, \xi\rangle = \phi(x)$ for all $x \in \mathcal{M}$.

Now, let $(H, \xi) = \bigotimes_{n\in\mathbb{N}}(H_i, \xi_i)$ be the tensor product of pointed Hilbert spaces and suppose that $\mathcal{M}_i \subseteq \mathcal{B}(H_i)$ is von Neumann algebra with ξ_i a cyclic and separating vector for $\mathcal{M}_i$. For $x = x_1 \otimes \cdots \otimes x_n \in \bigotimes_{i=1}^n \mathcal{B}(H_i)$, we have that x acts as bounded operator on $\bigotimes_{i=1}^K H_i$ for each $n \le K$ by

$$x(\eta_1 \otimes \cdots \otimes \eta_K) = (x_1\eta_1) \otimes \cdots \otimes (x_n\eta_n) \otimes \eta_{n+1} \otimes \cdots \otimes \eta_K.$$

It is clear that these actions of x commute with the connecting maps, so induce an isometric action of x on $\mathcal{B}(H)$. We note that $\langle x\xi\,, \xi\rangle = \prod_{i=1}^n \phi_i(x_i)$.

We denote by $\mathcal{M}$ the von Neumann algebra in $\mathcal{B}(H)$ generated by the operators $\bigcup_{n\in\mathbb{N}} \mathcal{M}_1 \otimes \cdots \otimes \mathcal{M}_n$. There is a normal state on $\mathcal{M}$ given by $\phi(x) = \langle x\xi\,, \xi\rangle$. We write

$$(\mathcal{M}, \phi) := \overline{\bigotimes}_{i\in\mathbb{N}}(\mathcal{M}_i, \phi_i).$$

We note that ξ is a separating vector for $\mathcal{M}$ as ξ is a cyclic vector vector for $\mathcal{M}'$ as $\bigotimes_{i=1}^n \mathcal{M}_i' \subseteq \mathcal{M}'$ and $\bigcup_{n\in\mathbb{N}}(\mathcal{M}_1' \otimes \cdots \otimes \mathcal{M}_n')\xi$ is dense in H. It is clear by the same reasoning that ξ is a cyclic vector for $\mathcal{M}$.

Definition 9.3. We denote by $\operatorname{Aut}(\mathcal{M}, \phi)$ the set of all $*$-automorphisms α so that $\phi \circ \alpha = \phi$.

LEMMA 9.4. *For any sequence $\alpha_i \in \mathrm{Aut}(\mathcal{M}_i, \phi_i)$, we have a unique induced $*$-automorphism $\alpha \in \mathrm{Aut}\left(\bigotimes_{i\in\mathbb{N}}(\mathcal{M}_i,\phi_i)\right)$ so that $\alpha(x_1\otimes\cdots\otimes x_n) = \alpha_1(x_1)\otimes\cdots\otimes\alpha_n(x_n)$ for all $x_i \in \mathcal{M}_i$.*

PROOF. Let $(\mathcal{M}_i, H_i, \xi_i)$ be the GNS construction associated to $(\mathcal{M}_i, \phi_i)$. For each $\alpha_i \in \mathrm{Aut}(\mathcal{M}_i, \phi_i)$, we construct a unitary operator $U_i \in \mathcal{B}(H_i)$ so that $U_i x U_i^* = \alpha_i(x)$ for all $x \in \mathcal{M}_i$. The operator is defined on $\mathcal{M}_i\xi_i$ by

$$U_i x\xi_i = \alpha_i(x)\xi_i.$$

Since α_i is an automorphism, it is clear that U_i is linear and well-defined. Further, we have that

$$\|U_i x\xi_i\|^2 = \langle \alpha_i(x)^*\alpha_i(x)\xi_i\,, \xi_i\rangle = \langle \alpha_i(x^*x)\xi_i\,, \xi_i\rangle = \phi_i(\alpha_i(x^*x)) = \phi(x^*x) = \|x\xi_i\|^2,$$

