THE MODEL THEORY OF METRIC LATTICES: PSEUDOFINITE PARTITION LATTICES

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Abstract. We initiate the study of general metric lattices in the context of the model theory of metric structures. As an application we develop a theory of pseudo-finite limits of partition lattices and connect this theory with the theory of continuous limits of partition lattices due to Björner and Lovász.

1. Introduction

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2. Model Theory of Metric Structures

In this paper we will use the model-theoretical framework as detailed in [15] and [3] for the case of one-sorted structures. In particular, we define the following terms.

- A language, or signature, $\mathfrak{L}$ contains the usual sets of function, relation and constant symbols with their respective arity, as in first order logic. Additionally, the language will associate to each relation symbol R a function $\Delta_R : [0,1] \to [0,1]$, such that $\lim_{\epsilon\to 0^+} \Delta(\epsilon) = 0$, which we call the uniform continuity modulus of R, and similarly for the function symbols.
- The logical symbols used in first order logic of $x = y$, $\forall_x$, and $\exists_x$ are replaced by $d(x, y) = 0$, $\sup_x$, and $\inf_x$, respectively. Connectives are now all continuous functions $u : [0,1]^n \to [0,1]$ for every n.
- Given a language $\mathfrak{L}$ we say that $\mathcal{M}$ is an $\mathfrak{L}$-*structure* if $\mathcal{M}$ consists of a complete metric space of diameter one along with a collection of functions and distinguished elements satisfying the following. For every function symbol $f \in \mathfrak{L}$ there is a corresponding uniformly continuous function $f^{\mathcal{M}} : M^{a(f)} \to M$ where $a(f)$ is the arity of f and Δ_f is a witness of the uniform continuity of $f^{\mathcal{M}}$. For every relation symbol $R \in \mathcal{L}$ there is a corresponding uniformly continuous function $R^{\mathcal{M}} : M^{a(R)} \to [0,1]$ where $a(R)$ is the arity of R and Δ_R is a witness of the uniform continuity of $R^{\mathcal{M}}$. For every constant symbol $c \in \mathfrak{L}$ there is a corresponding distinguished element $c^{\mathcal{M}} \in M$.
- The notions of substructure and embedding between $\mathcal{L}$-structures are defined in an analogous way to first order logic.
- The terms are constructed analogously to classical first order logic. Similarly, the formulas are constructed following the same process as in first order logic.

In addition, limits of formulas that converge uniformly over all $\mathcal{L}$-structures are also considered formulas. A formula with no unquantified variables is called a *sentence*. A formula $\varphi(x)$ is *satisfied* for some tuple $a \in M$ for $\mathcal{M}$ an $\mathcal{L}$-structure if $\varphi^{\mathcal{M}}(a) = 0$.

- Let Σ be a set of $\mathcal{L}$-sentences. We say that an $\mathfrak{L}$-structure *models* Σ, written $\mathcal{M} \models \Sigma$ if $\sigma^{\mathcal{M}} = 0$ for all $\sigma \in \Sigma$. The set Σ is consistent if it admits a model, in which case we write $\mathrm{Mod}(\Sigma)$ for the class of all models of Σ.
- An $\mathfrak{L}$-theory is a maximal collection of consistent $\mathfrak{L}$-sentences. If Σ is consistent, then all sentences satisfied by all models of Σ is an $\mathfrak{L}$-theory.
- A homomorphism $\rho : \mathcal{M} \to \mathcal{N}$ of $\mathfrak{L}$-structures is said to be *elementary* if it preserves the values of all sentences.

For a complete treatment of the model theory of metric structures the reader may consult [11], [15], or [3]. Besides the general notions mentioned before, we will also use the following model-theoretical results about definability.

Let $\mathfrak{L}$ be a signature and T an $\mathfrak{L}$-theory. We say that a functor $\mathcal{X} : \mathrm{Mod}(T) \to \mathsf{Met}$, where Met is the category of metric spaces with isometric embeddings, is a *T-functor* if $\mathcal{X}(\mathcal{M})$ is a closed subspace of M and if for every elementary map $\rho : \mathcal{M} \to \mathcal{N}$ of T-models, $\mathcal{X}(\rho) = \rho\restriction_{\mathcal{X}(\mathcal{M})}$.

Roughly, a T-functor is said to be definable if the distance to $\mathcal{X}(\mathcal{M})$ is computable in terms of a formula: see the first item in the theorem below. The following theorem found in [15] gives a full picture of the meaning of definability in the context of the model theory of metric structures.

Theorem 2.1 ([15], Theorem 8.2)**.** *Let's consider a signature $\mathcal{L}$, a $\mathcal{L}$-theory T and a T-functor $\mathcal{X}$. The following are equivalent:*

(1) *There is a T-formula $\varphi(\bar{x})$ such that for any model $\mathcal{M}$ of T and any $\bar{a} \in \mathcal{M}$,*
$$\varphi^{\mathcal{M}}(\bar{a}) = d(\bar{a}, \mathcal{X}(\mathcal{M})).$$
(2) *For any T-formula $\psi(\bar{x}, \bar{y})$, there is a T-formula $\sigma(\bar{y}$ such that for any model $\mathcal{M}$ of T and any $\bar{a} \in \mathcal{M}$,*
$$\sigma^{\mathcal{M}}(\bar{a}) = \inf_{\bar{x} \in \mathcal{X}(\mathcal{M})} \psi^{\mathcal{M}}(\bar{x}, \bar{a})$$
(or equivalently the analogous statement replacing inf *by* sup*).*
(3) *For every $\varepsilon > 0$, there is a T-formula $\varphi(\bar{x})$ and $\delta > 0$ such that, for all models $\mathcal{M}$ of T:*
   - *$\mathcal{X}(\mathcal{M}) \subset \{\bar{a} \in \mathcal{M} : \varphi^{\mathcal{M}}(\bar{a}) = 0\}$,*
   - *For all $\bar{a} \in \mathcal{M}$, if $\varphi^{\mathcal{M}}(\bar{a}) < \delta$, then $d(\bar{a}, \mathcal{X}(\mathcal{M})) \leq \varepsilon$.*
(4) *For every $\varepsilon > 0$, there is a basic $\mathfrak{L}$-formula $\varphi(\bar{x})$ and $\delta > 0$ such that, for all models $\mathcal{M}$ of T:*
   - *$\mathcal{X}(\mathcal{M}) \subset \{\bar{a} \in \mathcal{M} : \varphi^{\mathcal{M}}(\bar{a}) = 0\}$,*
   - *For all $\bar{a} \in \mathcal{M}$, if $\varphi^{\mathcal{M}}(\bar{a}) < \delta$, then $d(\bar{a}, \mathcal{X}(\mathcal{M})) \leq \varepsilon$.*

(5) *For any set I, family of $\mathfrak{L}$-structures $\mathcal{M}_i$ for $i \in I$, and ultrafilter $\mathcal{U}$ on I,*

$$\mathcal{X}(\mathcal{N}) = \prod_{\mathcal{U}} \mathcal{X}(\mathcal{M}_i), \text{ where } \mathcal{N} = \prod_{\mathcal{U}} \mathcal{M}_i.$$

The Beth Definability Theorem is a powerful tool in model theory since it offers a criterion to identify relations and predicates that can be expressed by formulas, as it establishes a connection between the explicit sense of definability, using formulas, to the implicit sense of definability, using models of a theory. The following formulation of the theorem appears in [15].

Theorem 2.2 (Beth Definability Theorem)**.** *Suppose that $\mathfrak{L}' \subseteq \mathfrak{L}$ are two continuous languages with the same sorts. Further, suppose T is an $\mathfrak{L}$-theory. If the forgetful functor F from models of T to $\mathfrak{L}'$-structures given by restriction is an equivalence of categories onto the image of F, then every $\mathfrak{L}$-formula is T-equivalent to an $\mathfrak{L}'$-formula.*

As it is claimed in [15], this result is even more important in the context of continuous logic given the challenge of identifying definable predicates as limits of explicit formulas. We will use this to prove the definability of the *meet* operation under certain circumstances. Specifically, we will use the following corollary of the Beth Definability Theorem found in [11, 4.2.3].

Corollary 2.3. *Suppose that $\mathcal{C}$ is an elementary class of structures in a language $\mathfrak{L}$ and, for every $A \in \mathcal{C}$, the structure A is expanded by a predicate P_A which is uniformly continuous with uniform continuity modulus independent of our choice of A. Let $\mathcal{C}' := \{(A, P_A) : A \in \mathcal{C}\}$ be a class of structures for an expanded language $\mathcal{L}'$ with a predicate for P. If $\mathcal{C}'$ is an elementary class for a theory T' in the language $\mathfrak{L}'$ then P is T'-equivalent to a definable predicate in $\mathfrak{L}$.*

3. Metric Lattices

A *semilattice* is a set L equipped with a commutative, associative, idempotent binary operation $+$. We can see that this operation defines a partial order $<$ on L by $x < y$ if $x + y = y$. This choice of partial order creates what is often referred to as an *upper semilattice* in the literature: when we refer to a partial order on a semilattice, we will always mean this one. Further, we will always assume that semilattices are equipped with (unique) elements 0 and 1 so that $0 + x = x$ and $1 + x = 1$ for all $x \in L$.

Definition 3.1. A tuple $\mathcal{L} = (L, <, +, \cdot, 0, 1)$ a *lattice* if:

(1) $(L, <)$ is a partially ordered set with a maximal element 1 and a minimal element 0;
(2) $+, \cdot$ are commutative, associative, idempotent binary operations;
(3) $x \leq y$ if and only if $x + y = y$ if and only if $xy = x$.

Following convention, we will refer to $+$ as the *meet* and $\cdot$ as the *join.* Note that the conditions imply that $\sum_{i=1}^{k} x_i$ is the least upper bound of $\{x_1, \ldots, x_k\}$ and $\prod_{i=1}^{k} x_i$ is the greatest lower bound.

If $\mathcal{L}$ is a finite semilattice, more generally, if the partial order is complete, then the meet operation xy can be defined in terms of this order as the maximal element simultaneously below both x and y. When generalizing beyond finite lattices, this poses an interesting choice of category in terms of whether to consider lattice or semilattice morphisms.

Definition 3.2. A *metric lattice* is a tuple $\mathcal{L} = (L, <, +, \cdot, 0, 1, d)$ so that $(L, <, +, \cdot, 0, 1)$ is a lattice and d is a metric on L satisfying:

(1) $d(1, 0) = 1$;
(2) $d(x + z, y + z) \leq d(x, y)$;
(3) $d(x, y) \leq d(x + y, 0) - d(xy, 0)$

for all $x, y, z, \in L$. If (L, d) is a complete metric space, we say that $\mathcal{L}$ is a *complete metric lattice*. We call a metric satisfying the last condition *semi-modular*.

Remark 3.3. Notice that condition (2) implies that

$$d(x + z, y + w) \leq d(x, y) + d(z, w) \tag{1}$$

by the triangle inequality, so $L \times L \ni (x, y) \mapsto x + y \in L$ is jointly uniformly continuous.

Given a metric lattice $\mathcal{L} = (L, <, +, \cdot, 0, 1, d)$, we can define a *rank function* $|\cdot| : L \to \mathbb{R}_{\geq 0}$ by

$$|x| := d(x, 0). \tag{2}$$

The semi-modular condition on the metric then becomes

$$d(x, y) \leq |x + y| - |xy|. \tag{3}$$

Proposition 3.4. *If $\mathcal{L} = (L, <, +, \cdot, 0, 1, d)$ is a metric lattice then:*

(1) *$x < y$ implies that $|x| < |y|$.*
(2) *$d(x, y) \in [0, 1]$ for all $x, y \in L$.*

Proof. If $x < y$ then $x = xy < x + y = y$, so $0 < d(x, y) \leq |y| - |x|$ by (3). Since $0 \leq x \leq 1$ for all $x \in L$ it follows that $0 \leq |x| \leq 1$. As a consequence

$$d(x, y) \leq |x + y| - |xy| \leq |x + y| \leq 1.$$ □

Remark 3.5. This definition differs from the definition given in [34]. It can be seen that a metric lattice as defined therein is also a metric lattice in our sense. Indeed, the only difference is the insistence that $d(x, y) = 2|x + y| - |x| - |y|$. For a metric lattice we define

$$d'(x, y) := d(x + y, x) + d(x + y, y). \tag{4}$$

Proposition 3.6. *For a metric lattice $\mathcal{L}$ the following are true.*

(1) *$||x| - |y|| \leq d(x, y)$.*
(2) *$d(x, y) = |y| - |x|$ for $x < y$.*
(3) *$d'(x, y) = 2|x + y| - |x| - |y|$.*
(4) *$|x + y| + |xy| \leq |x| + |y|$.*
(5) *If $z \leq x$ then $|z| + |y + x| \leq |z + y| + |x|$.*

(6) *d' is a metric and $\mathcal{L}' = (L, d')$ is a metric lattice.*
(7) $d(x, y) \leq d'(x, y) \leq 2\, d(x, y)$.

Proof. We prove each item sequentially.

(1) Since d is a metric by the triangle inequality we have that
$$d(x, 0) \leq d(x, y) + d(y, 0) \text{ and } d(x, 0) \leq d(x, y) + d(y, 0);$$
that is, $|x| \leq d(x, y) + |y|$ and $|y| \leq d(x, y) + |x|$. Combining this inequalities we conclude that $||x| - |y|| \leq d(x, y)$.

(2) If $x < y$ by the previous point and the semi-modular condition we conclude that
$$||y| - |x|| = |y| - |x| \leq d(x, y) \leq |x + y| - |xy| = |y| - |x|,$$
thus $d(x, y) = |y| - |x|$.

(3) From item (2) we have that
$$\begin{aligned} d'(x, y) = d(x + y, x) + d(x + y, y) &= (|x + y| - |x|) + (|x + y| - |y|) \\ &= 2|x + y| - |x| - |y|. \end{aligned}$$

(4) Again by item (2) we have that $d(x+y, x) = |x+y|-|x|$ and $d(y, xy) = |y|-|xy|$. Also from condition (2) in the definition of a metric lattice we have that
$$|x + y| - |x| = d(x + y, x) = d(x + y, x + xy) \leq d(y, xy) = |y| - |xy|;$$
thus, we get that $|x + y| + |xy| \leq |x| + |y|$.

(5) By item (4) we have that for $x, y, z \in L$,
$$|(z + y)x| + |z + y + x| \leq |z + y| + |x|.$$
Since $z \leq z + y, x$ we have that $z \leq (z + y)x$ and $z + y + x = y + x$; thus, by Proposition 3.4.1 we get that
$$|z| + |y + x| \leq |(z + y)x| + |z + y + x| \leq |z + y| + |x|.$$

(6) We first prove that d' is a metric. Since d is a metric is clear that $d'(x, y) \geq 0$, $d'(x, x) = 0$, and $d'(x, y) = d'(y, x)$.

Using our definition of a metric lattice and items (2) and (3) we can follow the strategy given in [34, Theorem 1.1] to prove the triangle inequality. Indeed, by item (3) and Proposition 3.4.1 we get that
$$\begin{aligned} d'(x, z) + d'(z, y) - d'(x, y) &= 2(|x + z| + |z + y| - |x + y| - |z|) \\ &\geq 2(|x + z| + |z + y| - |x + y + z| - |z|). \end{aligned}$$
Using (5) inequality with $z \leq x + z$ we conclude that
$$d'(x, z) + d'(z, y) - d'(x, y) \geq 2(|x + z| + |z + y| - |x + y + z| - |z|) \geq 0;$$
therefore, $d'(x, y) \leq d'(x, z) + d'(y, z)$.

Now, we need to check that (L, d') satisfies the axioms of being a metric lattice. Note that $d'(x,0) = |x|$, so d' induces the same rank function as d. We have that

$$\begin{aligned} &d'(x,y) - d'(x+z, y+z) \\ &= (2|x+y| - |x| - |y|) - (2|x+y+z| - |x+z| - |y+z|) \\ &= (|x+z| - |x|) + (|y+z| - |y|) - 2(|x+y+z| - |x+y|) \\ &= d(x+z,x) + d(y+z,y) - 2\,d(x+y+z, x+y) \geq 0 \end{aligned}$$

by item (2) and Definition 3.2.2. It follows from item (4) that

$$d'(x,y) \leq |x+y| - |xy|.$$

(7) By Definition 3.2.2 and the triangle inequality we have that

$$\begin{aligned} d(x,y) &\leq d(x, x+y) + d(y, x+y) \\ &= d(x+x, x+y) + d(x+y, y+y) \\ &\leq d(x,y) + d(x,y) = 2\,d(x,y). \end{aligned}$$

Thus by the definition of d' we conclude $d(x,y) \leq d'(x,y) \leq 2\,d(x,y)$. □

Proposition 3.7. *If $\mathcal{L}$ is a metric lattice, then we have that*

$$|x+y| + |z| \leq |x+z| + |y+z|$$

for all $x, y, z \in L$. As a consequence,

$$d(x,y) \leq d'(x,y) \leq |x+y| + |x+z| + |y+z| - |x| - |y| - |z|$$

for all $x, y, z \in L$.

Proof. Since $z \leq z + x$, we can apply Proposition 3.6.5 to get

$$|z| + |y+z+x| \leq |z+y| + |z+x|.$$

By Proposition 3.4 we know that $|x+y| \leq |y+z+x|$, so we conclude that

$$|x+y| + |z| \leq |x+z| + |y+z|$$

for all $x, y, z, \in L$. This result, along with Proposition 3.6.3 and Proposition 3.6.7, gives us

$$\begin{aligned} d(x,y) \leq d'(x,y) &= 2|x+y| - |x| - |y| \\ &= |x+y| + (|x+y| - |x| - |y|) \\ &\leq (|x+z| + |y+z| - |z|) + (|x+y| - |x| - |y|) \\ &= |x+y| + |x+z| + |y+z| - |x| - |y| - |z|. \end{aligned}$$

for all $x, y, z \in L$. □

Remark 3.8. We see that the metrics d and d' agree on any chain (totally ordered subset) of L. Further, by Proposition 3.6.2 and the proof of item (7) in the same, we have that if d and $\tilde{d}$ are metrics that induce a metric lattice structure on the same lattice and for which the rank functions agree, then $d(x,y) \leq 2\tilde{d}(x,y)$. In general, the set of metrics on a lattice can be quite large. Take, for example, the lattice of all Borel subsets of $[0,1]$. Any Borel regular measure μ of full support induces a metric lattice structure by $d_\mu(A,B) = \mu(A \cup B) - \mu(A \cap B)$.

