# A SHORT PROOF OF THE EXISTENCE OF THE INJECTIVE ENVELOPE OF AN OPERATOR SPACE

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Abstract. We use Ellis' lemma to give a simple proof of the existence of the injective envelope of an operator space first shown by work of Hamana and Ruan.

Ramsey-theoretic methods provide powerful tools in functional analysis, ergodic theory, and additive combinatorics: see [1, 9, 18] for many of these applications. The goal of this note is to give a short, elementary proof of the existence of the injective envelope of an operator space via Ellis' lemma on right topological semigroups. The theory of injective envelopes for the categories of C*-algebras and operator systems and their associated dynamical systems is developed in the seminal works of Hamana [14–17], while Ruan first showed the existence of injective envelopes for operator spaces [26]. We mention that the treatment of injective envelopes as in [23–25] is very close in spirit with our proof. In fact, in [24] the injective envelope of certain crossed products of a countable, discrete $G$ is related with idempotents in the Stone–Cech compactification $\beta G$, but it seems a formal connection with Ellis' lemma in the general context was never made.

Let $S$ be a nonempty Hausdorff topological space equipped with a semigroup operation so that $S \ni s \mapsto st$ is continuous for each $t \in S$ separately. We will say that $S$ is a *right topological semigroup*. Let $\mathcal{I}(S)$ be the (possibly empty) set of idempotent elements of $S$ which we equip with the natural partial order $e \preceq f$ if $ef = fe = e$. A *minimal idempotent* is an idempotent which is a minimal element of the poset $(\mathcal{I}(S), \preceq)$. Additionally we say that two idempotents $e, f \in \mathcal{I}(S)$ are *similar*, $e \sim f$, if $e = fe$ and $f = ef$. We note that similarity is an equivalence relation on $\mathcal{I}(S)$.

We now state Ellis' lemma [10], a fundamental and far-reaching result on the structure of compact right topological semigroups.

**Lemma 1.** *Every compact right topological semigroup contains an idempotent.*

In addition to Ellis' paper cited above, proofs can be found, for instance, as [9, Theorem 1.23], [11, Theorem 1.1], or [18, Theorem 2.5].

The following consequence of Ellis' lemma essentially appears, for instance, as Theorems 1.2-1.4 in [11] or as a combination of Theorem 1.60, Theorem 2.5, and Corollary 2.6 in [18].

**Lemma 2.** *If $S$ is a compact right topological semigroup, then for every idempotent $e$ there is a minimal idempotent $f \preceq e$.*

2010 *Mathematics Subject Classification.* 46L07, 46L55, 22A20.

*Key words and phrases.* topological semigroup, operator space, operator system, injective envelope.

We offer a streamlined proof for the convenience of the reader.

*Proof.* Let $e \in S$ be an idempotent, and consider the left ideal $Se$, which is closed by continuity of right multiplication. By compactness of $Se$ and Zorn's lemma, there is $J \subseteq Se$ a minimal closed left ideal. Noting that $J$ itself is a compact right topological semigroup, $J$ contains an idempotent $f$.

By minimality $J = Sx$ for all $x \in J$, thus $g = yh = yh^2 = gh$ for all $g, h \in J$ idempotent. If $g \in \mathcal{I}(S)$ and $g \preceq f$, then $g = gf$ implies $g \in J$. It follows that $g = fg = f$, so $f$ is minimal. Since $f \in Se$, $f = fe$, and we have $(ef)^2 = efef = ef^2 = ef$ is an idempotent in $J$, hence minimal. As $ef = eef = efe$, $ef \preceq e$. $\square$

**Remark 3.** If $S$ is a semigroup, $e, f \in \mathcal{I}(S)$, and $fe = e$, then $ef \preceq f$. Consequently, if $f$ is minimal, then $fe = e$ implies that $e \sim f$.

Let $S$ be a convex subset of a locally convex topological vector space $V$ which is equipped with a semigroup structure that is affine and continuous with respect to right multiplication. Following [4], we will refer to $S$ as a *affine right topological semigroup.*

The following result and its proof were communicated to the author by Matthew Kennedy. The result appears in essentially this form as [4, Theorem II.4.3].

