REMARK ON HARMONIC FUNCTIONS OF ALMOST SUBLINEAR GROWTH

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Abstract.

Let G be a finitely generated group, and let $S \subset G$ be a finite, unital, symmetric generating set. Given a measure $\mu \in \mathrm{Prob}(S)$ with full support and $\mu(s) = \mu(s^{-1})$, we consider the random walk on G generated by μ. The triple (G, S, μ) are fixed throughout. We recall the "long-range estimate" of Varopoulos which states that

$$\mu^{\star n}(g) \leq C\exp(-\alpha|g|^2/n) \tag{0.1}$$

for all n for some constants $C, \alpha > 0$. Here $|g|$ denotes the word length of g with respect to the generating set S. Really this is an application of the spectral theorem; see [LP, Theorems 13.4 and 13.2] for more on this.

The goal of this brief note is to show the following weakening of a result of Hebisch+Saloff-Coste [HSC]:

Theorem 0.1. *If G is of polynomial growth, then every $u : G \to \mathbb{R}$ which is μ-harmonic, large-scale lipschitz, and* **almost** *sublinear is constant.*

Recall from [CS] that a function $u : G \to \mathbb{R}$ is said to be *almost sublinear* if for every $\epsilon > 0$ the set of $g \in G$ such that $|u(g)| \geq \epsilon|g|$ has measure zero for any finitely additive, (left) G-invariant probability measure on G.

Definition 0.2. The random walk generated by μ is said to be *subgaussian* if there exists $\beta > 0$ so that

$$\limsup_{n\to\infty} \int \exp(\beta|g|^2/n)\,d\mu^{\star n}(g) < \infty. \tag{0.2}$$

It follows from the main result of [HSC] that if G has (strong) polynomial growth, then the random walk generated by μ is subgaussian. It is unclear to the author whether this is the case under the (a priori) weaker assumption that G is of weak polynomial growth, though it seems more than likely there is some connection between a random walk being subgaussian and it having slow entropy growth in the sense of [Oz]. In particular, this partially answers Question 2.14 in [CS] for the case of polynomial growth.

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Question 0.3. Are either of the properties of a random walk being subgaussian or having slow entropy growth implied by the other?

One could further conjecture that admitting a subgaussian random walk is equivalent to having (weak) polynomial growth.

Theorem 0.4. *If μ is a subgaussian random walk on G having slow entropy growth and $u : G \to \mathbb{R}$ is large-scale lipschitz, almost sublinear, and μ-harmonic, then μ is constant.*

Proof. We wil use a clever estimate of Ozawa [Oz, Section 4] which in turn draws inspiration from the analysis in [EK]. For ease of notation, let

$$H(n) := -\int \mu^{\star n}(g)\log(\mu^{\star n}(g))dg. \tag{0.3}$$

By the proof of the main result of [Oz], we have that for any $g \in G$, $s \in S$.

$$|u(gs) - u(g)|^2 \leq C_s(H(n+1) - H(n)) \int u(x)^2(\mu^{\star n}(gsx) + \mu^{\star n}(gx))dx \tag{0.4}$$

where C_s is a constant only depending on $s \in S$. Setting $\nu_n(x) = \mu^{\star n}(gsx) + \mu^{\star n}(gx)$, we have by Cauchy-Schwarz inequality and the subgaussian property of μ that

$$\int u(x)^2 d\nu_n(x) \leq C\left(\int u(x)^4 e^{-\beta|x|^2/n} d\nu_n(x)\right)^{1/2} \tag{0.5}$$

for some constant $C > 0$. Since $u(x)$ is large-scale lipschitz, it has at most linear growth, thus optimizing the right hand side of the equation, we see that u having almost sublinear growth implies that $\int u(x)^2 d\nu_n(x)$ grows sublinearly in n; hence, $|u(gs) - u(g)| = 0$.

□

References

[CS] I. Chifan and T. Sinclair; On the ergodic theorem for affine actions on Hilbert space, *Bull. Belg. Math. Soc. Simon Stevin* **22** (2015) 429-446.

[EK] A. Erschler and A. Karlsson; Homomorphisms to $\mathbb{R}$ constructed from random walks, *Ann. Inst. Fourier (Grenoble)* **60** (2010) 2095-2113.

[Gr] M. Gromov; Groups of polynomial growth and expanding maps, *Inst. Hautes Études Sci. Publ. Math.* **53** (1981) 53-73.

[HSC] W. Hebisch and L. Saloff-Coste; Gaussian estimates for Markov chains and random walks on groups, *Ann. Prob.* **21** (1993) 673-709.

[LP] R. Lyons and Y. Peres; *Probability on Trees and Networks*, Cambridge UP, to appear. Available at `http://pages.iu.edu/~rdlyons/prbtree/prbtree.html`.

[Oz] N. Ozawa; A functional analysis proof of Gromov's polynomial growth theorem, preprint, Oct. 2015.

[Sh] Y. Shalom; Harmonic anaylsis, cohomology, and the large-scale geometry of amenable groups, *Acta Math.* **192** (2004) 119-185.

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