Remark 3.9. Let's define

$$\sigma_d(x,y,z) = |x+y| + |x+z| + |y+z| - |x| - |y| - |z| \tag{5}$$

and note that σ_d is symmetric in the variables x, y, z and that $\sigma_d = \sigma_{d'}$. From Proposition 3.7 we see that

$$\max\{d(x,y), d(y,z), d(x,z)\} \leq \sigma_d(x,y,z)$$

for any metric lattice. Further, by Proposition 3.6.2 we have that

$$\begin{aligned} &d(x+y,0) - d(z,0) + d(x+z,x) + d(y+z,y) \\ &= |x+y| - |z| + (|x+z| - |x|) + (|y+z| - |y|) = \sigma_d(x,y,z). \end{aligned} \tag{6}$$

We thus have the following

Corollary 3.10. *Every metric lattice $\mathcal{L}$ satisfies the following strengthening of the semi-modular condition (3) from Definition 3.2:*

$$d(x,y) \leq |x+y| + d(x+z,x) + d(y+z,y) - |z| \tag{7}$$

for all $x, y, z \in L$.

Note that $d'(x,y) \leq |x+y| + d'(x+z,x) + d'(y+z,z) - |z|$ is equivalent to d' satisfying the triangle inequality.

3.1. **Modularity.** In a lattice L, a pair (x,y) of elements is called a *modular pair* if $xy + z = x(y+z)$ for all $z \leq x$. Note that this relation need not be symmetric: if it is, we will call the lattice *semi-modular*.

Definition 3.11. Given a metric lattice $\mathcal{L}$, we say that $(x,y) \in L^2$ is a *metrically modular pair* if

$$|x+y| + |xy| = |x| + |y|;$$

equivalently, if $d'(x,y) = |x+y| - |xy|$. Note that, unlike modularity of a pair, metric modularity is a symmetric relation. We say that $x \in L$ is *metrically modular* if (x,y) is a metrically modular pair for all $y \in L$ and that $\mathcal{L}$ is *metrically modular* if every pair $(x,y) \in L^2$ is a metrically modular pair.

Proposition 3.12. *For a metric lattice, every metrically modular pair (x,y) is a modular pair.*

Proof. Let's suppose that (x,y) is a metrically modular pair. To prove that they are a modular pair we need to show that for all $z \leq x, z + yx = (z+y)x$. We already know that for all $z \leq x$ we have the inequalities $z + yx \leq (z+y), x$, hence $z + yx \leq (z+y)x$.

Now using Proposition 1.6.4 and Proposition 1.6.5 with $yx \leq y$ we have that for all $z \leq x$

$$\begin{aligned} |(z+y)x| &\leq |z+y| + |x| - |z+y+x| \\ &= |z+y| + |x| - |y+x| \\ &= |z+y| + |x| - (|x| + |y| - |xy|) \\ &= |xy| + |z+y| - |y| \leq |z+yx|. \end{aligned}$$

We conclude that $|(z+y)x| = |z+yx|$, thus as $|\cdot|$ is strictly increasing, it follows that $(z+y)x = z+yx$. $\square$

Question 3.13. Is the condition of being a modular pair first-order axiomatizable in the theory of metric lattices?

Remark 3.14. We say that a metric lattice $\mathcal{L}$ is *strongly metrically semi-modular* if

$$|x| + |y| + |z| \leq |x+z| + |y+z| + |xy| \tag{8}$$

for all $x, y, z \in L$; equivalently, if

$$|x+y| - |xy| \leq \sigma_d(x, y, z) \tag{9}$$

for all $x, y, z \in L$. It is clear that $|x+y| - |xy| \geq \inf_z \sigma_d(x, y, z)$; thus, strong semi-modularity is equivalent to

$$|x+y| - |xy| = \inf_z \sigma_d(x, y, z) \tag{10}$$

for all $x, y \in \mathcal{L}$. In fact, we have that a metric lattice $\mathcal{L}$ is strongly metrically semi-modular if and only if it is metrically modular.

Indeed, assume $\mathcal{L}$ is strongly metrically semi-modular. Setting $y = z$, we get that $|x|+|y| \leq |x+y|+|xy|$ for all $x, y \in L$. By Proposition 3.6 we have the reverse inequality, so $\mathcal{L}$ is metrically modular. On the other hand, if $\mathcal{L}$ is metrically modular, we have $d'(x, y) = |x+y| - |xy|$; therefore, by Proposition 3.7 we have $|x+y| - |xy| \leq \sigma_d(x, y, z)$.

Definition 3.15. Let L be a lattice. We say that a function $\rho : L \times L \to \mathbb{R}_{\geq 0}$ is a *quasi-metric* if:

1. $\rho(x, y) = \rho(y, x)$;
2. $\rho(x, y) = 0$ only if $x = y$;
3. $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$ when $z \leq x$ or $z \leq y$.

Proposition 3.16. *Let $\mathcal{L}$ be a metric lattice, and let*

$$\delta(x, y) := |x| + |y| - 2|xy|.$$

The following are true.

1. *$\delta(x, y)$ is a quasi-metric.*
2. *$\delta(x, 0) = |x|$ for all $x \in L$.*
3. *$d'(x, y) \leq \delta(x, y)$.*

Moreover, we have that δ is a metric if and only if $\mathcal{L}$ is metrically modular.

Proof. We begin with the proofs of the numbered assertions.

(1) First note that since $xy \leq x, y$, we have that $|xy| \leq |x|, |y|$ thus $\delta(x,y) \geq 0$. It is clear that $\delta(x,y) = \delta(y,x)$ and that if $x = y$, then $\delta(x,y) = 0$. Now, if $x \neq y$ we have that $x \neq xy$ or $y \neq xy$, therefore $|x| > |xy|$ or $|y| > |xy|$ and we get that $\delta(x,y) > 0$. Thus we conclude that $\delta(x,y) = 0$ if and if $x = y$. Now to conclude that δ is a quasi-metric we need check the last condition.

If $z \in L$ is such that $z \leq x$ or $z \leq y$, then $|xz| \leq |xy|$ or $|yz| \leq |xy|$ and given that $|xz|, |yz| \leq |z|$ we conclude that $|xz| + |yz| \leq |z| + |xy|$. Using this inequality we get

$$|x| + |y| + 2|xz| + 2|yz| \leq |x| + |z| + |y| + |z| + 2|xy|,$$

hence

$$\begin{aligned}\delta(x,y) &= |x| + |y| - 2|xy| \\ &\leq |x| + |z| - 2|xz| + |y| + |z| - 2|yz| = \delta(x,z) + \delta(y,z).\end{aligned}$$

Thus δ is a quasi-metric.

(2) It is clear that $\delta(x,0) = |x| + |0| - 2|0| = |x| - 0 = |x|$ for all $x \in L$.

(3) By Proposition 2.6(4) we know that for all $x, y \in L$

$$|x+y| + |xy| \leq |x| + |y|.$$

We can reorder the terms to get

$$2|x+y| - |x| - |y| \leq |x| + |y| - 2|xy|,$$

thus $d'(x,y) \leq \delta(x,y)$.

For the last assertion, if δ satisfies the triangle inequality, then for all $x, y, z \in L$ we have that

$$|x| + |y| - 2|xy| \leq |x| + |z| - 2|xz| + |z| + |y| - 2|yx|,$$

which shows that

$$|xz| + |yz| \leq |z| + |xy|.$$

Taking $z = x + y$ we have that $|x| + |y| \leq |x+y| + |xy|$. We already know that $|x+y| + |xy| \leq |x| + |y|$, so we conclude that all pairs $x, y \in L$ are metrically modular pairs. In the other direction, if $\mathcal{L}$ is metrically modular then $\delta(x,y) = d'(x,y)$. $\square$

3.2. **Complete Metric Lattices.**

Lemma 3.17. *Let $\mathcal{L}$ be a metric lattice. The following are true.*

(1) *Any set $S \subset L$ has a least upper bound (respectively, greatest lower bound) if and only if each increasing sequence (resp., decreasing sequence) has a least upper bound (resp., greatest lower bound).*

(2) *If $\mathcal{L}$ is a complete metric lattice and S is closed under $+$ (respectively, closed under $\cdot$), then the least upper bound (resp., greatest lower bound) belongs to the closure of S.*

Proof. We first prove (1). The "only if" direction is clear. Let $(x_i)_{i\in I}$ be a chain. Since $x<y$ implies that $|x|<|y|$, $x_i \mapsto |x_i| \in \mathbb{R}$ is an order-preserving bijection, and we have that I is therefore countable. The proof that $S \subset L$ has a least upper bound (or greatest lower bound) then follows exactly as in [34, Lemma 2.2].

We now turn our attention to the proof of (2). Let $\mathcal{F}$ be the collection of all finite subsets of S. Consider the net $(x_F)_{F\in\mathcal{F}}$ where $x_F := \sum_{x\in F} x$. We have that x is an upper bound for S if and only if $x_F \le x$ for all $F \in \mathcal{F}$. We have that $|x_F|$ is increasing and bounded by 1; hence, $\alpha := \lim_{\mathcal{F}} |x_F|$ exists. Given $\epsilon > 0$, if $\alpha - \epsilon < |x_F| \le |x_{F'}| \le \alpha$ for $F \subset F'$, then $d(x_F, x_{F'}) \le |x_{F'}| - |x_F| < \epsilon$; thus, (x_F) is a Cauchy net and $x^* = \lim_{\mathcal{F}} x_F$ exists by metric completeness. Clearly $x_F + x_G = x_G$ if $F \subset G$, so passing to limits we have that $x_F + x^* = x^*$ which shows that x^* is an upper bound for S. If S is closed under $+$, then $x_F \in S$, so x^* belongs to the closure of S.

It now suffices to check that x^* is the least upper bound. If y is another upper bound for S, then so is x^*y, so without loss of generality we may only consider the case when $x_F \le y < x^*$ for all $F \in \mathcal{F}$. However, we would then have that by Proposition 3.6.2 that $d(x_F, x^*) = |x^*| - |x_F| \ge |x^*| - |y| > 0$, which is a contradiction.

The proof when S is closed under $\cdot$ with the greatest lower bound follows nearly identically. □

Proposition 3.18. *A complete metric lattice is complete as a lattice.*

Proof. The proof follows the strategy given in [34, Lemma 2.3]. By Lemma 3.17 we need only show that every increasing or decreasing sequence has a least upper bound or greatest lower bound, respectively.

We begin by assuming that (x_i) is increasing; hence, so is $(|x_i|)$. As $|x_i| \le 1$, it follows that $(|x_i|)$ is Cauchy, which implies (x_i) is Cauchy as well since $d(x_i, x_j) = |x_j| - |x_i|$ for $i \le j$. Since (L, d) is complete $x := \lim_i x_i$ exists. Note that $|x| = \lim_i |x_i|$ since $|x| = d(x, 0)$.

Our goal is to show that x is the least upper bound for (x_i). For all i, j with $i \le j$ we have that

$$d(x+x_i, x) \le d(x+x_i, x_j) + d(x, x_j) = d(x+x_i, x_j+x_i) + d(x, x_j) \le 2d(x, x_j) \tag{11}$$

which implies that $d(x+x_i, x) = 0$. Hence, $x + x_i = x$ or $x \ge x_i$ for all $i \in \mathbb{N}$. If there is z so that $x_i \le z \le x$ for all $i \in \mathbb{N}$, then

$$d(x_i, z) + d(z, x) \le (|z| - |x_i|) + (|x| - |z|) = |x| - |x_i| \to 0 \tag{12}$$

as $i \to \infty$, thus $d(z, x) = 0$. This shows that x is the least upper bound of (x_i).

Now, let (x_i) be decreasing and $x = \lim_i x_i$ as before. We have that for all $i \le j$

$$d(x+x_i, x_i) = d(x+x_i, x_j+x_i) \le d(x, x_j), \tag{13}$$

so $d(x+x_i, x_i) = 0$ or $x \le x_i$ for all $i \in \mathbb{N}$. That x is the greatest lower bound follows similarly. □

Lemma 3.19. *Let $\mathcal{L}$ be a complete metric lattice. If (x_i) is a sequence in L, then setting $z = \sum_i x_i$ we have that*

$$d(z + y, y) \leq \sum_i d(x_i + y, y)$$

if the sum converges.

Proof. Setting $z_n = \sum_{i=1}^n x_i$, we have that (z_n) is an increasing sequence, so (z_n) is Cauchy and $z_n \to z$ by the proof of Proposition 3.18. We have inductively that

$$\begin{aligned} d(z_n + y, y) &= d(z_{n-1} + x_n + y, y) \\ &\leq d(z_{n-1} + y + x_n, y + x_n) + d(x_n + y, y) \\ &\leq d(z_{n-1} + y, y) + d(x_n + y, y) \leq \sum_{i=1}^n d(x_i + y, y); \end{aligned}$$

hence, the result obtains in the limit by continuity. □

Proposition 3.20. *Let $\mathcal{L}$ be a complete metric lattice. A pair $x, y \in L$ is metrically modular if and only if for every $n \in \mathbb{N}$ there is $z \in L$ so that*

$$|x| + |y| \leq |x + y| + |z| + \frac{1}{n},\ d(x + z, x) < \frac{1}{n},\ d(y + z, y) < \frac{1}{n}. \tag{14}$$

Proof. The "only if" direction follows by choosing $z = xy$ and using Proposition 3.6.4. For the converse, fix $x, y \in L$. Choose $z_i \in L$ for $n = 2^i$ as above and set $t_k = \sum_{i=k}^\infty z_i$ with $t_k^N = \sum_{i=k}^N z_i$, noting that $t_k^N \to t_k$ as $N \to \infty$ by the proof of Proposition 3.18. For $k < l$ and $M > N$ we have that $d(t_k^M + t_l^N, t_k^M) = 0$, so

$$d(t_k + t_l^N, t_k) = \lim_M d(t_k^M + t_l^N, t_k^M) = 0$$

when $k < l$. Taking the limit in N, we see that $d(t_k + t_l, t_k) = 0$, or $t_l \leq t_k$ when $k < l$. Since (t_k) is therefore decreasing, it is Cauchy. Set $t = \lim_k t_k$. By Lemma 3.19, we have that $d(t_k + x, x), d(t_k + y, y) \leq \sum_{i=k}^\infty 2^{-i} \to 0$ as $k \to \infty$; thus, $d(t + x, x) = d(t + y, y) = 0$ or $t \leq xy$.

Since $t_k \geq z_k$, we have that

$$|x| + |y| \leq |x + y| + |t_k| + \frac{1}{2^k}$$

for all $k \in \mathbb{N}$ which implies that

$$|x| + |y| \leq |x + y| + |t| \leq |x + y| + |xy|.$$

The result is now apparent by Proposition 3.6.4. □

3.3. **An Alternate Axiomatization.** In the presence of metric completeness we can describe a metric lattice using only uniformly continuous operations, a point which will be crucial for the purpose of axiomatization as metric structures.

Definition 3.21. We define a *pseudo-metric semilattice* to be a structure $\mathcal{P} = (P, +, 0, 1, d)$ where the following hold:

(1) (P, d) is a pseudo-metric space;
(2) $0, 1 \in P$ are distinguished elements;
(3) $+ : P \times P \to P$ is a commutative, associative, idempotent ($x + x = x$) operation; and
(4) $0 + x = x$ and $1 + x = 1$ for all $x \in P$.

Further, we require the following axioms:

(1) $d(1, 0) = 1$;
(2) $d(x + z, y + z) \leq d(x, y)$; and
(3) $d(x, y) \leq d(x + y, 0) + d(x + z, x) + d(y + z, y) - d(z, 0)$

for all $x, y, z \in P$.

We say that $\mathcal{P}$ is a *metric semilattice* if (P, d) is a metric space and that $\mathcal{P}$ is *complete* if (P, d) is a complete metric space.

Remark 3.22. The property "$d(x, y) \leq d(x+y, 0)+d(x+z, x)+d(y+z, y)-d(z, 0)$ for all $x, y, z \in P$" implies "$d(x, y) \leq d(x + y, 0) - d(z, 0)$ for all $x, y, z \in P$ with $z \leq x, y$." Inspection of the proofs of Propositions 3.4 and 3.6 will convince the reader that these properties are equivalent.

Proposition 3.23. *Let $\mathcal{P}$ be a semilattice and let $f : P \to [0, 1]$ be a strictly increasing function with $f(0) = 0$ and $f(1) = 1$. We have that $d_f(x, y) := 2\, f(x + y) - f(x) - f(y)$ defines a semilattice metric on L if and only if f satisfies the "exchange relation"*

$$f(x + y) + f(z) \leq f(x) + f(y + z)$$

whenever $z \leq x$.

Proof. If d_f is a semilattice metric, then f satisfies the exchange relation by adapting the proof of Proposition 3.6.5. Conversely, d_f satisfies the triangle inequality if and only if $f(x + y) + f(z) \leq f(x + z) + f(y + z)$. Since $f(x + y + z) \geq f(x + y)$ and $z \leq x + z$ for all $x \in L$, we have that the exchange relation implies the stronger inequality

$$f(x + y + z) + f(z) \leq f(x + z) + f(y + z)$$

for all $x, y, z \in L$. We have that $d_f(x + z, y + z) \leq d_f(x, y)$ if and only if

$$2\, f(x + y + z) + f(x) + f(y) \leq 2\, f(x + y) + f(x + z) + f(y + z).$$

This may be obtained from two applications of the previous inequality, permuting the variables x, y, z appropriately. □

Remark 3.24. For a semilattice $\mathcal{P}$ and a map $f : P \to \mathbb{R}$, the key inequality used in the proof of Proposition 3.23 is that for all $x, y, z \in P$

$$f(x + y + z) + f(z) \leq f(x + z) + f(y + z).$$

This property is referred to as *strong subadditivity* in the literature [9, 30]. We direct the reader to Proposition 4.17 below for a related result. For a lattice, the exchange relation and strong subadditivity are equivalent to f being submodular, that is, $f(x+y)+f(xy) \leq f(x) + f(y)$.