so that U_i is an isometry. As $\mathcal{M}_i\xi_i$ is dense in H_i, we have that U_i extends uniquely to a bijective isometry on H_i. We now verify for all $x, y \in \mathcal{M}_i$

$$U_i x U_i^* y\xi_i = U_i x\alpha_i^{-1}(y)\xi_i = \alpha_i(x\alpha_i^{-1}(y))\xi_i = \alpha_i(x)y\xi_i,$$

which shows again by density that $U_i x U_i = \alpha_i(x)$ for all $x \in \mathcal{M}_i$. We note that $U_i\xi_i = \xi_i$.

We now see that each $U_1\otimes\cdots\otimes U_n$ is a unitary operator on $H_1\otimes\cdots\otimes H_n$ which fixes $\xi_1\otimes\cdots\otimes\xi_n$, so these unitaries commute with the connecting maps as in line (21). Thus the sequence induces a unitary U on $(H,\xi) = \bigotimes_{i\in\mathbb{N}}(H_i,\xi_i)$ so that $U\xi = \xi$. It is clear that

$$U(x_1\otimes\cdots\otimes x_n)U^* = \alpha_1(x_1)\otimes\cdots\otimes\alpha_n(x_n)$$

for all $x_i \in \mathcal{M}_i$. Since $x \mapsto UxU^*$ is SOT continuous and

$$U(\mathcal{M}_1\otimes\cdots\otimes\mathcal{M}_n)U^* = \mathcal{M}_1\otimes\cdots\otimes\mathcal{M}_n \subseteq \mathcal{M} = \overline{\bigotimes}_{i\in\mathbb{N}}(\mathcal{M}_i,\phi_i),$$

we have that $U\mathcal{M}U^* = \mathcal{M}$. $\square$

LEMMA 9.5. *For $(\mathcal{M},\phi) = \overline{\bigotimes}_{i\in\mathbb{N}}(\mathcal{M}_i,\phi_i)$, we have that the modular automorphism group σ_t^ϕ of $(\mathcal{M},\phi)$ is given by*

$$\sigma_t^\phi = \bigotimes_{i\in\mathbb{N}}\sigma_t^{\phi_i}.$$

PROOF. By the previous lemma, we have that $\alpha_t := \bigotimes_{i\in\mathbb{N}}\sigma_t^{\phi_i}$ belongs to $\mathrm{Aut}(\mathcal{M},\phi)$. It is clear that $\alpha_t\circ\alpha_s = \alpha_{t+s}$, so we must show that (α_t) is a strongly-continuous one-parameter group and that is satisfies the KMS condition with respect to $(\mathcal{M},\phi)$.

Let $\mathcal{A}$ be the C*-algebra generated by $\bigcup_{n\in\mathbb{N}}\mathcal{M}_1\otimes\cdots\otimes\mathcal{M}_n$. By Kaplansky's density theorem, we have that $(\mathcal{A})_1$ is SOT-dense in $(\mathcal{M})_1$. Since $\alpha_t(\mathcal{A}) = \mathcal{A}$ and

$$\|\alpha_t(x)-\alpha_t(y)\|_\phi = \|x-y\|_\phi$$

as $\phi\circ\alpha_t = \phi$, we have that it suffices to show that $\|\alpha_t(x) - x\|_\phi \to 0$ for all $x \in \mathcal{A}$. (Recall that by Lemma 8.3, $\|x-y\|_\phi$ metrizes the SOT on $(\mathcal{M})_1$.) In fact, since

$$\|x\|_\phi^2 = \phi(x^*x) \le \|x\|^2\phi(I) = \|x\|^2$$

it suffices to check this on the $*$-subalgebra $\bigcup_{n\in\mathbb{N}} \mathcal{M}_1\otimes\cdots\otimes\mathcal{M}_n$. For $x \in \mathcal{M}_1\otimes\cdots\otimes\mathcal{M}_n$, we have, using the same notations as in the proof of Lemma 9.4 that