**Lemma 4.** *Let $S$ be a compact affine right topological semigroup, and let $J$ be a minimal closed left ideal. We have that $J$ is a left zero semigroup, that is, $xy = x$ for all $x, y \in J$. In particular, every element of a minimal closed left ideal of $S$ is idempotent.*

*Proof.* We have that $J = Sy$ for any $y \in J$, thus $J$ is compact and convex. We have that $x \mapsto xy$ is a continuous affine map from $J$ to itself, hence by the Markov–Kakutani fixed point theorem [7, Theorem V.10.1] there is $x \in J$ so that $xy = x$. The set of all $x \in J$ so that $xy = x$ is thus nonempty, closed, and a left ideal, hence is equal to $J$ by minimality. $\square$

**Remark 5.** A related result appears in work of Marrakchi [22, Theorem 3.6], inspired by an earlier version of this manuscript. It states that if $S$ is a compact affine right topological semigroup and $e \in S$ is a minimal idempotent, then $exe = e$ for all $x \in S$. We now show that this result is equivalent to Lemma 4.

Since we have seen in the proof of Lemma 2 that every minimal closed left ideal in a right topological semigroup is of the form $Sf$ for some minimal idempotent, Marrakchi's result shows that $xy = xfyf = xf = x$ for all $x, y \in Sf$. Conversely, if $f$ is a minimal idempotent, then $Sf$ is a minimal closed left ideal. Indeed, if $J \subseteq Sf$ is a minimal closed left ideal, then for any (minimal) idempotent $g \in J$, $g = gf$, which shows that $fg$ is idempotent with $fg \preceq f$. Hence $fg = f$ by minimality, and it follows that $J \supseteq Sg = Sf$. By Lemma 4, $fxf = f(xf) = f$.

Let $X = Y^*$ be a dual Banach space. Denote $\mathcal{B}(X)$ to be the algebra of a bounded linear operators on $X$ and $\mathcal{C}(X)$ to be the affine subsemigroup of contractions.

**Lemma 6.** *We have that $\mathcal{C}(X)$ is a compact affine right topological semigroup under operator composition and convergence in the pointwise-weak* topology.*

*Proof.* We have that $f \mapsto f \circ g$ is clearly affine in $f$ for all $f, g \in \mathcal{C}(X)$ and further that if $f_n \to f$ in the pointwise-weak* topology, then $f_n \circ g \to f \circ g$ in the pointwise-weak* topology.

Since $X = Y^*$ is a dual Banach space, $\mathcal{B}(X)$ is isometrically isomorphic to the dual of the projective tensor product $X \otimes_\pi Y$ via the correspondence $\langle x \otimes y \,, \varphi\rangle \leftrightarrow \langle y \,, T_\varphi(x)\rangle$, [27, section 2.2]. In this way the weak* topology on $\mathcal{B}(X)$ can be seen to coincide with the topology of pointwise-weak* convergence on functions from $X$ to itself. The Banach–Alaoglu theorem then applies, establishing compactness. □

We now give the main applications to the theory of operator spaces. In the following, $H$ will denote a Hilbert space, and $\mathcal{B}(H)$ will be the algebra of all bounded linear operators on $H$. We recall that $\mathcal{B}(H)$ is a dual Banach space. It is then straightforward to check that $\mathcal{CC}(\mathcal{B}(H))$, the set of all completely contractive linear maps from $\mathcal{B}(H)$ to itself, it is a closed convex subsemigroup of $\mathcal{C}(\mathcal{B}(H))$ in the pointwise-weak* topology. Thus, $\mathcal{CC}(\mathcal{B}(H))$ is a compact affine right topological semigroup in its own right.

Let $E \subset \mathcal{B}(H)$ be an operator space. We say that $E$ is *injective* in the category of operator spaces if for all inclusions $A \subset B$ of operator spaces and all completely contractive maps $\varphi : A \to E$, there is a completely contractive extension of $\varphi' : B \to E$. Since $\mathcal{B}(H)$ is injective for any Hilbert space by Wittstock's extension theorem [23, Theorem 8.2], $E \subseteq \mathcal{B}(H)$ is injective if and only if there is a completely contractive idempotent $\varphi : \mathcal{B}(H) \to \mathcal{B}(H)$ with $E = \varphi(\mathcal{B}(H))$.

Following [23, Chapter 15], we will say that a pair $(F, \kappa)$ consisting of an operator space $F$ and a completely isometric embedding $\kappa : E \to F$ is an *injective envelope* for $E$ if $F$ is an injective object in the category of operator spaces and for all injective operator spaces $\kappa(E) \subseteq F_0 \subseteq F$, $F_0 = F$.