Lemma 3.25. *Let $\mathcal{P}$ be a pseudo-metric semilattice. Let (P', d') be the metric quotient of (P, d) and define $\mathcal{P}' = (P', +', [0], [1], d')$ where $[x] +' [y] := [x + y]$. Then $\mathcal{P}'$ is a metric semilattice.*

Proof. It suffices to check that $+' : P' \times P' \to P'$ is well defined. Letting $[x_1] = [x_2]$ and $[y_1] = [y_2]$, we have that

$$\begin{aligned} d'([x_1 + y_1], [x_2 + y_2]) &= d(x_1 + y_1, x_2 + y_2) \\ &\leq d(x_1 + y_1, x_2 + y_1) + d(x_2 + y_1, x_2 + y_2) \\ &\leq d(x_1, x_2) + d(y_1, y_2) = 0. \end{aligned}$$

Therefore $[x_1 + y_1] = [x_2 + y_2]$ and the join operation is well defined. □

Lemma 3.26. *If $\mathcal{P}$ is a pseudo-metric semilattice, then declaring $x \leq y$ if $y = x + y$ defines a partial order on P where 0 and 1 are the minimal and maximal elements, respectively, and where $x + y$ is the least upper bound of $\{x, y\}$. Additionally, $d(x, y) \leq |y| - |x|$ when $x \leq y$. (We define $|x| = d(x, 0)$ as above in the lattice case.)*

Proof. First let's prove that $\leq$ is a partial order. It is clear that for all $x \in \mathcal{P}, x \leq x$ since $+$ is idempotent, $\leq$ is reflexive. If for a pair $x, y \in \mathcal{P}, x \leq y$ and $y \leq x$ then $y = x + y = y + x = x$, so $\leq$ is antisymmetric. Lastly, for all $x, y, z \in \mathcal{P}$ if $x \leq y$ and $y \leq z$ then $z = y + z$ and $y = x + y$ therefore $z = y + z = (x + y) + z = x + (y + z) = x + z$ so $x \leq z$ and we conclude that $\leq$ is transitive. In conclusion, it is a partial order.

By definition 1.21 (4) it is clear that for all $x \in \mathcal{P}, 0 \leq x \leq 1$ thus 0 and 1 are the minimal and maximal elements, respectively. By idempotency, commutativity and associativity of $+$, we get $x + y = x + (x + y) = y + (x + y)$ thus $x, y \leq x + y$. If $x, y \leq z$, then $(x + y) + z = x + (y + z) = x + z = z$, so $x + y \leq z$. Finally if $x \leq y$ we can use the axiom (3) about d in Definition 1.21 to obtain

$$\begin{aligned} d(x, y) &\leq d(x + y, 0) + d(x + x, x) + d(y + x, y) - d(x, 0) \\ &= d(y, 0) + d(x, x) + d(y, y) - d(x, 0) = |y| - |x|. \end{aligned}$$ □

The next proposition follows in exactly the same way as Lemma 3.19 and Proposition 3.18.

Proposition 3.27. *If $\mathcal{P}$ is a complete metric semilattice, then every set $S \subset P$ has a least upper bound. If S is closed under $+$, then the least upper bound belongs to the closure of S.*

By Corollary 3.10 we have that every metric lattice is an example of a metric semilattice. We now prove a partial converse.

Proposition 3.28. *Every complete metric semilattice $\mathcal{P}$ is a complete metric lattice with $0, 1$ being the minimal and maximal element, respectively, and $+$ being the lattice join operation.*

Proof. Let us consider

$$\phi_{x,y}(z) := d(z + x, x) + d(z + y, y).$$

Then $\phi_{x,y}(z) = 0$ if and only if z is a lower bound for $\{x, y\}$. The sum of two common lower bounds for x and y is again a common lower bound, as $z + x = z' + x = x$ implies that $z + z' + x = x$. Moreover, the set of all lower bounds is closed by continuity of $\phi_{x,y}$, so the set of common lower bounds for x and y must have a maximal element by Proposition 3.27. We define xy as this element. The function $(x, y) \mapsto xy$ is clearly commutative and idempotent. We have that $t \leq (xy)z$ if and only if $t \leq xy, z$ if and only if $t \leq x, y, z$ which shows that it is also associative. The remaining parts of Definition 3.1 are easy to check. $\square$

The following result was proved in [34, Section 3]. We present a more direct approach here.

Corollary 3.29. *Let $\mathcal{L}$ be a metric lattice. We have that the metric completion $\overline{\mathcal{L}}$ is a metric lattice and the canonical inclusion $L \hookrightarrow \overline{L}$ is an embedding of metric semilattices.*

Proof. If $\mathcal{L}$ is a metric lattice, then it is also a metric semilattice. The metric completion of a semilattice is again a metric semilattice as the $+$ operation is uniformly continuous. The inclusion $L \to \overline{L}$ is isometric, so clearly order preserving. $\square$

Proposition 3.30. *Let $\mathcal{L}$ and $\mathcal{L}'$ be complete metric lattices and $\phi : \mathcal{L} \to \mathcal{L}'$ a continuous semilattice homomorphism, that is $\phi(0) = 0$, $\phi(1) = 1$, and $\phi(x + y) = \phi(x) + \phi(y)$ for all $x, y \in L$. We have that the subsemilattice $\phi(L) \subset L'$ is a complete metric lattice.*

Proof. By Proposition 3.27 we need only show that $\phi(L)$ is closed. First suppose that $(\phi(x_n))$ is an increasing sequence, so that $\phi(x_n) \to z$ for some $z \in L'$ by the proof of Lemma 3.17. Setting $y_n = \sum_{k=1}^{n} x_n$ and $y = \sum_{k=1}^{\infty} x_k$, we have that $y_n \to y$ and so $\phi(x_n) = \phi(y_n) \to \phi(y)$ by continuity. Therefore $z = \phi(y)$.

Now let $\phi(x_n) \to z$ be arbitrary. By passing to a subsequence, we may assume that $d(\phi(x_n), z) < \frac{1}{2^n}$. For $n < L$, we set $y_{n,L} = \phi(\sum_{k=n}^{L} x_k) = \sum_{k=n}^{L} \phi(x_k)$ and $y_n = \phi(\sum_{k=n}^{\infty} x_k)$ so that by the previous paragraph $y_{n,L} \to y_n$ as $L \to \infty$ and by Lemma 3.19 we have that $d(y_n + z, z) \to 0$ as $n \to \infty$.

Setting $w_n = \sum_{k=n}^{\infty} x_k \in L$, we have that (w_n) is a decreasing sequence, so has a limit $w \in L$. We have $y_n = \phi(w_n) \to \phi(w) =: y \in \phi(L)$ by continuity. Therefore, by the previous paragraph we have that $d(y + z, z) = 0$ or $y \leq z$. On the other hand, $y_n \geq \phi(x_k)$ for all $k > n$, so $y_n = y_n + \phi(x_k) \to y_n + z$ as $k \to \infty$. Thus $y = y + z$, which implies that $y \geq z$. Altogether this shows that $z \in \phi(L)$, which establishes the result. $\square$

4. Examples

In this section we detail various constructions of lattice metrics. For the case of finite Boolean lattices, there is some overlap with results obtained by Lovász in [21, section 9] which the authors were unaware of during the completion of the manuscript. However, the perspective presented here is more general and in some ways complementary to those results.

Example 4.1. For a finite set E, let $\mathcal{B}(E)$ be the lattice of subsets of E under the normalized Hamming metric $d(A,B) := \frac{|A\Delta B|}{|E|}$ for $A, B \in 2^E$.

Example 4.2. Let E be a finite set. Given x, y partitions of E we can define a partial order by $x \leq y$ if and only x refines y, that is, if for all $B \in x$, exists a $B' \in y$ such that $B \subseteq B'$. Let $\mathcal{P}(E)$ be the lattice of partitions of E with the metric

$$d(x,y) = \frac{\#x + \#y - 2\#(x+y)}{|E| - 1},$$

where for a partition $\#x$ denotes the number of blocks. When $E = [[n]] := \{1, \ldots, n\}$, we denote $P(E)$ by P_n.

Example 4.3. Let E be a finite set. We say that $r : 2^E \to \mathbb{N}_0$ is a *rank function* if the following hold for all $A, B \in 2^E$:

(1) $r(A) \leq |A|$;
(2) $A \subset B$ implies that $r(A) \leq r(B)$;
(3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$.

There is a one-to-one correspondence between rank functions on a set E and *matroids* on the same set: see [31, Section 1.3].

For a rank function define the pseudo-metric $D_r(A,B) := 2\,r(A \cup B) - r(A) - r(B)$.

Lemma 4.4. *If r is a rank function on E, then $D_r(A,B) \leq |A\Delta B|$.*

Proof. It follows from submodularity that $D_r(A,B) \leq r(A \cup B) - r(A \cap B)$. Using submodularity and disjointness, we have that

$$r(A \cup B) \leq r(A \setminus B) + r(A \cap B) + r(B \setminus A),$$

so altogether $D_r(A,B) \leq r(A \setminus B) + r(B \setminus A) \leq |A \setminus B| + |B \setminus A| = |A\Delta B|$. $\square$

Definition 4.5. We say that a rank function $r : 2^E \to \mathbb{N}_0$ is *k-sparse* if

$$\#E \leq k \cdot r(E).$$

Example 4.6. If $G = (V,E)$ is a k-regular graph and $r : 2^E \to \mathbb{N}_0$ is given by $r(A)$ is the maximal cardinality of the edges of a forest in (V,A), then r is k-sparse.

For a rank function r on E, let's define the normalized rank distance by $d_r(A,B) = D_r(A,B)/r(E)$. Let $\mathcal{L}_r(E) = (L_r(E), <, +, \cdot, d_r)$ be the quotient metric lattice associated to the pseudo-metric lattice $(2^E, d_r)$. Since 2^E is a finite lattice, the fact that $\mathcal{L}_r(E)$ is a metric lattice follows from Lemma 3.25 and Proposition 3.28. The lattice $\mathcal{L}_r(E)$ is referred to as the *lattice of flats* of the associated matroid. The following proposition is evident.

Proposition 4.7. *For a rank function r on E which is k-sparse, we have that quotient map $q : 2^E \to L_r(X)$ induces a surjective, k-Lipschitz metric semilattice homomorphism $\mathcal{B}(E) \to \mathcal{L}_r(E)$.*

Definition 4.8. Let $\mathcal{P}$ be a semilattice. We say that a function $\rho : P \to \mathbb{R}$ is *positive definite* (respectively, *positive semidefinite*) if for all $x_1, \ldots, x_n \in P$, the matrix

$$R_{ij} := [\rho(x_i + x_j)]$$

is positive definite (resp., positive semidefinite).

Example 4.9. For $x \in P$, define $\rho_x : P \to \{0, 1\}$ by $\rho_x(y) = 1$ if $y \leq x$ and $\rho_x(y) = 0$ otherwise. Then ρ_x is positive semidefinite.

The next lemma begins to show how positive (semi)definiteness imposes strong restrictions on the function ρ.

Lemma 4.10. *If $\rho : P \to \mathbb{R}$ is positive semidefinite then ρ is positive and nonincreasing, with ρ being strictly decreasing if ρ is positive definite.*

Proof. We have $\rho(x) = \rho(x + x) \geq 0$ by positive definiteness. If $y > x$, then the matrix

$$\begin{bmatrix} \rho(x) & \rho(y) \\ \rho(y) & \rho(y) \end{bmatrix}$$

is positive semidefinite, which implies that $\rho(x) \geq \rho(y)$ with strict inequality in the positive definite case. $\square$

Let $\mathcal{P} = (P, +, 0, 1)$ be a complete semilattice (thus, every subset of P has a least upper bound). We define the *order topology* on P to be the topology generated by the collection of closed sets

$$C_x := \{y \in P : x + y = y\} = \{y \in P : y \geq x\}$$

for each $x \in L$. Let $\mathcal{C}$ be the σ-algebra generated by the sets $\{C_x : x \in P\}$.

Example 4.11. If μ is a measure on $(P, \mathcal{C})$ then $\rho(x) := \mu(C_x)$ is positive semidefinite since, using that $x + y$ is the least upper bound of $\{x, y\}$,

$$\rho(x + y) = \mu(C_{x+y}) = \mu(C_x \cap C_y) = \int 1_{C_x} 1_{C_y} \, d\mu.$$

Indeed, choosing $x_1, \ldots, x_n$ in P and $g_1, \ldots, g_n \in \mathbb{R}$. Setting $g = \sum_{i=1}^n g_i 1_{C_{x_i}}$, we have that

$$\sum_{i,j} g_j g_j \rho(x_i + x_j) = \int \sum_{i,j} g_i g_j 1_{C_x} 1_{C_y} \, d\mu = \int g^2 \, d\mu \geq 0.$$

If $\mathcal{P}$ is a finite semilattice, then for every function $f : P \to \mathbb{R}$, there is a unique function $f^\mu : P \to \mathbb{R}$ so that

$$f(x) = \sum_{y \in C_x} f^\mu(y).$$

The function f^μ is known as the *Möbius inversion* of f. We refer the reader to [1, Section IV.2] for a detailed treatment of this topic. Setting $n = |P|$, we fix an enumeration $x_1, x_2, \ldots, x_n$ of P so that $x_i \leq x_j$ only if $i \leq j$. We define the $n \times n$ *zeta matrix* of $\mathcal{P}$ by

$$\zeta_{ij} := \delta_{x_j, x_i + x_j}$$

where δ_{ab} is the Kronecker delta. That is, $\zeta_{ij} = 1$ if $x_i \le x_j$ and 0 otherwise. Since ζ is upper triangular with ones on the main diagonal, it is invertible. For a sequence $a = (a_1, \ldots, a_n)$, let D_a be the diagonal matrix with entries $[D_a]_{ii} = a_i$. The following identity is due independently to Lindström [18] to Wilf [35]:

$$[f(x_i + x_j)]_{i,j=1}^n = \zeta D_{f^\mu} \zeta^t. \tag{15}$$

The following result which provides a converse to Example 4.11 was first observed by Mattila and Haukkanen in the positive definite case [24]. We give a simple proof that extends to the positive semidefinite case as well.

Proposition 4.12. *Let $\mathcal{P}$ be a finite semilattice. We have that $\rho : P \to \mathbb{R}$ is positive semidefinite (resp., positive definite) if and only if there is a measure m_ρ on P (resp., a measure of full support on P) so that*

$$\rho(x) = m_\rho(C_x).$$

Proof. It is a standard fact that if $A \in M_n(\mathbb{R})$ is positive semidefinite, then BAB^t is as well for any $B \in M_n(\mathbb{R})$, with the same holding for positive definiteness in the case that B is invertible. Thus $\zeta^{-1}[\rho(x_i + x_y)](\zeta^{-1})^t = D_{\rho^\mu}$ is a positive (semi)definite diagonal matrix by equation (15). This is equivalent to $\rho^\mu \ge 0$ in the positive semidefinite case and $\rho^\mu > 0$ in the positive definite case. For $A \subseteq P$, we then define $m_\rho(A) := \sum_{x \in A} \rho^\mu(x)$. The result now follows by Móbius inversion. $\square$

Remark 4.13. If $\mathcal{P}$ is a complete metric lattice which is first countable, then standard sampling theory arguments can be used to show that the same is true for any continuous positive semidefinite function with regard to a σ-finite Radon measure on $\mathcal{C}$.

Example 4.14. Let $K \in M_n(\mathbb{C})$ be an invertible Hermitian matrix with spectrum in $(0, 1]$. And let $L = 2^{[[n]]}$ be the lattice of subsets of $[[n]] = \{1, \ldots, n\}$. By a result of Lyons [23] there is a measure μ_K on L so that

$$\mu_K(C_S) = \det(K_S)$$

for all $S \subseteq [[n]]$ where K_S is the principal minor of K given by deleting all entries in rows or columns not indexed by S. Note that $1 = \mu(L) = \mu(C_\emptyset) = \det(K_\emptyset)$ by convention.

In light of the previous proposition, it would be interesting to find a direct proof of positive semidefiniteness of $S \mapsto \det(K_S)$ which avoids explicitly constructing the measure.

Definition 4.15. We say that a function $\eta : P \to \mathbb{R}$ is *conditionally negative definite* if

$$\sum_{i,j} c_i c_j \cdot \eta(x_i + x_j) \le 0$$

when $\sum_i c_i = 0$ for all $x_1, \ldots, x_n \in P$.

Remark 4.16. By Schoenberg's theorem [32, Proposition 11.2], η is conditionally negative definite if and only if $x \mapsto \exp(-t\eta(x))$ is positive semidefinite for all $t \ge 0$. Therefore, we have that every conditionally negative definite function is increasing.

For η conditionally negative definite, define

$$d_\eta(x,y) := 2\eta(x+y) - \eta(x) - \eta(y),$$

which is again conditionally negative definite with $d_\eta(x,x) = 0$ and $d_\eta(x,y) \geq 0$ for all $x, y \in P$. We note that

$$|x|_\eta := d_\eta(x,0) = \eta(x) - \eta(0) \geq 0$$

is again conditionally negative definite and that

$$d_\eta(x,y) = 2|x+y|_\eta - |x|_\eta - |y|_\eta = d'_\eta(x,y).$$

Proposition 4.17. *For η conditionally negative definite we have that $(\mathcal{P}, d_\eta)$ is a pseudo-metric semilattice. Moreover, d_η is a semilattice metric on $\mathcal{P}$ if and only if* $\exp(-\eta)$ *is positive definite.*

Proof. We write $|x|$ for $|x|_\eta$ and $d(x,y)$ for $d_\eta(x,y)$ for brevity. If $\exp(-\eta)$ is positive definite, then $|x|$ is strictly increasing, from which it easily follows that $d(x,y) > 0$ if $x \neq y$.