$$\begin{aligned}
\|\alpha_t(x) - x\|_\phi^2 &= \langle (\alpha_t(x) - x)^*(\alpha_t(x) - x)\xi\,,\xi\rangle \\
&= \langle \alpha_t(x^*x)\xi\,,\xi\rangle - 2\,\mathrm{Re}\,\langle \alpha_t(x^*)x\xi\,,\xi\rangle + \langle x^*x\xi\,,\xi\rangle \\
&= 2\,\phi(x^*x) - 2\,\mathrm{Re}\,\langle U_t x^* U_t^* x\xi\,,\xi\rangle \\
&= 2\,\phi(x^*x) - 2\,\mathrm{Re}\,\langle U_t^* x\xi\,, x\xi\rangle \\
&= 2\,\phi(x^*x) - 2\,\mathrm{Re}\,\langle (U_{1,t}\otimes\cdots\otimes U_{n,t})x(\xi_1\otimes\cdots\otimes\xi_n)\,, x(\xi_1\otimes\cdots\otimes\xi_n)\rangle
\end{aligned}$$

where we have used $U_t^*\xi = \xi$. The last expression clearly converges to 0 as $U_{1,t}\otimes\cdots\otimes U_{n,t}$ is SOT continuous on $\bigotimes_{i=1}^n H_i$.

Now, if $(\alpha_{i,t})$ is a strongly-continuous, one-parameter group satisfying the KMS condition for $(\mathcal{M}_i, \phi_i)$ with $\alpha_{i,t}(x) = U_{i,t} x U_{i,t}^*$ for $(U_{i,t})$ the corresponding SOT-continuous, one-parameter group of unitaries for the GNS space (H_i, ξ_i) $\square$

Remark 9.6. Although naively the tensor product $(\mathcal{M}, \phi) = \overline{\bigotimes}_{i\in\mathbb{N}}(\mathcal{M}_i, \phi_i)$ seems like it ought to behave similalry to the direct product $\prod_{i\in\mathbb{N}} \mathcal{M}_i$ acting on $\bigoplus H_i$, this is very far from the truth. Indeed a bounded sequence $x_i \in \mathcal{M}_i$ defines an element $x_1 \otimes x_2 \otimes \cdots$ in $\mathcal{M}$ if and only if the sequence $y_n := x_1 \otimes \cdots \otimes x_n$ is Cauchy in the $\|\cdot\|_\phi$-norm, that is, if and only if

$$\prod_{i=1}^n \phi_i(x_i^* x_i)\left[1 - 2\,\mathrm{Re}\prod_{i=n+1}^{n+k}\phi_i(x_i) + \prod_{i=n+1}^{n+k}\phi_i(x_i^* x_i)\right] \to 0$$

as $n, k \to \infty$. For this reason an infinite tensor product of inner automorphism need not be inner. In fact, the next result gives a necessary and sufficient condition for this to occur.

Proposition 9.7. *Let $(\mathcal{M}_i, \phi_i)$ be a sequence of von Neumann algebras which are factors equipped with faithful, normal states. Suppose $u_i \in \mathcal{M}_i^{\phi_i}$ are unitaries so that $\alpha_i(x) := u_i x u_i^* \in \mathrm{Aut}(\mathcal{M}_i, \phi_i)$ for each $i \in \mathbb{N}$. We have that $\alpha = \bigotimes \alpha_i$ is an inner automorphism of $\overline{\bigotimes}_{i\in\mathbb{N}}(\mathcal{M}_i, \phi_i)$ if and only if*

$$\sum_{i=1}^\infty 1 - |\phi_i(u_i)| < \infty.$$

Proof. By multiplying by a phase we may assume without loss of generality that $\phi_i(u_i) > 0$. Let

$$v_k := u_1 \otimes \cdots \otimes u_k \otimes 1 \otimes 1 \otimes \cdots \in \overline{\bigotimes}_{i\in\mathbb{N}}(\mathcal{M}_i, \phi_i) =: (\mathcal{M}, \phi).$$

We will first show under the convergence assumption that (v_k) converges in the S*OT. By Lemma 8.3 it suffices to check that $\|v_k - v_l\|_\phi$ and $\|v_k^* - v_l^*\|_\phi$ both tend to 0 as $k, l \to \infty$. Since the unitary group of a von Neumann algebra is closed in the S*OT (exercise), we have that $v_k \to v$ for some unitary $v \in \mathcal{M}$. Computations similar to those in Lemma 9.5 then show that $\alpha(x) = vxv^*$.