**Theorem 7.** *Any operator space has an injective envelope.*

*Proof.* Let $E \subset \mathcal{B}(H)$ be an operator space. Let $\mathcal{S}$ be the set of all $\varphi \in \mathcal{CC}(\mathcal{B}(H))$ so that $\varphi(x) = x$ for all $x \in E$. It is easy to check that $\mathcal{S}$ is a closed convex subsemigroup of $\mathcal{CC}(\mathcal{B}(H))$ in the topology of pointwise-weak* convergence. Therefore, by Lemma 2 there is a minimal idempotent $\varphi \in \mathcal{S}$. Let $F = \varphi(\mathcal{B}(H))$. Suppose that $E \subseteq F_0 \subseteq F$ is injective, as witnessed by $\psi \in \mathcal{I}(\mathcal{S})$. We have $\varphi \circ \psi = \psi$, hence $\varphi \sim \psi$ by Remark 3. This implies that $\varphi$ and $\psi$ have the same range as linear operators, thus $F_0 = F$. □

**Remark 8.** This theorem effectively characterizes the injective envelopes of $E \subseteq \mathcal{B}(H)$ as the images of minimal idempotents in the semigroup of completely contractive self-maps of $\mathcal{B}(H)$ which pointwise fix $E$.

The next result was brought to the author's attention by Matthew Kennedy, as a consequence of Lemma 4.

**Theorem 9.** *Let $E$ be an operator space, and let $E \subseteq F$ be an injective envelope. Then $E \subseteq F$ is* rigid*, that is, the only completely contractive map $\theta : F \to F$ with $\theta(x) = x$ for all $x \in E$ is the identity map.*

*Proof.* Let $F \subseteq \mathcal{B}(H)$. Let $\mathcal{S}$ be the compact affine right topological semigroup of all completely contractive self-maps of $\mathcal{B}(H)$ that pointwise fix $E$. We have that $F = \varphi(\mathcal{B}(H))$ for some minimal idempotent $\varphi \in \mathcal{S}$. By Wittstock's extension theorem, $\theta$ extends to a completely contractive map $\theta' \in \mathcal{S}$. Since $\theta\varphi = \theta'\varphi$ is in the minimal right ideal of $\mathcal{S}$ generated by $\varphi$, we have that $\varphi\theta\varphi = \varphi^2 = \varphi$ by Lemma 4, thus $\theta$ must fix $F$ pointwise. □

We discuss a slight modification which seems to have connections with the theory of noncommutative Poisson boundaries: [2, 8, 19]. Let $\mathcal{M}$ be a von Neumann algebra. For $\varphi \in \mathcal{CC}(\mathcal{M})$ we let $F_\varphi := \{x \in \mathcal{M} : \varphi(x) = x\}$ denote the norm-closed subspace of fixed points of $\varphi$. Suppose $\varphi$ is continuous in the relative weak* topology (that is, the ultraweak topology) as a map of the unit ball of $\mathcal{B}(H)$ to itself (that is, $\varphi$ is normal) and that $E \subseteq F_\varphi$ is a norm-closed subspace. We consider

$$\mathcal{T}_{E,\varphi} := \{\theta \in \mathcal{CC}(\mathcal{M}) : \varphi\theta = \theta,\ E \subseteq F_\theta\}.$$

It is straightforward to check that $\mathcal{T}_{E,\varphi}$ is an affine subsemigroup of $\mathcal{CC}(\mathcal{M})$ which is closed in the pointwise-ultraweak topology, with closure being where the normality of $\varphi$ is necessary. Moreover, $\mathcal{T}_{E,\varphi}$ is nonempty as any pointwise-ultraweak cluster point of the sequence

$$\tau_N := \frac{1}{N}\sum_{k=1}^{N} \varphi^k$$

belongs to it. (Any such cluster point may even be seen to be idempotent, for instance, by [2, Proposition 5.2].) Notice that if $\theta(x) = x$, then $\varphi(x) = \varphi\theta(x) = \theta(x) = x$, so $F_\theta \subseteq F_\varphi$. Thus for any idempotent $e \in \mathcal{I}(\mathcal{T}_{E,\varphi})$, we have that the range of $e$ is contained in $F_\varphi$. If $\varphi \in \mathcal{CC}(\mathcal{B}(H))$, then the foregoing applies to $\varphi^{**} \in \mathcal{CC}(\mathcal{B}(H)^{**})$ and $E = F_\varphi \subseteq (F_\varphi)^{**} \subseteq F_{\varphi^{**}}$. Note, however, that $\mathcal{B}(H)^{**}$ is not injective, as $\mathcal{B}(H)$ isn't nuclear [6].