By Proposition 3.23 and Remark 3.24, we need only check that strong subadditivity holds, that is,

$$|x+y+z| + |z| \leq |x+z| + |y+z|.$$

Consider the matrix

$$K := \begin{bmatrix} |x+z| & |x+y+z| & |x+z| & |x+y+z| \\ |x+y+z| & |y+z| & |y+z| & |x+y+z| \\ |x+z| & |y+z| & |z| & |x+y+z| \\ |x+y+z| & |x+y+z| & |x+y+z| & |x+y+z| \end{bmatrix}$$

which is $|a_i + a_j|$ where $a_1 = x+z$, $a_2 = y+z$, $a_3 = z$, and $a_4 = x+y+z$. Since $|x|$ is conditionally negative definite we have $\xi K \xi^t \leq 0$ where $\xi = [1, 1, -1, -1]$. We calculate

$$\begin{aligned} \xi K \xi^t &= \\ &= |x+z| + |x+y+z| - |x+z| - |x+y+z| \\ &+ |x+y+z| + |y+z| - |y+z| - |x+y+z| \\ &- |x+z| - |y+z| + |z| + |x+y+z| \\ &- |x+y+z| - |x+y+z| + |x+y+z| + |x+y+z| \\ &= -|x+z| - |y+z| + |z| + |x+y+z|, \end{aligned}$$

which proves the assertion. ∎

Example 4.18. If $\rho : P \to \mathbb{R}$ is positive semidefinite, then $x \mapsto a - \rho(x)$ for $a \in \mathbb{R}$ is clearly conditionally negative definite. Thus, if $(\Omega, \mathcal{B}, \mu)$ is a probability space, then $X \mapsto \mu(X^c)$ is positive semidefinite, and $X \mapsto \mu(X) = 1 - \mu(X^c)$ is conditionally negative definite.

Accordingly, let $(\Omega, \mathcal{B})$ be a Borel space, and let Φ be a finite measure on $(\Omega\times\Omega, \mathcal{B}\otimes\mathcal{B})$. The map $\rho_\Phi : \mathcal{B} \ni X \mapsto \Phi(X^c \times X^c)$ is positive semidefinite, since

$$\rho_\Phi(X \cup Y) = \Phi\left((X \cup Y)^c \times (X \cup Y)^c\right) = \Phi((X^c \times X^c) \cap (Y^c \times Y^c)).$$

Therefore,

$$\eta_\Phi : \mathcal{B} \ni X \mapsto \Phi(\Omega \times \Omega) - \rho_\Phi(X) = \Phi(\Omega \times X) + \Phi(X \times X^c)$$

is conditionally negative definite and $\exp(-t\eta_\Phi)$ is positive semidefinite for $t \geq 0$. Moreover, we can see that $d_\Phi(X, Y) = \eta_\Phi(X \cup Y) - \eta_\Phi(X) - \eta_\Phi(Y)$ gives the lattice of Borel sets the structure of a pseudo-metric pre-lattice.

Question 4.19. In [21, section 9] it is shown that for finite Boolean lattices conditional negative definiteness coincides with a sequence of inequalities which strengthen submodularity first considered by Choquet [9]. These inequalities already imply, in the metric case, that elements of the lattice are uniquely complemented: see Remark 5.14 below. It would be interesting to determine if there is a similar characterization of conditionally negative definite functions on semilattices in general.

Let $\mathcal{P}$ be a finite semilattice and let $x_1, \ldots, x_n$ be an enumeration of the elements of P such that $x_i \leq x_j$ only if $i \leq j$. Given a function $f : P \to \mathbb{R}$, we say that a subset $S \subseteq [[n]]$ is a *chain* if $\{x_i : i \in S\}$ is totally ordered. We say that an $n \times n$ matrix A is *totally $\mathcal{P}$-non-negative* if for all chains S, T with $|S| = |T|$ we have that $\det(A_{S,T}) \geq 0$, where $A_{S,T}$ is the submatrix of A consisting of the rows S and columns T. If $\mathcal{P} = [[n]]$ with $i + j = \max\{i, j\}$, then total $\mathcal{P}$-positivity corresponds with the usual notion of k-total non-negativity of matrices [2].

We say that $f : P \to \mathbb{R}_{\geq 0}$ is *totally $\mathcal{P}$-non-negative* if the associated matrix

$$[f(x_i + x_j)]_{i.j=1}^n$$

is totally $\mathcal{P}$-non-negative. We note that even though such a matrix is symmetric, it is not clear that a variant of Sylvester's criterion would apply to show that a totally $\mathcal{P}$-non-negative matrix is positive semidefinite. However, considering the 3×3 principal submatrix corresponding to a triple $z \leq x, y$,

$$\begin{bmatrix} f(z) & f(x) & f(y) \\ f(x) & f(x) & f(x+y) \\ f(y) & f(x+y) & f(y) \end{bmatrix},$$

We see that total $\mathcal{P}$-non-negativity implies that f is non-increasing and consequently f satisfies the celebrated "FKG inequality" [12]

$$f(x + y)f(z) \geq f(x)f(y)$$

whenever $z \leq x, y$. Note that the determinant of the 2×2-upper right block is necessarily non-positive if f is non-increasing, but is not considered unless $x < y$, in which case it is 0. Evidently, if f is the characteristic function of $P \setminus C_x$ for any $x \in P$, then f is totally non-negative as the non-zero entries are constant and form a rectangle in the upper left corner of the associated matrix.

It would be interesting to completely characterize such matrices. For $\mathcal{P} = [[n]]$ with $i+j = \max\{i,j\}$ this is true for all $f : [[n]] \to \mathbb{R}_{\geq 0}$ which are non-increasing [2, example 7.I(f)]. However, the example of $2^n \ni S \mapsto \det(K_S)$ above shows that this is false in general as $\det(K_{S\cup T})\det(K_{S\cap T}) \leq \det(K_S)\det(K_T)$ for all matrices K that are positive semidefinite by Kotelyanskii's inequality [16, Theorem 7.8.9].

5. Model Theoretic Aspects

We now turn to studying the model theory of metric lattices. We define a language $\mathfrak{L}$ as follows.

(1) There is a single sort (X,d) consisting of a complete metric space of diameter 1.
(2) There are constant symbols 0 and 1.
(3) A function symbol $+ : X \times X \to X$ which is a contraction where $X \times X$ is equipped with the ℓ^1-metric, $d_1((x,y),(w,z)) = d(x,w) + d(y,z)$.

For axioms we require the following collection of sentences T_{ML}:

(1) $d(0,1) - 1$;
(2) $\sup_x d(x+0,x) + d(x+1,1)$;
(3) $\sup_x d(x+x,x)$;
(4) $\sup_{x,y} d(x+y,y+x)$;
(5) $\sup_{x,y,z} d((x+y)+z, x+(y+z))$;
(6) $\sup_{x,y,z}(d(x+z,y+z) \dot{-} d(x,y))$;
(7) $\sup_{x,y,z}([d(x,y)+d(z,0)] \dot{-} [d(x+y,0)+d(x+z,x)+d(y+z,y)])$.

Clearly, every model of T_{ML} is complete metric semilattice. The following proposition summarizes the main results of section 3.3.

Proposition 5.1. *We have that $\mathfrak{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are models of T_{ML} if and only if $\mathcal{M}$ and $\mathcal{N}$ are complete metric lattices. Furthermore, the $\mathfrak{L}$-structure homomorphisms $\phi : \mathcal{M} \to \mathcal{N}$ are exactly the contractive maps so that $\phi(0) = 0$, $\phi(1) = 1$, and $\phi(x+y) = \phi(x) + \phi(y)$ for all $x, y \in \mathcal{M}$ and $\phi(\mathcal{M}) \subseteq \mathcal{N}$ is a model of T_{ML}.*

Proof. This follows directly from Propositions 3.28 and 3.30. □

Consider the sentence

$$\sigma_{mod} := \sup_{x,y}\inf_{z} \max\{(|x|+|y|) \dot{-} (|x+y|+|z|), d(x+z,x), d(y+z,y)\}. \tag{16}$$

and the theory

$$T_{MML} := T_{ML} \cup \{\sigma_{mod}\}. \tag{17}$$

Proposition 5.2. *We have that an $\mathfrak{L}$-structure $\mathcal{M}$ models T_{MML} if and only if it is an metrically modular complete metric lattice.*

Proof. This follows directly from Proposition 3.20. □

Our objective will be to show that the *meet* operation is definable in T_{MML}, which in our context means that we need to prove that the predicate $P(x,y,z) := d(xy,z)$ is definable.

Lemma 5.3. *If $\mathcal{M}$ is a metric lattice and (x,z) and (y,z) are metrically modular pairs, then $d'(xz,yz) \leq d'(x,y)$ and $d(xz,yz) \leq 2\,d(x,y)$.*

Proof. We follow the proof given in [33, Lemma 21]. Let's consider $z(x+y)$ and zy, by Proposition 2.6.4 we get that $|z(x+y)| \leq |z| + |x+y| - |z+x+y|$. Since (z,y) is a metrically modular pair $|zy| = |z| + |y| - |z+y|$ it follows that

$$\begin{aligned}|z(x+y)| - |zy| &\leq |z| + |x+y| - |z+x+y| - |z| - |y| + |z+y| \\ &= |x+y| - |y| + |z+y| - |z+x+y| \leq |x+y| - |y|.\end{aligned}$$

Using a similar process we can show that $|z(x+y)| - |zx| \leq |x+y| - |x|$, hence

$$2|z(x+y)| - |zx| - |zy| \leq 2|x+y| - |x| - |y| = d'(x,y).$$

Now observe that as $zx, zy \leq z(x+y)$, we have $zx + zy \leq z(x+y)$, thus $|zx+zy| \leq |z(x+y)|$. Therefore $d'(zy,zx) = 2|zx+zy| - |zx| - |zy| \leq d'(x,y)$. This proves the first part of the lemma.

For the second part we use Prop 2.6.7 to get that

$$d(xz,yz) \leq d'(xz,yz) \leq d'(x,y) \leq 2\,d(x,y).$$ □

Remark 5.4. Let

$$\varphi(x,y) := \inf_z \max\{(|x|+|y|) \dot{-} (|x+y|+|z|), d(x+z,x), d(y+z,y)\}, \tag{18}$$

noting that for complete metric lattices $\varphi(x,y) = 0$ is equivalent to (x,y) being a metrically modular pair by Proposition 3.20. In light of the previous lemma, it is natural to ask whether $\varphi(x,z), \varphi(y,z) < \epsilon$ implies that $d(xz,yz) \leq 2d(x,y) + K\epsilon$ for some fixed constant K. The answer turns out to be 'no'. Indeed, we will show that for every K there is a finite partition lattice $P_\bullet$, a real number $\varepsilon > 0$, and partitions $x,y,z \in P_\bullet$ such that $\varphi(x,z), \varphi(y,z) < \varepsilon$ and $d(xz,yz) > 2d(x,y) + K\varepsilon$.

Given K, take a natural number $n > K+2$, set $\varepsilon = \frac{1}{3n-2}$, and consider the finite partition lattice of the set $\{1,2,3,...,3n\}$, which we will note as P_{3n}. Here the norm of a partition x is $|x| := \frac{3n - \#x}{3n-1}$ where $\#x$ is the number of blocks in the partition x, and the distance of two partitions is $d(x,y) := 2|x+y| - |x| - |y| = d'(x,y)$.

Consider the following partitions

$$\begin{aligned} x &:= \{\{1,2,3,...,2n\},\{2n+1,...,3n\}\} \\ y &:= \{\{1,2,3,...,n\},\{n+1,n+2,...,3n\} \\ z &:= \{\{i,i+n,i+2n\}\}_{i=1}^n. \end{aligned}$$

We compute the partitions $x+z, y+z, x+y, xz, yz$ and $xz+yz$ which we will need for later.

$$\begin{aligned} x+z &= \{\{1,2,...,3n\}\} = y+z = x+y \\ xz &= \{\{i,i+n\}\}_{i=1}^n \cup \{\{j\}\}_{j=2n+1}^{3n} \\ yz &= \{i\}_{i=1}^n \cup \{\{j,j+n\}\}_{j=n+1}^{2n} \\ xz+yz &= z. \end{aligned}$$

It is clear that $|x+z| = |y+z| = |x+y| = 1$. Further,

$$|x| = |y| = \frac{3n-2}{3n-1}, \quad |z| = \frac{2n}{3n-1}, \quad |xz| = |yz| = \frac{n}{3n-1}.$$

Now, let's check that $\varphi(x,z), \varphi(y,z) < \varepsilon$. Observe that $|x| + |z| \leq |x+z| + |z|$,

$$|x+z| - |x| = 1 - \frac{3n-2}{3n-1} = \frac{1}{3n-1},$$

and $|z+z| - |z| = 0$, thus $\varphi(x,z) < \frac{1}{3n-1} < \epsilon$. Similarly $\varphi(y,z) < \varepsilon$. However, we have that

$$d(xz, yz) = 2|xz + yz| - |xz| - |yz| = \frac{4n}{3n-1} - \frac{n}{3n-1} - \frac{n}{3n-1} = \frac{2n}{3n-1}$$

and

$$\begin{aligned} 2d(x,y) + K\varepsilon &= 2(2|x+y| - |x| - |y|) + K\varepsilon \\ &= 2\left(2 - \frac{3n-2}{3n-1} - \frac{3n-2}{3n-1}\right) + K\varepsilon \\ &= \frac{4}{3n-1} + \frac{K}{3n-2}. \end{aligned}$$

Since $n > K+2$ we conclude that

$$d(xz, yz) = \frac{2n}{3n-1} > \frac{4}{3n-1} + \frac{2K}{3n-1} > \frac{2}{3n-1} + \frac{K}{3n-2} = 2d(x,y) + K\varepsilon.$$

In general picking $n > K + S$ we can use this counterexample to get that $\varphi(x,z) < \varepsilon, \varphi(y,z) < \varepsilon$ and $d(xz, yz) > Sd(x,y) + K\varepsilon$.

Lemma 5.5. *Let $\{\mathcal{M}_i : i \in I\}$ be a collection of metrically modular complete metric lattices, and U a non-principal ultrafilter on I. Let $\mathcal{M} = \prod_{i \in U} \mathcal{M}_i$ be the ultraproduct. For $x, y \in \mathcal{M}$ with representatives $x = (x_i)$ and $y = (y_i)$ we have that $xy = (x_i y_i)$.*

Proof. Clearly $x_i y_i \leq x_i, y_i$, hence $(x_i y_i) \leq x, y$ and $(x_i y_i) \leq xy$. In the other direction, suppose that $z \leq x, y$ with representative $z = (z_i)$. We therefore have that $d_i(x_i + z_i, x_i) \to 0$ and $d(y_i + z_i, y_i) \to 0$ as $i \to U$. By metric modularity, we have that

$$d_i(x_i z_i, z_i) = |z_i| - |x_i z_i| = |x_i + z_i| - |x_i| = d_i(x_i + z_i, z_i)$$

with the corresponding equations holding for $d_i(y_i z_i, z_i)$. By Lemma 5.3 and the triangle inequality

$$d_i(x_i y_i z_i, z_i) \leq d_i(x_i y_i z_i, y_i z_i) + d_i(y_i z_i, z_i) \leq 2(d_i(x_i z_i, z_i) + d_i(y_i z_i, z_i)) \to 0$$

as $i \to U$. Therefore $z = (x_i y_i z_i) \leq (x_i y_i)$ so $xy \leq (x_i y_i)$. $\square$

In order to show the next result we will invoke the Beth Definability Theorem which we discussed in the preliminaries.

Proposition 5.6. *The predicate $P(x,y,z) = d(xy, z)$ is definable for the elementary class* $\mathrm{Mod}(T_{MML})$ *of metrically modular complete metric lattices.*

Proof. First, we prove that for all $\mathcal{M} \in \mathrm{Mod}(T_{MML})$ the predicate $P^{\mathcal{M}}(x, y, x) := d(xy, z)$ satisfies the conditions of Corollary 2.3. Observe that given $x, y, z, w \in \mathcal{M}$ we can apply the Lemma 5.3 plus the triangle inequality to get

$$d(xy, zw) \leq d(xy, yz) + d(yz, zw) \leq 2d(x, z) + 2d(y, w).$$

Therefore the meet operation is uniformly continuous. Now note that if we take

$$x_1, x_2, y_1, y_2, z_1, z_2 \in \mathcal{M},$$

using the triangle inequality we obtain

$$d(x_1y_1, z_1) \leq d(x_1y_1, z_2) + d(z_1, z_2) \leq d(x_1y_1, x_2y_2) + d(x_2y_2, z_2) + d(z_1, z_2)$$

and

$$d(x_2y_2, z_2) \leq d(x_2y_2, z_1) + d(z_1, z_2) \leq d(x_1y_1, x_2y_2) + d(x_1y_1, z_1) + d(z_1, z_2),$$

therefore

$$|d(x_1y_1, z_1) - d(x_2y_2, z_2)| \leq d(x_1y_1, x_2y_2) + d(z_1, z_2).$$

Finally we have that

$$|P^{\mathcal{M}}(x_1, y_1, z_1) - P^{\mathcal{M}}(x_2, y_2, z_2)| \leq 2\left(d(x_1, x_2) + d(y_1, y_2)\right) + d(z_1, z_2).$$

Since the predicates $P^{\mathcal{M}}$ are thus uniformly continuous with common modulus of continuity for every $\mathcal{M} \in \mathrm{Mod}(T_{MML})$, we can take the class

$$\mathcal{C}' := \{(\mathcal{M}, P^{\mathcal{M}}) : \mathcal{M} \in \mathrm{Mod}(T_{MML})\}$$

of structures for our expanded language with a predicate for P, which is interpreted in each $\mathcal{M} \in \mathrm{Mod}(T_{MML})$ as $P^{\mathcal{M}}$. The class $\mathcal{C}'$ is axiomatizable by the following theory in our expanded language

$$T' := T_{ML} \cup \left\{ \sup_x \sup_y \left(d(0, x+y) + P(x, y, 0) - d(x, 0) - d(y, 0)\right) \right\},$$

so by Corollary 2.3 we conclude that the meet function is axiomatizable in the theory of metrically modular complete metric lattices. □

Since the predicate $d(xy, z)$ is T_{MML}-equivalent to a definable predicate or formula in $\mathfrak{L}$ (due to the general definition of formula we are using) we know there is formula ϕ_m such that for every model $\mathcal{M}$ of T_{MML} and all $x, y, z \in M$, $\phi_m(x, y, z) = d(xy, z)$. Using this we can axiomatize the theory of distributive metrically modular complete metric lattices.