We now compute. Assuming $l > k$ we have that

$$\begin{aligned}\|v_k - v_l\|_\phi^2 &= \phi((v_k - v_l)^*(v_k - v_l))\\ &= 2 - \phi(v_k^* v_l) - \phi(v_l^* v_k)\\ &= 2 - 2\operatorname{Re}\phi(v_k^* v_l) = 2\left(1 - \prod_{i=k+1}^{l} \phi_i(u_i)\right)\end{aligned}$$

Now we have $0 \le a_1, \ldots, a_N \le 1$, then using a telescoping sum we have that

$$1 - \prod_{i=1}^{N} a_i = (1 - a_1) + a_1(1 - a_2) + a_1 a_2 (1 - a_3) + \cdots \le \sum_{i=1}^{N} (1 - a_i).$$

Hence $\|v_k - v_l\|_\phi^2 \le 2\left(\sum_{i=k+1}^{l} 1 - \phi_i(u_i)\right) \to 0$ as $k, l \to \infty$. A similar estimate works for $\|v_k^* - v_l^*\|_\phi$.

Next, suppose that α is inner, so there is $v \in \mathcal{M}$ so that $\alpha(x) = vxv^*$. For $N \in \mathbb{N}$, let $\mathcal{M}_N := \overline{\bigotimes}_{i=1}^{N} (\mathcal{M}_i, \phi_i)$ and $\mathcal{M}^N = 1 \otimes \cdots \otimes 1 \otimes \overline{\bigotimes}_{i=N+1}^{\infty} (\mathcal{M}_i, \phi_i)$. Since each $\mathcal{M}_i$ is a factor, we have that $\mathcal{M}_N' \cap \mathcal{M} = \mathcal{M}^N$. Using the same notation as before, we have that $\alpha(x) = v_N x v_N^*$ for all $x \in \mathcal{M}_N$. Setting $w_N := v^* v_N$, it follows that $w_N x w_N^* = x$ for all $x \in \mathcal{M}_N$, thus $w_N \in \mathcal{M}^N$.

However, since $v \in \mathcal{M}$ there exists $L \in \mathbb{N}$ and $x \in \mathcal{M}_L$ so that $\phi(x^* v) > 0$. Without loss of generality we can take $x = x_1 \otimes \cdots \otimes x_L$ so that for all $N > L$

$$\kappa := \phi(x^* v) = \left[\prod_{i=1}^{L} \phi_i(x_i^* v)\right] \cdot \left[\prod_{j=L+1}^{N} \phi_i(u_i)\right] \cdot \phi(w_N) > 0.$$

Since $|\phi(w_N)| \le 1$, we have that

$$1 \ge \prod_{j=L+1}^{N} \phi_i(u_i) \ge \frac{\kappa}{\left|\prod_{i=1}^{L} \phi_i(x_i^* v)\right|} =: C > 0$$

for all $N > L$. Thus we have that

$$\sum_{i=L+1}^{N} -\log \phi_i(u_i) \le -\log C < \infty$$

for all $N > L$. Since $-\log \alpha > 1 - \alpha$ for all $0 < \alpha < 1$, it follows that $\sum_{i=1}^{\infty} 1 - \phi_i(u_i)$ converges. $\square$