**Corollary 10.** *Let $\mathcal{M}$ be an injective von Neumann algebra, $\varphi : \mathcal{M} \to \mathcal{M}$ a completely contractive normal map, and $E \subseteq F_\varphi$ a norm-closed subspace. If $e \in T_{E,\varphi}$ is a minimal idempotent with range $F$, then $E \subseteq F$ is a rigid inclusion of operator spaces.*

*Proof.* Let $\theta : F \to F$ be a completely contractive map which pointwise fixes $E$. The proof follows the same lines as the previous theorem, noting that if $\theta' : \mathcal{M} \to \mathcal{M}$ is an extension of $\theta$, we may replace it with an extension $\theta''$ satisfying $\varphi\theta'' = \theta''$ by taking $\theta''$ to be a pointwise-ultraweak cluster point of $\tau_N\theta'$. □

These arguments can be adapted to a wide variety of related categories: see [5] for a detailed treatment of categorical considerations. We outline a few of these.

(1) If $X \subset \mathcal{B}(H)$ is a weakly closed injective subspace, then the set $\mathcal{CC}(X)$ of completely contractive maps $\phi : X \to X$ is a closed affine right topological subsemigroup of $\mathcal{C}(\mathcal{B}(H))$.

(2) If there is a $G$-action by complete isometries on an operator space $E$ and $E \subset X \subset \mathcal{B}(H)$ is weakly closed, injective, and $G$-invariant, then the set of $G$-equivariant maps in $\mathcal{CC}(X)$ which pointwise fix $E$ is easily seen to be a closed subsemigroup of $\mathcal{S}$, and the proof Theorem 7 shows the existence of a (relative) $G$-injective envelope for $E$. See [20, 21] for more on this and applications to the theory of reduced group C*-algebras and C*-algebras of groupoids.

(3) If $\pi : G \to \mathcal{U}(H)$ is a unitary representation, then there is a minimal unital completely positive $\pi$-invariant projection $E : \mathcal{B}(H) \to \mathcal{B}(H)$. In general there should be many such projections. If $\pi$ is the left regular representation of $G$ on $\ell^2(G)$, then the image of one such projection lies in $\ell^\infty(G)$, thus corresponds with the Furstenberg–Hamana boundary. If $\pi$ is amenable in the sense of Bekka [3], it is trivial to see that $\mathbb{C}1_{\mathcal{B}(H)}$ is one such subspace.

## Acknowledgements

The author thanks Mehrdad Kalantar for thoughtful comments and for pointing out Paulsen's work [25]. This note is a reworked version of an informal note [28] which the author publicly posted to his research webpage in October 2015. The author is thankful to Adam Dor-On and Matthew Kennedy for the encouragement to turn that note into a formal manuscript and to the anonymous reviewer for helpful comments. The author was partially supported by NSF grant DMS-2055155.

## References

[1] Spiros A. Argyros and Stevo Todorcevic, *Ramsey methods in analysis*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel, 2005. MR2145246

[2] William Arveson, *The asymptotic lift of a completely positive map*, J. Funct. Anal. **248** (2007), no. 1, 202–224, DOI 10.1016/j.jfa.2006.11.014. MR2329688

[3] Mohammed E. B. Bekka, *Amenable unitary representations of locally compact groups*, Invent. Math. **100** (1990), no. 2, 383–401, DOI 10.1007/BF01231192. MR1047140

[4] J. F. Berglund, H. D. Junghenn, and P. Milnes, *Compact right topological semigroups and generalizations of almost periodicity*, Lecture Notes in Mathematics, vol. 663, Springer, Berlin, 1978. MR0513591

[5] Arianna Cecco, *A categorical approach to injective envelopes*, Ann. Funct. Anal. **15** (2024), no. 3, Paper No. 49, 28, DOI 10.1007/s43034-024-00350-z. MR4736304

[6] Man Duen Choi and Edward G. Effros, *Nuclear $C^*$-algebras and injectivity: the general case*, Indiana Univ. Math. J. **26** (1977), no. 3, 443–446, DOI 10.1512/iumj.1977.26.26034. MR0430794