To do so, we define the sentence

$$\sigma_{dist} := \sup_{x,y,z} \inf_{t,w} \max\{\phi_m(x, y, t), \phi_m(x, z, w), \phi_m(x, y+z, t+w)\} \tag{19}$$

and consider the theory

$$T_{DML} := T_{MML} \cup \{\sigma_{dist}\} = T_{ML} \cup \{\sigma_{mod}, \sigma_{dist}\}. \tag{20}$$

Proposition 5.7. *Let $\mathcal{M}$ be a $\mathfrak{L}$-structure then $\mathcal{M}$ models T_{DML} of and only if it is a distributive metrically modular complete metric lattice.*

Proof. It is clear that if $\mathcal{M}$ is distributive metrically modular complete metric lattice, then it is a model of T_{DML}. Now let's suppose that $\mathcal{M}$ is a model of T_{DML}. By Proposition 5.2 we know that $\mathcal{M}$ is a metrically modular complete metric lattice, thus we just need to prove that $\mathcal{M}$ has the distributive property.

Let us fix $x, y, z \in \mathcal{M}$. Since $\mathcal{M}$ models T_{DML} it satisfies σ_{dist}, so for every $n \in \mathbb{N}$ there are a t_n and a w_n such that

$$\max\{d(xy, t_n), d(xz, w_n), d(x(y+z), t_n + w_n)\} < \frac{1}{n}.$$

Therefore $t_n \to xy$, $w_n \to xz$, and $t_n + w_n \to xy + xz$ as $n \to \infty$, where the first two limits are immediate and the last follows from these by uniform continuity of the join operation (Remark 3.3). Finally we observe that for every $n \in \mathbb{N}$

$$d(x(y+z), xz + xz) \leq d(x(y+z), t_n + w_n) + d(t_n + w_n, xy + xz),$$

thus $d(x(y+z), xy + xz) = 0$, or equivalently $x(y+z) = xy + xz$. This implies that $x + yz = (x+y)(x+z)$, thus $\mathcal{M}$ is a distributive metrically modular complete metric lattice. □

Definition 5.8. For $\mathcal{L}$ a lattice, we say that $x \in \mathcal{L}$ is *complemented* if there exist a $y \in \mathcal{L}$ such that $x + y = 1$ and $xy = 0$. We refer to y as a complement of x. We say that $\mathcal{L}$ is a *complemented lattice* if every element of $\mathcal{L}$ is complemented.

Lemma 5.9. *Let $\mathcal{L}$ be a metric lattice, and fix $x \in L$. If $d'(x, y) = 1$ then y is a complement of x. If $y \in L$ is metrically modular, then the converse is also true.*

Proof. If $d'(x, y) = 1$ then $1 = d'(x, y) = 2|x + y| - |x| - |y| \leq |x + y| - |xy|$, thus $1 + |xy| \leq |x + y| \leq 1$. We conclude that $|x + y| = 1$ and $|xy| = 0$, therefore $x + y = 1$ and $xy = 0$.

Since y is metrically modular, then $d'(x, y) = |x+y| - |xy|$. Thus if y is a complement of x, $d'(x, y) = |x + y| - |xy| = 1 - 0 = 1$. □

Remark 5.10. For the converse statement in the previous result, the assumption of metric modularity is necessary. To see this, consider the finite partition lattice of a set with 4 elements, P_4, and the partitions

$$x := \{\{1, 2\}, \{3, 4\}\}, \quad y := \{\{1, 3\}, \{2, 4\}\}.$$

Since $x + y = 1, xy = 0, |x|, |y| = \frac{4-2}{3} = \frac{2}{3}$, and $|x + y| + |xy| = 1 < \frac{4}{3} = |x| + |y|$, we observe that P_4 is not metrically modular and y is a complement of x. However, $d(x, y) = 2|x + y| - |x| - |y| = 2 - \frac{2}{3} - \frac{2}{3} = \frac{2}{3} < 1$.

For the following result we need this lemma.

Lemma 5.11. *Let $\mathcal{M}$ be a distributive metrically modular complete metric lattice. If (x_i) is a sequence in $\mathcal{M}$, then setting $z = \Pi_i x_i$ we have that*

$$d(y + z, 1) \leq 2 \sum_i d(y + x_i, 1)$$

and

$$d'(y+z,1) \leq \sum_i d'(y+x_i,1)$$

if the sum converges.

Proof. Setting $z_n = \Pi_{i=1}^n x_i$, we have that (z_n) is a decreasing sequence, so (z_n) is Cauchy and $z_n \to z$ by the proof of Proposition 3.18. Using Lemma 5.3 and the distributive property we have inductively that

$$\begin{aligned}
d(z_n+y,1) &= d(z_{n-1}x_n+y,1) \\
&= d((z_{n-1}+y)(x_n+y),1) \\
&\leq d((z_{n-1}+y)(x_n+y),z_{n-1}+y) + d(z_{n-1}+y,1) \\
&\leq d(z_{n-1}+y,1) + 2d(x_n+y,1) \leq 2\sum_{i=1}^n d(x_i+y,1)
\end{aligned}$$

hence, the result obtains in the limit by continuity. The argument works the same with d' replacing d. ☐

Definition 5.12. Let $\mathcal{L}$ be a metric lattice. We say that $x \in L$ is *weakly complemented* if $\sup_y d'(x,y) = 1$. We say that $\mathcal{L}$ is *weakly complemented* if $\inf_x \sup_y d'(x,y) = 1$.

Proposition 5.13. *Let $\mathcal{M}$ be a distributive, metrically modular complete metric lattice. We have that $x \in \mathcal{M}$ is weakly complemented if and only if x is complemented. Since we are in a distributive lattice the complement is unique.*

Proof. We know that complemented implies weakly complemented by Lemma 5.9, so we only need to prove the converse.

Since $x \in \mathcal{M}$ is weakly complemented we can choose for every $i \in \mathbb{N}$ a $z_i \in \mathcal{M}$ such that $1-2^{-i} < d'(x,z_i) \leq 1$. Given that for every metric lattice the inequality $d'(x,y) \leq |x+y| - |xy| \leq 1$ holds, we can conclude that for every $i \in \mathbb{N}, 1-2^{-i} < |x+z_i| - |xz_i|$, thus

$$0 \leq d'(1,x+z_i) = 1-|x+z_i| < 2^{-i} - |xz_i| = 2^{-i} - d'(xz_i,0) \leq 2^{-i}.$$

Therefore for every $n \in \mathbb{N}$, we have that $d'(x+z_n,1)$, $d'(xz_n,0) < 2^{-n}$.

We define $t_k^N := \Pi_{i=k}^N z_i$ and $t_k := \Pi_{i=k}^\infty z_i$, noting that $t_k^N \to t_k$ as $N \to \infty$ by the proof of Proposition 3.18. For $k < l$ and $M > N$ we have that $d'(t_k^M t_l^N, t_k^M) = 0$, so $\lim_{M\to\infty} d'(t_k^M t_l^N, t_k^M) = d'(t_k t_l^N, t_k) = 0$ when $k < l$ by Lemma 5.3. Now, taking the limit as N tends to ∞, we have that $d'(t_k t_l, t_k) = 0$ when $k < l$, thus (t_k) is increasing, therefore Cauchy. Set $t = \lim_k t_k$, from $t \geq t_k$ and Lemma 5.11 we observe that for every $k \in \mathbb{N}$ we have that

$$d'(x+t,1) \leq d'(x+t_k,1) \leq \sum_{i=k}^\infty d'(x+z_i,1) < \sum_{i=k}^\infty 2^{-i}.$$

Using that $t_k \leq z_k$ for every k we get

$$d'(xt,0) \leq d'(xt,xt_k) + d'(xt_k,0) \leq d'(t,t_k) + d'(xz_k,0).$$

In either case the right side of the inequality tends to 0 as k tends to ∞, therefore we conclude that $d(x+t,1)=d(xt,0)=0$. In other words $x+t=1$ and $xt=0$, which shows that t is a complement of x. To prove uniqueness suppose t' is another complement of x. Using the distributive property we have that

$$t = 1t = (x+t')t = xt + t't = t't = t't + xt' = t'(x+t) = 1t' = t',$$

and the result obtains. □

Remark 5.14. Consider the inequality

$$|x+y+z| + |x| + |y| + |z| \geq |x+y| + |x+z| + |y+z|,$$

which we may rephrase as $2|x+y+z| \geq d'(x,z) + d'(y,z) + d'(x,y)$. It is easy to see that this implies that if both x and y complement z, then $x=y$.

Uniqueness of complements does not hold for metric lattices in general, as witnessed by the partition lattice P_3. In this case the inequality also fails to hold. Consider $x_1 := \{\{1\},\{2,3\}\}, x_2 := \{\{2\},\{1,3\}\}$ and $x_3 := \{\{3\},\{1,2\}\}$, observe that $|x_1| = |x_2| = |x_3| = \frac{1}{2}$ and $|x_1 + |x_2| = |x_1 + x_3| = |x_2 + x_3| = |x_1 + x_2 + x_3| = 1$. Therefore,

$$|x_1 + x_2 + x_3| + |x_1| + |x_2| + |x_3| = \frac{5}{2} < 3 = |x_1 + x_2| + |x_2 + x_3| + |x_3 + x_1|.$$

Question 5.15. For each $S \subseteq [[n]]$, we write $x_S := \sum_{i \in S} x_i$, with $x_\emptyset := 0$. For which lattices does the "inclusion/exclusion" inequality hold that

$$\sum_{S \in 2^{[[n]]}} (-1)^{|S|} |x_S| \geq 0?$$

Remark 5.16. A lattice which is distributive and complemented is referred to as a *Boolean algebra.* Thus it follows from the last result that a distributive metrically modular metrically complemented complete metric lattice is a Boolean algebra.

Notation 5.17. For a Boolean algebra $\mathcal{A}$ we will denote the (unique) complement of $x \in \mathcal{A}$ by x^c.

Lemma 5.18. *Let $\mathcal{M}$ be a distributive, metrically modular complete metric lattice. For every $x, y \in \mathcal{M}$ we have that $d'(x^c, y^c) = d'(x,y)$.*

Proof. By Remark 5.16 we know that $\mathcal{M}$ is a Boolean algebra. For $x \in \mathcal{M}$ we observe by metric modularity that $1 = |x + x^c| - |xx^c| = |x| + |x^c|$. Therefore, by metric modularity and De Morgan's laws we get that

$$\begin{aligned} d'(x,y) &= |x+y| - |xy| = (1-|xy|) - (1-|x+y|) = \\ &= |(xy)^c| - |(x+y)^c| = |x^c + y^c| - |x^c y^c| = d'(x^c, y^c). \end{aligned}$$ □

Now let's consider the sentences

$$\sigma_{wcom} := \inf_x \sup_y (1 - d(x,y)) \tag{21}$$

and

$$\sigma_{d=d'} := \sup_{x,y} |d(x,y) - d'(x,y)| \tag{22}$$

and the theory

$$T_{BML} := T_{DML} \cup \{\sigma_{wcom}, \sigma_{d=d'}\}. \tag{23}$$

Next, we will prove that this theory and the following L^{pr}-theory PR found in [5, Section 4] are equivalent.

Definition 5.19. The continuous signature L^{pr} is defined as follows.

- μ as an unitary predicate symbol with modulus of uniform continuity $\Delta_\mu(\varepsilon) = \varepsilon$.
- $\cap$ and $\cup$ as binary operations with modulus of uniform continuity $\Delta_\cap(\varepsilon) = \Delta_\cup(\varepsilon) = \frac{\varepsilon}{2}$. $\cdot^c$ as an unitary operation symbol with modulus of uniform continuity $\Delta_c(\varepsilon) = \varepsilon$.
- 1 and 0 as constant symbols.

The L^{pr}-theory is formed by the following L^{pr}-conditions:

- Boolean algebra axioms: As stated in [5, Section 4], each one of these is the $\forall$-closure of an equation between terms, so it can be expressed in our setting as a condition. Example: From the axiom $\forall x, y(x \cap y = \cap x)$ we get the condition $\sup_{x,y} d(x \cap y, y \cap x)$.
- Measure axioms:
  - $\mu(0) = 0$ and $\mu(1) = 1$
  - $\sup_{x,y}(\mu(x \cap y) \dot{-} \mu(x))$
  - $\sup_{x,y}(\mu(x) \dot{-} \mu(x \cup y))$
  - $\sup_{x,y} |(\mu(x) \dot{-} \mu(x \cap y)) - (\mu(x \cup y) \dot{-} \mu(y))|$
- Identification of d and μ: $\sup_{x,y} |d(x,y) - \mu(x \Delta y)|$

We are considering $\mathfrak{L} \subset L^{pr}$ where the constant symbols are the same and $+$ corresponds to the $\cup$ symbol.

Proposition 5.20. *If $\mathcal{M}$ is a $\mathfrak{L}$-structure, then $\mathcal{M}$ is a model of T_{BML} if and only if M (viewed as a L^{pr}-structure) is a model of* PR.

Proof. Let's suppose that $\mathcal{M} \models T_{BML}$. First let's check that $\mathcal{M}$ can be seen as a L^{pr}-structure. It is clear that $\mathcal{M}$ has a single sort, diameter 1, a couple of distinguished elements $0, 1$ and a function $+$ with modulus of uniform continuity $\Delta(\varepsilon) = \varepsilon/2$, which is the interpretation of $\cup$. We need to find interpretations for $\cap, \cdot^c$ and the measure μ. Using Proposition 3.6.1, Lemma 5.3, and Lemma 5.18 it is easy to check that $|\cdot|$, $\cdot$, and $\cdot^c$ each have the modulus of continuity necessary to take them as interpretations of μ, $\cap$, and $\cdot^c$ respectively. Thus we can view $\mathcal{M}$ as a L^{pr} structure.

We now need to check that $\mathcal{M}$ interpreted as an L^{pr} structure satisfies the axioms of PR. Since $\mathcal{M}$ is a Boolean algebra it is clear that it satisfies the axioms about Boolean algebras, it also satisfies the axiom about the connection between μ and d because by

$\sigma_{d'}$ we get that for every $x, y \in \mathcal{M}$

$$d(x,y) = d'(x,y) = |x+y| - |xy| = |x\Delta y|,$$

where $x\Delta y := xy^c + x^c y$

For the measure axioms, we can check that $|0| = 0$ and $|1| = 1$. By Proposition 2.4, the sentence σ_{mod} and the behavior defined for $+$ and $\cdot^c$ it follows that $\mathcal{M}$ satisfies the axioms of measure. Therefore it is a model of PR.

Now let's suppose that $\mathcal{M}$ is a L^{pr}-structure. It is clear that we can view $\mathcal{M}$ as a $\mathfrak{L}$ structure, when we forget about the interpretations of $\cap, \cdot^c$ and μ. (Note, however, that from the axiom (3) of PR it follows that $\mu = |\cdot|$.) Now let's check that it satisfies the axioms of T_{BML}. We proceed sequentially.

- T_{ML}) The first five axioms follow easily from the axioms of PR, the sixth axiom follows from the inequality $(x+z)\Delta(y+z) \leq x\Delta y$ in Boolean algebras plus the measure and metric axioms of PR theory. The last axiom follows from the modularity expressed by the measure axioms and Proposition 2.7. Hence $\mathcal{M}$ satisfies T_{ML}.
- T_{MML}) From the measure axioms it is clear that $\mathcal{M}$ is metrically modular thus by Proposition 3.2 it models T_{MML}
- T_{DML}) This follows from Proposition 3.11 and the Boolean algebra axioms.
- T_{BML}) This follows from metric modularity, the Boolean algebra axioms, and Proposition 3.13. □

Proposition 5.21. *Let $\mathcal{M}$ be a complete metric lattice. If $\mathcal{N} \subseteq \mathcal{M}$ is a sublattice which is either: (1) metrically modular, distributive, and metrically complemented; or (2) Boolean, then $\overline{\mathcal{N}}$ is a sublattice with the same properties.*

Proof. First, let's prove that every $\overline{\mathcal{N}}$ is metrically modular. Given $x, y \in \overline{\mathcal{N}}$, we know by uniform continuity of the $\mathfrak{L}$-formula φ, that for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $d(x,x'), d(y,y') < \delta$ then $|\varphi(x,y) - \varphi(x',y')| < \varepsilon$. Since $x, y \in \overline{\mathcal{N}}$ and $\mathcal{N}$ is metrically modular, for each $\varepsilon > 0$ there are some $x_\varepsilon, y_\varepsilon \in \mathcal{N}$ such that $d(x,x_\varepsilon), d(y,y_\varepsilon) < \delta$ and $\varphi(x_\varepsilon, y_\varepsilon) = 0$. It follows from this that $|\varphi(x,y) - \varphi(x_\varepsilon, y_\varepsilon)| = |\varphi(x,y)| < \varepsilon$ for every $\varepsilon > 0$, so we conclude that $\varphi(x,y) = 0$ for any $x, y \in \overline{\mathcal{N}}$, as desired.