Definition 9.8. For $\lambda \in (1/2, 1)$, let $\rho_\lambda := \begin{pmatrix} \lambda & 0 \\ 0 & 1 - \lambda \end{pmatrix}$ and $\phi_\lambda(x) := \operatorname{tr}(x \rho_\lambda)$ for all $x \in M_2(\mathbb{C})$, noting that ϕ_λ is faithful. We define the *Powers Factor* $(\mathcal{R}_\lambda, \tau_\lambda)$ as

$$(\mathcal{R}_\lambda, \tau_\lambda) := \overline{\bigotimes}_{\mathbb{N}} (M_2(\mathbb{C}), \phi_\lambda).$$

PROPOSITION 9.9. *We have that* $T(\mathcal{R}_\lambda) = \frac{2\pi}{\log(\frac{\lambda}{1-\lambda})}\mathbb{Z}$.

PROOF. By Lemma 9.5 we have that

$$\sigma_t^{\tau_\lambda}(x_1 \otimes x_2 \otimes \cdots) = (\sigma_t^{\phi_\lambda}(x_1) \otimes \sigma_t^{\phi_\lambda}(x_2) \otimes \cdots) = (\rho_\lambda^{it} x_1 \rho_\lambda^{-it} \otimes \rho_\lambda^{it} x_2 \rho_\lambda^{-it} \otimes \cdots).$$

Thus by Proposition 9.7 we have that $\sigma_t^{\tau_\lambda}$ is inner if and only if

$$\sum_{k=1}^{\infty} 1 - |\operatorname{tr}(\rho_\lambda^{it}\rho_\lambda)| \leq \infty,$$

that is, if and only if

$$|\operatorname{tr}(\rho_\lambda^{1+it})| = |\lambda^{1+it} + (1-\lambda)^{1+it}| = 1.$$

Since $|\lambda^{1+it} + (1-\lambda)^{1+it}| \leq \lambda + (1-\lambda)1$, this occurs if and only if λ^{it} and $(1-\lambda)^{it}$ have the same phase, that is, if and only if $(\frac{\lambda}{1-\lambda})^{it} = 1$. This is true exactly when $t\log(\frac{\lambda}{1-\lambda}) \in 2\pi\mathbb{Z}$. □

As a consequence of Corollary 8.9 we have the following

COROLLARY 9.10. $\mathcal{R}_\lambda$ *is not* $*$*-isomorphic to* $\mathcal{R}_\mu$ *for* $\lambda \neq \mu$. *Hence, there is a continuum of non-isomorphic factors.*

10. Dynamical Systems and Krieger's Asymptotic Ratio Set (to do)

IN PROGRESS

11. Connes' Classification of Type III Factors (to do)

IN PROGRESS

CHAPTER 3

Algebraic Quantum Field Theory (to do)

Bibliography

The following works were used as resources in compiling these notes.

[1] Alain Connes, *Une classification des facteurs de type* III, Ann. Sci. École Norm. Sup. (4) **6** (1973), 133–252 (French). MR0341115

[2] John B. Conway, *A course in functional analysis*, 2nd ed., Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1990. MR1070713

[3] Uffe Haagerup, *Tomita–Takesaki Theory for Pedestrians*, lecture notes (1988).

[4] Marc A. Rieffel and Alfons Van Daele, *The commutation theorem for tensor products of von Neumann algebras*, Bull. London Math. Soc. **7** (1975), no. 3, 257–260, DOI 10.1112/blms/7.3.257. MR0383096

[5] Marc A. Rieffel and Alfons van Daele, *A bounded operator approach to Tomita-Takesaki theory*, Pacific J. Math. **69** (1977), no. 1, 187–221. MR0438147

[6] Walter Rudin, *Functional analysis*, McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973. MR0365062

[7] Masamichi Takesaki, *Theory of operator algebras. I*, Springer-Verlag, New York-Heidelberg, 1979. MR0548728

[8] M. Takesaki, *Theory of operator algebras. II*, Encyclopaedia of Mathematical Sciences, vol. 125, Springer-Verlag, Berlin, 2003. Operator Algebras and Non-commutative Geometry, 6. MR1943006