[7] John B. Conway, *A course in functional analysis*, 2nd ed., Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1990. MR1070713

[8] Sayan Das and Jesse Peterson, *Poisson boundaries of* $\mathrm{II}_1$ *factors*, Compos. Math. **158** (2022), no. 8, 1746–1776, DOI 10.1112/S0010437X22007539. MR4493239

[9] Mauro Di Nasso, Isaac Goldbring, and Martino Lupini, *Nonstandard methods in Ramsey theory and combinatorial number theory*, Lecture Notes in Mathematics, vol. 2239, Springer, Cham, 2019. MR3931702

[10] Robert Ellis, *Distal transformation groups*, Pacific J. Math. **8** (1958), 401–405. MR0101283

[11] H. Furstenberg and Y. Katznelson, *Idempotents in compact semigroups and Ramsey theory*, Israel J. Math. **68** (1989), no. 3, 257–270, DOI 10.1007/BF02764984. MR1039473

[12] Eli Glasner, *Ergodic theory via joinings*, Mathematical Surveys and Monographs, vol. 101, American Mathematical Society, Providence, RI, 2003. MR1958753

[13] Don Hadwin and Vern I. Paulsen, *Injectivity and projectivity in analysis and topology*, Sci. China Math. **54** (2011), no. 11, 2347–2359, DOI 10.1007/s11425-011-4285-7. MR2859698

[14] Masamichi Hamana, *Injective envelopes of $C^*$-algebras*, J. Math. Soc. Japan **31** (1979), no. 1, 181–197, DOI 10.2969/jmsj/03110181. MR0519044

[15] ———, *Injective envelopes of operator systems*, Publ. Res. Inst. Math. Sci. **15** (1979), no. 3, 773–785, DOI 10.2977/prims/1195187876. MR0566081

[16] ———, *Injective envelopes of $C^*$-dynamical systems*, Tohoku Math. J. (2) **37** (1985), no. 4, 463–487, DOI 10.2748/tmj/1178228589. MR0814075

[17] ———, *Injective envelopes of dynamical systems*, Toyama Math. J. **34** (2011), 23–86. MR2985658

[18] Neil Hindman and Dona Strauss, *Algebra in the Stone-Čech compactification*, extended edition, De Gruyter Textbook, Walter de Gruyter & Co., Berlin, 2012. Theory and applications. MR2893605

[19] Masaki Izumi, *$E_0$-semigroups: around and beyond Arveson's work*, J. Operator Theory **68** (2012), no. 2, 335–363. MR2995726

[20] Mehrdad Kalantar and Matthew Kennedy, *Boundaries of reduced $C^*$-algebras of discrete groups*, J. Reine Angew. Math. **727** (2017), 247–267, DOI 10.1515/crelle-2014-0111. MR3652252

[21] Matthew Kennedy, Se-Jin Kim, Xin Li, Sven Raum, and Dan Ursu, *The ideal intersection property for essential groupoid C*-algebras*, arXiv e-prints, posted on 2021, DOI 10.48550/arXiv.2107.03980.

[22] Amine Marrakchi, *On the weak relative Dixmier property*, Proc. Lond. Math. Soc. (3) **122** (2021), no. 1, 118–123, DOI 10.1112/plms.12347. MR4210259

[23] Vern Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, vol. 78, Cambridge University Press, Cambridge, 2002. MR1976867

[24] Vern I. Paulsen, *A dynamical systems approach to the Kadison-Singer problem*, J. Funct. Anal. **255** (2008), no. 1, 120–132, DOI 10.1016/j.jfa.2008.04.006. MR2417811

[25] ———, *Weak expectations and the injective envelope*, Trans. Amer. Math. Soc. **363** (2011), no. 9, 4735–4755, DOI 10.1090/S0002-9947-2011-05203-7. MR2806689

[26] Zhong-Jin Ruan, *Injectivity of operator spaces*, Trans. Amer. Math. Soc. **315** (1989), no. 1, 89–104, DOI 10.2307/2001374. MR0929239

[27] Raymond A. Ryan, *Introduction to tensor products of Banach spaces*, Springer Monographs in Mathematics, Springer-Verlag London, Ltd., London, 2002. MR1888309

[28] Thomas Sinclair, *A very short proof of the existence of an injective envelope for an operator space*, `https://www.math.purdue.edu/~tsincla/injective-note-1.pdf`.

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