Now, observe that for any $x, y \in \overline{\mathcal{N}}$, there are Cauchy sequences $(x_n), (y_n)$ such that $x_n \to x$ and $y_n \to y$. From the uniform continuity of the join operation on $\mathcal{M}$ it follows that $x_n + y_n \to x + y$ on M, in addition $x_n + y_n \in \mathcal{N}$ for each n since $\mathcal{N}$ is a sublattice. So we can conclude that $x + y \in \overline{\mathcal{N}}$. Though the meet operation isn't necessarily uniformly continuous, by 5.6 we know that it is uniformly continuous on metrically modular sets. Since $\overline{\mathcal{N}}$ is metrically modular and $x, y, x_n, y_n \in \overline{\mathcal{N}}$, we have that $x_n y_n \to xy$ and $x_n y_n \in \mathcal{N}$ for each n, concluding that $xy \in \overline{\mathcal{N}}$. So $\overline{\mathcal{N}}$ is a sublattice.

Finally, let's prove that it is distributive and metrically complemented. Taking $x, y, z \in \overline{\mathcal{N}}$ and sequences $(x_n), (y_n), (z_n)$ on $\mathcal{N}$ such that $x_n \to x, y_n \to y$ and $z_n \to z$ it is clear by the uniform continuity of the join and meet operations that $x_n(y_n + z_n) \to x(y+z)$ and $z_n y_n + x_n z_n \to xy + xz$. However, as $\mathcal{N}$ is boolean and $x_n, y_n, z_n \in \mathcal{N}$ for each n, so it follows that $x_n(y_n + z_n) = x_n y_n + x_n z_n$ and we conclude that $x(y+z) = xy + xz$

for any $x, y, z \in \overline{\mathcal{N}}$, or in other words that $\overline{\mathcal{N}}$ is distributive. On the other hand, we know that $\sup_y d(x, y) = 1$ for every $x \in \mathcal{N}$, as that expression is uniformly continuous, we get that $\sup_y d(x, y) = 1$ for every $x \in \overline{\mathcal{N}}$. We conclude that $\overline{\mathcal{N}}$ is also metrically complemented, and therefore the proposition is proved. □

6. Partition Lattices

We begin by fixing some notation and terminology which will be crucial for this section.

Definition 6.1. Let P_n denote the lattice of partitions of the set $[[n]] := \{1, \dots, n\}$ with the following order: Given x, y partitions of $[[n]]$, $x \le y$ if and only if for all $A \in x$ there is a $B \in y$ such that $A \subset B$. Given a partition $x \in P_n$ we define

(1) $\langle x \rangle$ as the number of blocks of x that are singletons,
(2) $[x]$ as the number of blocks of x that have at least two elements,
(3) $\#x = \langle x \rangle + [x]$ as the number of blocks of x,
(4) $|x| = \frac{n - \#x}{n-1}$ as the norm of P_n, and
(5) for $S \subseteq [[n]]$, $i(x, S)$ as the number of blocks of x which have non-empty intersection with S. (Note $i(x, [[n]]) = \#x$.)

If $[x] \le 1$, then x is called a *singular partition*. Every non-zero singular partition is defined by its unique block with more than two elements, which we will call the *basic block* of x, in which case we will define $B_x \subseteq [[n]]$ to be the basic block of x. By convention $B_0 = \{1\}$ is the basic block of 0. We will denote the set of singular partitions in P_n by Σ_n.

Recall that P_n can be viewed as a metric lattice when equipped with the metric

$$d(x, y) := \frac{\#x + \#y - 2\#(x + y)}{n - 1} = d'(x, y).$$

We will implicitly assume that P_n is equipped with this metric.

The following formulas are readily apparent.

Lemma 6.2. *For $x, y \in P_n$ with y singular, we have that*

$$\#(x + y) = \#x - i(x, B_y) + 1$$

and

$$\#xy = \#y + i(x, B_y) - 1.$$

Lemma 6.3. *For any partition $x \in P_n$ we have that*

$$d(x, \Sigma_n) := \inf_{y \in \Sigma_n} d(x, y) = \frac{[x] - 1}{n - 1}.$$

Proof. We fix $x \in P_n$. If y is a singular partition, then by Lemma 6.2 we get that

$$d(x, y) = \frac{\#y - \#x + 2i(x, B_y) - 2}{n - 1}.$$

We will use this formula to show that a closest singular partition to x is the one whose basic block is the union of all of the non-singleton blocks of the partition x.

Let's denote this partition by π_x. Since $x \leq \pi_x$ it is clear that

$$d(x, \pi_x) = \frac{\#x - \#\pi_x}{n-1} = \frac{[x]-1}{n-1}.$$

Setting B_{π_x} to be the basic block of π_x, for another singular partition y with basic block B_y, let k be the number of elements in $B_{\pi_x} \setminus B_y$ and l be the number of elements in $B_y \setminus B_{\pi_x}$. Since B_{π_x} is the disjoint union of blocks of x of size at least two, the removal of the elements $B_{\pi_x} \setminus B_y$ decreases $i(x, B_{\pi_x})$ by at most $\lfloor k/2 \rfloor$, while the addition of elements from $B_y \setminus B_{\pi_x}$ increases $i(x, B_{\pi_x})$ by exactly l since every block of x intersecting $B_y \setminus B_{\pi_x}$ is a singleton. Therefore we have that

$$i(x, B_y) \geq i(x, B_{\pi_x}) + l - \lfloor k/2 \rfloor.$$

It follows that

$$\begin{aligned} d(x,y) &= \frac{\#y - \#x + 2i(x, B_y) - 2}{n-1} \\ &= \frac{(\#\pi_x + k - l) - \#x + 2i(x, B_y) - 2}{n-1} \\ &\geq \frac{\#\pi_x + (k-l) - \#x + 2i(x, B_{\pi_x}) + 2(l - \lfloor k/2 \rfloor) - 2}{n-1} \\ &\geq \frac{\#\pi_x - \#x + 2i(x, B_{\pi_x}) - 2}{n-1} = d(x, \pi_x). \end{aligned}$$

In this way the result obtains. □

Lemma 6.4. *For all partitions $x, y \in P_n$ with $x \leq y$ we have that*

$$\langle x \rangle - \langle y \rangle \leq 2\,(\#x - \#y)$$

Proof. We begin by noting that $x \leq y$ implies that $\#x \geq \#y$ and $\langle x \rangle \geq \langle y \rangle$, however $[x] - [y]$ can be negative. Next, observe that

$$[y] \leq [x] + \frac{\langle x \rangle - \langle y \rangle}{2} \tag{24}$$

since a block of size at least two of y either contains a block of size at least two of x or is obtained by merging at least two singleton blocks of x. As $\langle y \rangle$ singleton blocks must remain, there are at most $\langle x \rangle - \langle y \rangle$ blocks which can be merged. Rearranging this equation we get that

$$-\left(\frac{\langle x \rangle - \langle y \rangle}{2}\right) \leq [x] - [y],$$

hence

$$\frac{\langle x \rangle - \langle y \rangle}{2} \leq \big(\langle x \rangle - \langle y \rangle\big) + \big([x] - [y]\big) = \#x - \#y.$$

The result thus obtains. □

Recall the formula

$$\varphi(x,y) := \inf_z \max\{(|x|+|y|) \dot{-} (|x+y|+|z|), d(x+z,x), d(y+z,y)\},$$

which measures how far (x,y) is from being a modular pair.

Proposition 6.5. *For any $x \in P_n$, we have that*

$$d(x,\Sigma_n) \leq 48 \sup_{y\in P_n} \varphi(x,y).$$

Proof. Given a partition $x \in P_n$, by choosing two distinct elements from any block of size at least two, we can write

$$x = \{\{a_i,b_i\} \sqcup R_i\}_{i=1}^{[x]} \cup \{\{c_i\}\}_{i=1}^{\langle x\rangle}.$$

Using this notation, we define a partition x^* in the following way. If $[x]$ is even, then

$$x^* := \{\{a_{2i-1},b_{2i}\} \sqcup R_{2i}, \{a_{2i},b_{2i-1}\} \sqcup R_{2i-1}\}_{i=1}^{[x]/2} \cup \{\{c_i\}\}_{i=1}^{\langle x\rangle}.$$

If $[x]$ is odd, then

$$x^* := \{\{a_{2i-1},b_{2i}\} \sqcup R_{2i}, \{a_{2i},b_{2i-1}\} \sqcup R_{2i-1}\}_{i=1}^{([x]-1)/2} \cup \{\{a_{[x]},b_{[x]}\} \cup R_{[x]}\} \cup \{\{c_i\}\}_{i=1}^{\langle x\rangle}.$$

We observe that

$$\langle x^*\rangle = \langle x\rangle, \quad \langle x+x^*\rangle = \langle x\rangle, \quad [x^*] = [x], \quad [x+x^*] = \lceil [x]/2\rceil. \tag{25}$$

Using this notation we will write

$$C_i := \{\{a_{2i-1},b_{2i-1}\} \sqcup R_{2i-1}, \{a_{2i},b_{2i}\} \sqcup R_{2i}\}$$

and

$$C_i^* := \{\{a_{2i-1},b_{2i}\} \sqcup R_{2i}, \{a_{2i},b_{2i-1}\} \sqcup R_{2i-1}\}$$

and refer to these as the *i-th pair* in x and x^*, respectively.

We claim that

$$\varphi(x,x^*) \geq \frac{1}{48}\frac{[x]-1}{n-1}, \tag{26}$$

which establishes the result by Lemma 6.3.

Since

$$|p| = d(p,0) = \frac{n-\#p}{n-1}$$

for any partition $p \in P_n$, the previous condition is equivalent to showing that for all $t \in P_n$, we have that

$$\max\{\#(x+x^*)+\#t-\#x-\#x^*, \#x-\#(x+t), \#x^*-\#(x^*+t)\} \geq \frac{[x]-1}{48}.$$

We assume that $\#x - \frac{[x]-1}{48} < \#(x+t)$ and $\#x^* - \frac{[x]-1}{48} < \#(x^*+t)$. Setting

$$m := \left\lceil \frac{[x]-1}{48} \right\rceil \geq 1$$

we have that

$$\#x - m \leq \#(x+t), \quad \#x^* - m \leq \#(x^*+t) \tag{27}$$

as all other terms are integers.

Since $\langle p\rangle \geq \langle q\rangle$ whenever $p \leq q$ we have that

$$\langle x\rangle + [x] - m = \#x - m \leq \#(x+t) = \langle x+t\rangle + [x+t] \leq \langle x\rangle + [x+t].$$

By similar reasoning for x^*, we conclude using line (25) that

$$[x] - m = [x^*] - m \leq [x+t],\ [x^*+t]. \tag{28}$$

Applying Lemma 6.4 to the pairs $x \leq x+t$ and $x^* \leq x^*+t$ and using (25) and (27), we have that

$$\langle x\rangle - 2m = \langle x^*\rangle - 2m \leq \langle x+t\rangle,\ \langle x^*+t\rangle. \tag{29}$$

Next, note that by the inequalities (29) that there are at least $\langle x\rangle - 2m$ singleton blocks of x which still appear as singleton blocks of $x+t$. It follows that these must also be singleton blocks of t, so

$$\langle t\rangle \geq \langle x\rangle - 2m = \langle x^*\rangle - 2m. \tag{30}$$

Finally, we note that from line (28) that there are $[x]-m$ blocks of x of size at least two and $[x^*]-m$ blocks of x^* of size at least two which belong to pairwise distinct blocks of $x+t$ and x^*+t, respectively. We denote these sets of blocks by D and D^*, respectively. For each index $i \in \{1, \dots, \lfloor [x]/2\rfloor\}$ such that both blocks of the pair C_i of x belong to D and both blocks of the pair C_i^* belong to D^*, the elements $\{a_{2i-1}, b_{2i-1}, a_{2i}, b_{2i}\}$ must belong to distinct blocks of t which aren't among the $\langle x\rangle - 2m$ singleton blocks chosen in the previous paragraph. We will call these pairs the "repeated pairs". For x, at most most m blocks of size at least two do not belong to D, so there are at least

$$\left\lfloor \frac{[x]-2m}{2} \right\rfloor = \lfloor [x]/2\rfloor - m$$

pairs C_i among the blocks of D. Similarly there are at least $\lfloor [x^*]/2\rfloor - m = \lfloor [x]/2\rfloor - m$ pairs C_i^* among the blocks of D^*. Therefore the number of repeated pairs is at least $\lfloor [x]/2\rfloor - 2m$. Using this along with the estimate (29) we conclude that

$$\begin{aligned}\#t &\geq 4\left(\frac{[x]-1}{2} - 2m\right) + \langle x\rangle - 2m \\ &= 2[x] + \langle x\rangle - 10m - 2 \geq 2[x] + \langle x\rangle - 12m,\end{aligned} \tag{31}$$

since $m \geq 1$. It follows from lines (25) and (31) that

$$\begin{aligned}
&\#(x + x^*) + \#t - \#x - \#x^* \\
&\geq \frac{[x]-1}{2} + \langle x \rangle + \#t - 2([x] + \langle x \rangle) \\
&\geq \frac{[x]-1}{2} + \langle x \rangle + \Big[2[x] + \langle x \rangle - 12m\Big] - 2([x] + \langle x \rangle) \\
&= \frac{[x]-1}{2} - 12m \\
&\geq \frac{[x]-1}{2} - 12\,\frac{[x]-1}{48} = \frac{[x]-1}{4} > \frac{[x]-1}{48},
\end{aligned}$$

which suffices to show the result. □

Corollary 6.6. *If P_n is a finite partition lattice, then $x \in P_n$ is metrically modular if and only if x is singular.*

Proof. The "if" direction follows directly from Proposition 6.5. For the "only if" direction, we fix a singular partition $x \in P_n$. By Lemma 6.2 we have that $\#(x + y) + \#xy = \#x + \#y$, hence $|x + y| + |xy| = |x| + |y|$. □

Corollary 6.7. *Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and $P = \prod_{\mathcal{U}} P_n$ be an ultraproduct of finite partition lattices. We have that $x \in P$ is metrically modular if and only if x is an ultraproduct of singular partitions.*

Proof. Suppose $x \in P$ is metrically modular, and let $(x_n) \in \prod_{n\in\mathbb{N}} P_n$ be a representative sequence for x. By Łos' Theorem we have that $\lim_{n\to\mathcal{U}} \sup_{y\in P_n} \varphi(x_n, y) = 0$, hence by Proposition 6.5 we have that $\lim_{n\to\mathcal{U}} d(x_n, \Sigma_n) = 0$. Thus we can find $x'_n \in \Sigma_n$ so that $\lim_{n\to\mathcal{U}} d(x_n, x'_n) = 0$.

In the other direction, by Corollary 6.6 every singular element of Σ_n is metrically modular, hence so is every element of $\prod_{n\in\mathcal{U}} \Sigma_n$ by Proposition 6.5 and Łos' Theorem. □

Definition 6.8. Let P_n be the finite partition lattice of a set with n elements. A sublattice A is called *singular* if every $x \in A$ is a singular partition.

Lemma 6.9. *Let A be a maximal singular distributive sublattice of P_n. The following claims are true:*

1. *There must exist an element $a^* \in [[n]]$ such that a^* belongs to the basic block of every non-zero partition of A.*
2. *The atoms of A are the singular partitions x_e for $e \in [[n]] - \{a^*\}$, where x_e is the singular partitions with basic block $\{a^*, e\}$.*
3. *$A \models T_{BML}$.*
4. *A is a maximal boolean sublattice of P_n.*
5. *For every singular partition $x \in P_n$, we have that $d(x, A) \leq \frac{1}{n-1}$.*

Proof. We prove each claim in turn.

(1) First, observe that since all the elements of A are singular partitions, for any non-zero partitions $x, y \in A$ the intersection $B_x \cap B_y$ of their basic blocks must be non-empty, otherwise $x + y$ wouldn't be a singular partition. Let's suppose the claim is false. Then there must be non-zero partitions $x, y, z \in A$ with basic blocks B_x, B_y, and B_z, respectively, such that $B_x \cap B_y, B_x \cap B_z, B_y \cap B_x \neq \emptyset$ and $B_x \cap B_y \cap B_z = \emptyset$. Taking $a \in B_x \cap B_y$ and $b \in B_x \cap B_z$ we observe that $x(y + z) \neq xy + xz$ because the pair $\{a, b\}$ belongs to the basic block of the partition $x(y + z)$, but since $(B_x \cap B_y) \cap (B_x \cap B_z) = \emptyset$, we get that a and b are in different blocks of the partition $xy+xz$. Hence A wouldn't be distributive.

(2) Let A^* be the sublattice generated by all such x_e. It's clear that A^* is Boolean as it can be seen to be isomorphic to the lattice $2^{[[n]]-\{a^*\}}$ by associating to each $X \subseteq [[n]] - \{a^*\}$ the singular partition with basic block $X \cup \{a^*\}$. By the previous claim there is an a^* which belongs to the basic block of every non-zero singular partition in A, hence $A \subseteq A^*$. Equality then follows by maximality of A.

(3) It is clear that the isomorphism of A^* with the Boolean lattice $2^{[[n]]-\{a^*\}}$ given in the previous item is isometric with respect to the canonical lattice metric on $2^{[[n]]-\{a^*\}}$ induced by the counting measure.

(4) For any partition x not belonging to A there is a block which contains a pair of elements $\{e, e'\} \subseteq [[n]] - \{a^*\}$. We have that $x(x_e + x_{e'}) \neq xx_e + xx_{e'}$ because the pair $\{e, e'\}$ is included in a block of $x(x_e + x_{e'})$ but e and e' are in different blocks of the partition $xx_e + xx_{e'}$. Therefore $A \cup \{x\}$ cannot generate a Boolean sublattice by failure of distributivity, hence A is maximal Boolean.

(5) Let $x \in P_n$ be a singular partition with basic block B_x. We choose $y \in A$ to be the singular partition with basic block $B_x \cup \{a^*\}$ and observe that

$$d(x, y) = \frac{\#x - \#y}{n - 1} = \frac{\#x - (\#x - 1)}{n - 1} = \frac{1}{n - 1}. \qquad \square$$

Now, we can use this lemma to prove our result.

Proposition 6.10. *Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. For each $n \in \mathbb{N}$ choose $A_n \subseteq P_n$ a maximal singular sublattice. We have that $\prod_{n\in\mathcal{U}} A_n$ is a Boolean sublattice of $\prod_{n\in\mathcal{U}} P_n$ and is equal to the set of metrically modular elements of $\prod_{n\in\mathcal{U}} P_n$.*

Proof. We begin by showing that every metrically modular element of $\prod_{n\in\mathcal{U}} P_n$ belongs to $\prod_{n\in\mathcal{U}} A_n$. Indeed, if x is metrically modular, by Corollary 6.7 there is representative sequence (x_n) with $x_n \in P_n$ singular for n $\mathcal{U}$-generic. Hence, by Lemma 6.9.5 we have that x has a representative sequence (x'_n) with $x'_n \in A_n$ for n $\mathcal{U}$-generic, and $x \in \prod_{n\in\mathcal{U}} A_n$. On the other hand it is clear that if $x \in \prod_{n\in\mathcal{U}} A_n$ then x is singular again by Lemma 6.7. Finally $\prod_{n\in\mathcal{U}} A_n$ is a Boolean algebra because by Lemma 6.9 we know that each $A_n \vDash T_{BML}$ so $\prod_{n\in\mathcal{U}} A_n \models T_{BML}$ and therefore it is a boolean sublattice of $\prod_{n\in\mathcal{U}} P_n$. □

6.1. **Pseudofinite partition lattices.**

Definition 6.11. We write T_{FPL} as the theory of the set $\{P_n : n \in \mathbb{N}\}$, that is $\mathcal{M} \models T_{FPL}$ if and only if $\sigma^{\mathcal{M}} = 0$ for any sentence so that $\sup_n \sigma^{P_n} = 0$. We then say that $\mathcal{M}$ is a *pseudofinite partition lattice.* Note that any ultraproduct of finite partition lattices is a pseudofinite partition lattice.

In this section we investigate the properties of T_{FPL}. Let $\mathcal{M} \models T_{ML}$, and consider the functor $\mathcal{M} \mapsto \mu(\mathcal{M})$ which sends $\mathcal{M}$ to its collection of metrically modular elements. (Note that $\mu(\mathcal{M})$ is always non-empty since 0 and 1 are modular.) We begin with the following observation which is a direct consequence of Proposition 6.5 and Corollary 6.6.

Lemma 6.12. *We have that $\mathcal{M} \mapsto \mu(\mathcal{M})$ is a definable functor in T_{FPL}.*

Proof. Indeed, the quoted results show that every "approximately modular" element of any finite partition lattice is uniformly close, independent of the size of the partition lattice, to a singular partition and that singular partitions are modular. □

Remark 6.13. By Theorem 2.1, it follows that we may quantify over modular elements in T_{FPL}.

Proposition 6.14. *If $\mathcal{M} \models T_{FPL}$ and has density character at least $\aleph_0$, then $\mu(\mathcal{M})$ is a Boolean sublattice.*

Proof. It is well known (see, for instance, [13, Lemma 16.2.4]) that if $\mathcal{M} \models T_{FPL}$, then it admits an elementary embedding into an ultraproduct $\prod_{i\in U} P_{n_i}$. As $\mathcal{M}$ has density character at least $\aleph_0$, we must have that $\lim_{i\in U} n_i = \infty$. Since modularity and being a Boolean sublattice are definable properties by Lemma 6.12 and Proposition 5.20, respectively, the result follows by Proposition 6.10. □

Question 6.15. Is $\mu(\mathcal{M})$ definable for an arbitrary model of T_{ML}?

Definition 6.16. Let $\mathcal{L}$ be a metric lattice, and let us fix $x \in L$. We say that an element $y \in \mu(\mathcal{L})$ is a *selector* for x if y complements x.

The following is a straightforward consequence of Lemma 5.9.

Lemma 6.17. *We have that y is a selector for x if and only if $x+y = 1$ and $|x|+|y| = 1$.*

Lemma 6.18. *Let P_n be a finite partition lattice. We have that every element $x \in P_n$ admits a selector. The set of selectors is the set of all singular partitions whose main block intersects each block of x at exactly one element.*

Proof. Given a partition x and a singular partition y of P_n whose main block B_y intersects each block of x at exactly one element we observe that B_y contains all singletons of x and since it intersects every other block of x it is clear that $x + y = \{[[n]]\} = 1$.

On the other hand, we observe that $|B_x| = \langle x\rangle + [x] = \#x$, therefore $\#y = n - \#x + 1$. Consequently,

$$|x| + |y| = \frac{n-\#x}{n-1} + \frac{n-(n-\#x+1)}{n-1} = \frac{n-\#x+\#x-1}{n-1} = \frac{n-1}{n-1} = 1,$$

so the result follows by Lemma 6.17. $\square$

The main result of this section is that the same is true for any pseudofinite partition lattice.

Proposition 6.19. *If $\mathcal{M} \models T_{FPL}$, then every element $x \in M$ admits a selector.*

In order to prove this proposition, we will first require a few intermediate results.

Lemma 6.20. *Let $\mathcal{M} \models T_{FPL}$, and let $x \in \mathcal{M}$ and $y \in \mu(\mathcal{M})$. We have that*

$$\inf_{z\in\mu(\mathcal{M})} \max\left\{d(z,y) \mathbin{\dot{-}} d(x+y,1), d(x+z,1) \mathbin{\dot{-}} \frac{1}{4}d(x+y,1)\right\} = 0.$$

Proof. We claim that for any finite partition lattice P and any $x \in P$ and $y \in \mu(P)$, there is $z \in \mu(P)$ so that $d(y,z) \le d(x+y,1)$ and $d(x+z,1) = 0$. From this, the result easily follows.

Let P_{n+1} be the finite partition lattice on $n+1$ elements, and let us fix a partition x and a singular partition y. By Corollary 6.6 we know that every element of $\mu(P_{n+1})$ is a singular partition. Now, using Lemma 6.2 we can compute

$$d(x+y,1) = 1 - |x+y| = 1 - \frac{n+1-\#(x+y)}{n} = \frac{\#x - i(x,B_y)}{n}.$$

Defining $k := \#x - i(B_y,x)$, we have that $d(x+y,1) = \frac{k}{n}$. Observe that from Lemma 6.2 it also follows that $x+y = 1$ if and only if $i(x,B_y) = \#x$. By the definition of $i(x,B_y)$ there are k blocks of x that do not intersect B_y; denote them by $C_1,...,C_k$. Taking $c_i \in C_i$ for $1 \le i \le k$ we define a new singular partition z where $B_z := B_y \cup \{c_i\}_{i=1}^k$. We see that $i(x,B_z) = \#x, y \le z$, and $\#z = \#y - k$, so we conclude that $x+z = 1$ and

$$d(z,y) = |z| - |y| = \frac{n+1-\#z}{n} - \frac{n+1-\#y}{n} = \frac{\#y-(\#y-k)}{n} = \frac{k}{n} \le d(x+y,1),$$

finishing the proof. $\square$

Corollary 6.21. *Let $\mathcal{M} \models T_{FPL}$. For any $x \in M$ and $y \in \mu(\mathcal{M})$, there is $z \in \mu(M)$ so that $x+z=1$ and $d(y,z) \le 4\,d(x+y,1)$.*

Proof. Given such $x,y \in M$, by Lemma 6.20 we may inductively construct a sequence (z_n) in $\mu(\mathcal{M})$ with $z_0 = y$, so that $d(z_{n+1},z_n) \le \frac{1}{2^{n-1}}d(x+y,1)$ and $d(x+z_{n+1},1) \le \frac{1}{2^{n+1}}d(x+y,1)$. By the triangle inequality, (z_n) is Cauchy and converges to $z \in \mu(\mathcal{M})$ with the required properties. $\square$

We define the following $\mathfrak{L}$-formula for models of T_{FPL}:

$$\chi_z(y) := \inf_{w\in\mu(\mathcal{M})} \max\{d(z,w) \dot{-} |1-|y|-|z||, d(z+w,z), \\ d(w+y,1), |1-|w|-|y|| \dot{-} \frac{1}{4}|1-|y|-|z||\}$$

Lemma 6.22. *The following formula holds in T_{FPL}:*

$$\sup_y \sup_{z\in\mu(\mathcal{M})} \chi_z(y) \dot{-} 2d(y+z,1) = 0.$$

Proof. We claim that for any finite partition lattice $P = P_{n+1}$ and any $y \in P$ and $z \in \mu(P)$, there is $w \in \mu(P)$ so that

$$\begin{aligned} d(w,z) &\leq 2\,d(y+z,1) + |1-|y|-|z||, \\ d(z+w,z) &= d(y+z,1), \text{ and} \\ d(w+y,1) &= |1-|w|-|y|| = 0. \end{aligned}$$

From this the result easily follows.

We fix y and z as above, and define the numbers

$$k := \#y - i(y,B_z), \quad l := |B_z| - i(y,B_z).$$

Note that

$$d(y+z,1) = \frac{\#(y+z)-1}{n} = \frac{\#y - i(y,B_z)}{n} = \frac{k}{n}$$

and

$$|1-|y|-|z|| = \left|\frac{n-(n+1-\#y)-(n+1-\#z)}{n}\right| = \left|\frac{\#y-1-(|B_z|-1)}{n}\right| = \frac{|k-l|}{n}.$$

By the definition of $i(y,B_z)$ we get that there are k blocks, $C_1,\ldots,C_k$ of y which do not intersect B_z and l additional elements, $x_1,\ldots,x_l$ of B_z which belong to blocks whose intersection with B_z contains at least one other element. Taking an element $c_i \in C_i$ from each of the k blocks, as we did in the proof of Lemma 6.20, we let w be the singular partition for which $B_w = (B_z \setminus \{x_1,\ldots,x_l\}) \cup \{c_1,\ldots,c_k\}$. By construction we have that $|B_w| = \#y = |B_z| + k - l = y + w = 1$ and $1 = |y| + |w|$. Finally we note that

$$\begin{aligned} d(z,w) = |z+w| - |zw| &= \frac{|B_{z+w} - B_{zw}|}{n} \\ &= \frac{|B_z| + k - (|B_z| - l)}{n} \\ &= \frac{2k+l-k}{n} \leq 2\,d(y+z,1) + |1-|y|-|z||, \end{aligned}$$

and

$$d(z+w,z) = |z+w| - |z| = \frac{|B_{z+w}| - |B_z|}{n} = \frac{|B_z| + k - |B_z|}{n} = \frac{k}{n} = d(y+z,1),$$

as desired. $\square$

Lemma 6.23. *Let $\mathcal{L}$ be a metric lattice and $\sum_{n=0}^{\infty} a_n$ a convergent series. If (x_n) is a sequence in $\mathcal{L}$ such that*

$$d(x_{n+1} + x_n, x_n) \le a_n$$

for all $n \in \mathbb{N}$, then (x_n) is Cauchy.

Proof. Let $x_{(n,k)} := x_n + x_{n+1} + \cdots + x_{n+k}$. We have by the triangle inequality and the property that $d(x+z, y+z) \le d(x,y)$ that

$$\begin{aligned}
&d(x_n, x_n + x_{n+k})\\
&\le \sum_{i=0}^{k-1} d(x_{(n,i)}, x_{(n,i+1)}) + \sum_{j=1}^{k-1} d(x_{(n,k-j)} + x_{n+k}, x_{(n,k-j-1)} + x_{n+k})\\
&\le \sum_{i=0}^{k-1} d(x_{n+i}, x_{n+i} + x_{n+i+1}) + \sum_{j=1}^{k-1} d(x_{n+k-j} + x_{n+k-j-1}, x_{n+k-j-1})\\
&\le \sum_{i=0}^{k-1} a_{n+i} + \sum_{j=1}^{k-1} a_{n+k-j-1} \le 2\sum_{i=n}^{\infty} a_n.
\end{aligned}$$

We conclude that

$$0 \le |x_{n+k} + x_n| - |x_n| \le d(x_n, x_n + x_{n+k}) \le 2\sum_{i=n}^{\infty} a_n$$

hence,

$$|x_{n+k}| \le |x_{n+k} + x_n| \le |x_n| + 2\sum_{i=n}^{\infty} a_n \tag{32}$$

for all $n, k \in \mathbb{N}$. Setting $\alpha = \liminf_n |x_n|$, it follows that $\lim_n |x_n| = \alpha$, thus $(|x_n|)$ is a Cauchy sequence. It now follows from (32) that

$$\lim_n d'(x_{n+k}, x_n) = \lim_n (2|x_{n+k} + x_n| - |x_{n+k}| - |x_n|) = 0$$

for all $k \in \mathbb{N}$. Thus (x_n) is Cauchy in d', hence for d as well. □

Notation 6.24. For $\mathcal{M} \models T_{FPL}$ and $x \in M$, we write $\Gamma(x)$ for the set of all selectors of x in M.

The next proposition is a quantitative version of Proposition 6.19, so easily implies that result.

Proposition 6.25. *Let $\mathcal{M} \models T_{FPL}$. If $y \in \mathcal{M}, z \in \mathrm{Mod}(\mathcal{M})$ and $y + z = 1$ then exists $w \in \mu(\mathcal{M})$ such that $w \in \Gamma(y)$ and $d(z, w) \le 4|1 - |y| - |z||$.*

Proof. Since $y+z = 1$, by Lemma 6.17 we have that $z \in \Gamma(y)$ if and only if $|1-|y|-|z|| = 0$. Thus without loss of generality we may assume that $|1 - |y| - |z|| \neq 0$. It also follows from Lemma 6.22 that $\chi_z(y) = 0$; therefore, for every $n \in \mathbb{N}$ we can find $w_n \in \mu(\mathcal{M})$ so that

$$\begin{aligned} d(z, w_n) &\leq \left(1 + \frac{1}{2^n}\right) |1 - |z| - |y||, \\ d(z + w_n, z) &\leq \frac{1}{2^n}|1 - |z| - |y|| \leq \frac{1}{2^n}, \\ d(w_n + y, 1) &\leq \frac{1}{2^n}|1 - |z| - |y|| \leq \frac{1}{2^n}, \text{ and} \\ |1 - |w_n| - |y|| &\leq (\frac{1}{4} + \frac{1}{2^n})(|1 - |z| - |y|). \end{aligned}$$

By Corollary 6.21 we can find a modular element w_n' so that $w_n' + y = 1$ and

$$d(w_n, w_n') \leq 4\, d(w_n + y, 1) \leq \frac{1}{2^{(n-2)}}\, |1 - |z| - |y|| \leq \frac{1}{2^{(n-2)}}.$$

Thus, for any $z \in \mu(\mathcal{M})$ with $y + z = 1$ and all $n \in \mathbb{N}$ sufficiently large we can find $w \in \mu(\mathcal{M})$ so that $y + w = 1$,

$$d(z, w) \leq 2|1 - |y| - |z||,\ d(w + z, z) \leq \frac{1}{2^n}, \text{ and } |1 - |w| - |y|| \leq \frac{1}{2}|1 - |z| - |y||.$$

Starting with $w_0 = z$, we may thus inductively construct a sequence (w_n) in $\mu(\mathcal{M})$ so that $y + w_n = 1$,

$$\begin{aligned} d(w_n + w_{n-1}, w_{n-1}) &\leq \frac{1}{2^n}, \\ |1 - |y| - |w_n|| &\leq \frac{1}{2^n}|1 - |y| - |w_0||, \text{ and} \\ d(w_n, w_{n-1}) &\leq \frac{1}{2^{(n-2)}}||1 - |y| - |w_0||. \end{aligned} \tag{33}$$

By Lemma 6.23, the sequence (w_n) is Cauchy; thus, by completeness (w_n) converges to a selector w of y so that

$$d(z, w) = d(w_0, w) \leq \sum_{n=0}^{\infty} d(w_n, w_{n+1}) \leq \sum_{n=0}^{\infty} 2^{-(n-1)}|1 - |y| - |w_0|| = 4|1 - |y| - |z||.$$

In this way the result obtains. □

Corollary 6.26. *Let $\mathcal{M} \models T_{FPL}$, and consider $x, y \in M$. If $z \in \Gamma(x)$, then there is $w \in \Gamma(y)$ so that $d(z, w) \leq 24\, d(x, y)$.*

Proof. Using that $z \in \Gamma(x)$ and Corollary 6.21 we know there exist a z' such that $y + z' = 1$ and $d(z, z') \leq 4\, d(z + y, 1) \leq 4(d(z + y, z + x) + d(z + x, 1) \leq 4\, d(x, y)$. Now, using Proposition 6.25 we know there is a $w \in \Gamma(y)$ such that

$$\begin{aligned}
d(z',w) &\leq 4|1-|y|-|z'|| \\
&\leq 4|1-|x|-|z|+|x|-|y|+|z|-|z'|| \\
&\leq 4(|1-|x|-|z||+||x|-|y||+||z|-|z'||) \\
&\leq 4(0+d(x,y)+d(z,z')) \leq 20\,d(x,y)
\end{aligned}$$

So we can conclude that

$$d(z,w) \leq d(z,z')+d(z',w) \leq 4\,d(x,y)+20\,d(x,y) = 24\,d(x,y). \qquad \square$$

The next result proves that the set of selectors of a given element is definable.

Proposition 6.27. *Given $\mathcal{M} \models T_{FPL}$ and $x \in M$. For every $y \in M$ we have that*

$$d(y,\Gamma(x)) \leq 4|1-|x|-|y|| + 20\,d(y+x,1) + 1200 \sup_w \varphi(y,w),$$

where φ is defined as in line (18).

Proof. Applying the results from Proposition 6.5, Corollary 6.21 and Proposition 6.25 we have that:

- There is a $y' \in \mathrm{Mod}(\mathcal{M})$ such that $d(y,y') \leq 48 \sup_w \varphi(y,w)$,
- There is a $z \in \mathrm{Mod}(\mathcal{M})$ such that $x+z=1$ $d(y',z) \leq 4\,d(y'+x,1)$,
- There is a $z' \in \Gamma(x)$ such that $d(z,z') \leq 4|1-|x|-|z||$.

We observe that $d(y,\Gamma(x)) \leq d(y,z') \leq d(y,y')+d(y',z)+d(z,z')$. Developing this inequality using the mentioned results we have that

$$\begin{aligned}
d(y,\Gamma(x)) &\leq d(z,z')+d(y',z)+d(y,y') \\
&\leq 4|1-|x|-|z||+4d(y'+x,1)+d(y,y') \\
&\leq 4|1-|x|-|y||+4||y|-|y'||+4||y'|-|z||+4d(y'+x,1)+d(y,y') \\
&\leq 4|1-|x|-|y||+4d(y,y')+4d(y',z)+4d(y'+x,1)+d(y,y') \\
&\leq 4|1-|x|-|y||+16d(y'+x,1)+4d(y'+x,1)+5d(y,y') \\
&\leq 4|1-|x|-|y||+20d(y'+x,1)+5d(y,y') \\
&\leq 4|1-|x|-|y||+20(d(y+x,1)+d(y+x,y'+x))+5d(y,y') \\
&\leq 4|1-|x|-|y||+20d(y+x,1)+20d(y,y')+5d(y,y') \\
&\leq 4|1-|x|-|y||+20d(y+x,1)+25d(y,y') \\
&\leq 4|1-|x|-|y||+20d(y+x,1)+25(48\sup_w \varphi(y,w)) \\
&\leq 4|1-|x|-|y||+20d(y+x,1)+1200\sup_w \varphi(y,w)),
\end{aligned}$$

as was desired. $\square$

Corollary 6.28. *For all $\mathcal{M} \models T_{FPL}$ and $x \in M$, $\Gamma(x)$ is definable.*

Proof. This follows from the previous result by Theorem 2.1.3. $\square$

We recall that the Hausdorff distance between two subsets A, B of a metric space (X, d) is defined as

$$d_{Haus}(A,B) := \max\left\{\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,A)\right\}.$$

Proposition 6.29. *Given a finite partition lattice $P_n, x, y \in P_n$ and $z \in \Gamma(x), w \in \Gamma(y)$, we have that*

$$d(z,w) = \frac{\#x + \#y - 2\ \max\{1, |B_z \cap B_w|\}}{n-1}$$

and,

$$d_{Haus}(\Gamma(x), \Gamma(y))) = \frac{\#x + \#y - 2\min\{\gamma(x,y), \gamma(y,x)\}}{n-1}$$

where $\gamma(x,y) := \min_{z\in\Gamma(x)} \max_{w\in\Gamma(y)} |B_z \cap B_w|$.

Proof. By Corollary 6.6, we can claim that for every pair of singular partitions $z, w \in P_n$ that

$$d(z,w) = |z+w| - |zw| = \frac{\#zw - \#(z+w)}{n-1} = \frac{|B_z| + |B_w| - 2\ \max\{1, |B_z \cap B_w|\}}{n-1}.$$

Indeed, if $B_z \cap B_w \neq \emptyset$ then

$$\#zw = n - |B_z \cap B_w| + 1, \quad \#(z+w) = n - |B_z \cup B_w| + 1,$$

and if and if $B_z \cap B_w = \emptyset$, then

$$\#zw = n, \quad \#(z+w) = n - |B_z| - |B_w| + 2.$$

Fixing $z \in \Gamma(x)$ we see that

$$\begin{aligned} d(z, \Gamma(y)) &= \min_{w\in\Gamma(y)} \frac{|B_z| + |B_w| - 2\ \max\{1, |B_z \cap B_w|\}}{n-1} \\ &= \min_{w\in\Gamma(y)} \frac{\#x + \#y - 2\ \max\{1, |B_z \cap B_w|\}}{n-1} \\ &= \frac{\#x + \#y - 2\max\{1, \max_{w\in\Gamma(y)} |B_z \cap B_w|\}}{n}. \end{aligned}$$

Observe that we can always find a $w \in \Gamma(y)$ such that $|B_z \cap B_w| \geq \#(x+y) \geq 1$ because we can eliminate elements of B_z until we get a selector of $x+y$ which will have a main block of cardinality $\#(x+y)$: we can then add elements to get a selector w of y. Therefore, we can conclude that, for every $z \in \Gamma(x)$,

$$d(z, \Gamma(y)) = \frac{\#x + \#y - 2\ \max_{w\in\Gamma(y)} |B_z \cap B_w|}{n}.$$

The last claim about $d_{Haus}(\Gamma(x), \Gamma(y))$ follows easily. ☐

Corollary 6.30. *Given $\mathcal{M} \models T_{FPL}$ and $x, y \in M$, we have that*

$$d_{Haus}(\Gamma(x), \Gamma(y)) \leq d(x,y)$$

Proof. This follows from the proof of Proposition 6.29 since we observed therein that $\gamma(x,y), \gamma(y,x) \geq \#(x+y)$. □

Remark 6.31. Equality generally fails to hold. Consider partitions x, y of P_6 defined as follows: $x := \{\{1,4\},\{2,3\},\{5,6\}\}$ and $y := \{\{1,2\},\{4,5\},\{3\},\{6\}\}$. Note that

$$d(x,y) = \frac{\#x + \#y - 2\#(x+y)}{5} = \frac{3+4-2}{5} = 1.$$

However, we can check that for any $z \in \Gamma(x)$ there exists a $w \in \Gamma(y)$ such that $|B_z \cap B_w| = 2$, similarly for any $w \in \Gamma(y)$. Therefore, by Proposition 6.29 we conclude that $d_{Haus}(\Gamma(x),\Gamma(y)) = \frac{4+3-2\cdot 2}{5} = \frac{3}{5} < 1 = d(x,y)$.

However, we can, in fact, bound $d(x,y)$ using a multiple of $d_{Haus}(\Gamma(x),\Gamma(y))$. Before presenting the result, we introduce the following notation and remarks.

Definition 6.32. In P_n, given $x \in P_n$ and $S \subset [[n]]$ we define the restriction of x to S as $x_S := \{B \cap S : B \in x\}$.

If we impose some conditions on S, the restrictions will have a good behavior:

Remark 6.33. Consider $x, y, z \in P_n$ with $z := \{B_1, ..., B_m\} \geq x, y$. Then the following claims are true:

- $\#x = \sum_{i=1}^{m} \#x_{B_i}$,
- $t \in \Gamma(x)$ if and only if there exists $t_i \in \Gamma(x_{B_i})$ for $i = 1, ..., m$ such that $B_t = \sqcup_{i=1}^{m} t_i$,
- $\gamma(x,y) = \sum_{i=1}^{m} \gamma(x_{B_i}, y_{B_i})$.

Proposition 6.34. *Given $\mathcal{M} \models T_{FPL}$ and $x, y \in M$, we have that*

$$d(x, x+y) \leq 2\, d_{Haus}(\Gamma(x),\Gamma(y)).$$

Consequently,

$$d(x,y) \leq 4\, d_{Haus}(\Gamma(x),\Gamma(y)) \leq 4\, d(x,y)$$

for all $x, y \in M$.

Proof. Firstly, observe that for each $B \in x+y$ if $x_B \leq y_B$, then $\#y_B = 1$, thus

$$\gamma(x,y) = \gamma(y,x) = 1.$$

Considering the partition $\{H, H^c\} \geq x+y$, where $H = \bigcup\{B \in x+y : x_B \leq y_B \vee y_B \leq x_B\}$, and using Proposition 6.29 we note that

$$\gamma(x_H, y_H) = \gamma(y_H, x_H) = \#(x+y)_H,$$

$$\begin{aligned} d(x, x+y) &= \frac{\#x - \#(x+y)}{n-1} \\ &= \frac{\#x_H + \#x_{H^c} - \#(x+y)_H - \#(x+y)_{H^c}}{n-1}, \end{aligned}$$

and

$$\begin{aligned} d_{Haus}(\Gamma(x), \Gamma(y)) &= \frac{\#x + \#y - 2\min\{\gamma(x,y), \gamma(y,x)\}}{n-1} \\ &= \frac{\#x_H + \#y_H - 2\#(x+y)_H}{n-1} \\ &\quad + \frac{\#x_{H^c} + \#y_{H^c} - 2\min\{\gamma(x_{H^c}, y_{H^c}), \gamma(y_{H^c}, x_{H^c})\}}{n-1} \end{aligned}$$

Since it is clear that $\#x_H - \#(x+y)_H \leq 2(\#x_H + \#y_H - 2\#(x+y)_H)$, we need to focus on the H^c part. We can assume without loss of generality that for all $B \in x+y$, $x_B \nleq y_B$ and $y_B \nleq x_B$. We need to show

$$d(x, x+y) \leq \frac{2\#x + 2\#y - 4\min\{\gamma(x,y), \gamma(y,x)\}}{n-1} = 2\, d_{Haus}(\Gamma(x), \Gamma(y)).$$

Recalling that $d(x, x+y) = (\#x - \#(x+y))/(n-1)$, this is equivalent to showing

$$4\min\{\gamma(x,y), \gamma(y,x)\} \leq \#x + 2\#y + \#(x+y). \tag{34}$$

We will establish this through a series of claims. Before proceding further, let us introduce some definitions.

Under the assumption that $x_B \nleq y_B$ and $y_B \nleq x_B$ for all $B \in x+y$, we define a graph $\mathcal{G}_x = (\mathcal{V}_x, \mathcal{E}_x)$ as follows. The vertices of $\mathcal{G}_x$ are the blocks of x. Two blocks $A, B \in x$ are connected by an edge if there is some block $C \in y$ so that $A \cap C$ and $B \cap C$ are both nonempty, and $A \in x$ has a self loop if there is $Y \in y$ so that $Y \subseteq A$. By our assumptions, every vertex is incident to at least one edge that is not a self loop. We note that this graph is not simple in the sense that we allow two blocks to be joined by multiple edges, one for each block of y which intersects each of the blocks non-trivially. We thus have an edge labeling function $\ell_x : \mathcal{E}_x \to y$ which sends each edge to the block of y that induces it. For an edge $e \in \mathcal{E}_x$, we refer to $\ell_x(e)$ as the *label* of e.

We fix a maximum matching $P := \{\{A_i, B_i\}\}_{i=1}^N$ of the vertices of $\mathcal{G}_x$ and let $\{\{C_j\}\}_{j=1}^S$ enumerate the unmatched vertices. We consider the quotient graph $\widetilde{\mathcal{G}}_x = (\widetilde{\mathcal{V}}_x, \widetilde{\mathcal{E}}_x)$ of $\mathcal{G}_x$ obtained by identifying the vertices corresponding to A_i and B_i for each $i = 1, \ldots, N$. We partition $\widetilde{\mathcal{V}}_x$ into two sets $U_x = \{u_1, \ldots, u_N\}$ and $V_x = \{v_1, \ldots, v_S\}$ corresponding, respectively, to the sets of matched vertex pairs and unmatched vertices. We define the function $L_x : \widetilde{\mathcal{V}}_x \to 2^y$ by sending each vertex $v \in V_x$ to the set of all labels of edges incident to it, and each vertex $u = \{A, B\} \in U_x$ to the set of all labels of edges incident to $\{A, B\}$. For each subset $S \subseteq \widetilde{\mathcal{V}}_x$, we define its support $\mathrm{supp}(S) \subseteq 2^x$ to be the collection of the representative blocks of the elements of S. For example, if $u = \{A, B\} \in U_x$, $\mathrm{supp}(\{u\}) = \{A, B\}$. Note that there can be multiple edges with the same label incident to a vertex v.

Interchanging the roles of x and y, we can define the graph $\widetilde{\mathcal{G}}_y$ similarly, with U_y, V_y, L_y, etc., having the same interpretations.

Claim 6.35. With the same notation as above, the following statements are true.

(1) For $\{A, B\} := u \in U_x$ and $v := \{C\}, v' := \{C'\} \in V_x$, distinct. If there is an edge $e \in \mathcal{E}_x$ incident to $\{A, C\}$ then there is no edge $e' \in \mathcal{E}_x$ incident to $\{B, C'\}$.
(2) For $v, v' \in V_x$, distinct, we have that $L_x(v) \cap L_x(v') = \emptyset$.
(3) For $u, u' \in U_x$ and $v, v' \in V_x$, all distinct, if there are edges e, e' so that e is incident to $\{u, v\}$ and e' is incident to $\{u', v'\}$, then $L_x(u) \cap L_x(u') = \emptyset$.

Proof. We prove each assertion in turn.

(1) If there was an edge $e' \in \mathcal{E}_x$ incident to $\{B, C'\}$ then the set
$$(P \setminus \{\{A, B\}\}) \cup \{\{A, C\}, \{B, C'\}\}$$
would be a matching of $\mathcal{G}_x$ of greater cardinality, contradicting that P is a maximum matching.
(2) Let us denote by C and C' the blocks of x corresponding to $v, v' \in V_x$. If $L_x(v) \cap L_x(v') \neq \emptyset$, then there exist edges e, e' incident to v and v' respectively, such that $\ell_x(e) = \ell_x(e') = Y \in y$. By definition of $\mathcal{G}_x$ this means that $Y \cap C \neq \emptyset$ and $Y \cap C' \neq \emptyset$, so there is an edge e^* incident to $\{v, v'\}$ and $P \cup \{v, v'\}$ would be a matching of $\mathcal{G}_x$. This is a contradiction, since P is a maximum matching.
(3) Let us denote by $\{A, B\}, \{A', B'\}$ the sets of blocks of x corresponding to u, u', respectively, and by C, C' the blocks of x corresponding to v, v' respectively. If $L_x(u) \cap L_x(u') \neq \emptyset$ then there exists edges f, f' in $\mathcal{G}_x$ incident to $\{A, B\}$ and $\{A', B'\}$, respectively, such that $\ell_x(f) = \ell_x(f') = Y$. Therefore A, A', B, B' all have non-empty intersection with Y. As e is incident to $\{u, v\}$ and e' is incident to $\{u', v'\}$, we can assume without loss of generality that $\{A, C\}$ and $\{A', C'\}$ are incident in $\mathcal{G}_x$. Therefore, the set
$$(P \setminus \{\{A, B\}, \{A', B'\}\}) \cup \{\{C, A\}, \{B, B'\}, \{C, A'\}\}$$
is a matching of $\mathcal{G}_x$, again contradicting maximality. □

Claim 6.36. Keeping the same notation, the following statement are true.

(1) For $u \in U_x$ and $v, v' \in V_x$, distinct. If $L_x(u) \cap L_x(v) \neq \emptyset$ then there is no edge e incident to $\{v', u\}$.
(2) For $u \in U_x$ and $v_{i_1}, ..., v_{i_n} \in V_x$, all distinct. If there are edges $e_1, ..., e_n$ incident to $\{u, v_{i_1}\}, ..., \{u, v_{i_n}\}$, respectively, then there exists $X \in u \subset x$ such that $X \cap \ell_x(e_i) \neq \emptyset$ for every $i = 1, ..., n$.

Proof. The results follow from Claim 6.35.1 □

For each $v_j \in V_x$ we define the set $C(v_j) := \{a \in \widetilde{\mathcal{G}}_x : L_x(a) = L_x(v_j)\}$ and the set $\mathfrak{v}_j := \operatorname{supp} C(v_j)$. Let $\overline{\mathfrak{v}}_j := \bigcup_{A \in \mathfrak{v}_j} A$. Note that
$$C(v_j) = \{v_j\} \cup \{u \in U_x : L_x(u) = L_x(a)\}$$
and that $j \neq j'$ implies $C(v_j) \cap C(v_{j'}) = \emptyset$, by Claim 6.35.2. We define a new partition x' to be finest partition greater than x so that each $\overline{\mathfrak{v}}_j$ is a block of x'. Now, we define
$$P' := \{\{A, B\} \in P : \forall j \in \{1, .., S\}, \{A, B\} \notin C(v_j)\}$$
and note that $|P'| = |P| - \sum_{j=1}^{S} |C(v_j) - 1|$.

Claim 6.37. P' is a maximum matching of $\mathcal{G}_{x'}$.

Proof. Let us suppose that there exists a matching Q' of x' with $|Q'| > |P'|$. Since, by the previous observations, for all $j, j' \in \{1, ..., S\}$, with $j \neq j'$, there are no edges incident to $\{\overline{\mathfrak{v}}_j, \overline{\mathfrak{v}}_{j'}\}$; therefore, any pair $\{A, B\} \in Q'$ is of the form $A, B \in x - \bigcup_{j=1}^{S} \mathfrak{v}_j$, or $A \in x - \bigcup_{j=1}^{S} \mathfrak{v}_j$ and $B \in \{\overline{\mathfrak{v}}_j\}_{j=1}^{S}$. Let us denote by Q'_1 the set of pairs of Q' corresponding to the first case, and by Q'_2 the set of pairs corresponding to the second case. We can define a matching Q of $\mathcal{G}_x$ the following way:

- Take all pairs in Q'_1.
- For each pair $\{A, B\} \in Q_2$, taking $A = \overline{\mathfrak{v}}_j$, for some j, there exists $B' \in x \cap \mathfrak{v}_j$ such that $\{B, B'\}$ is an incident edge of $\mathcal{G}_x$. Note that by construction $|\mathfrak{v}_j| = 2|C(v_j)| - 1$, and there exists $Y \in y$ such that $Y \cap X \neq \emptyset$ for all $X \in \mathfrak{v}_j$. Therefore, we can find a perfect matching of $\{B\} \cup \mathfrak{v}_j$ as a subgraph of $\mathcal{G}_x$ with size $|C(v_j)|$.
- Otherwise, we take all pairs in $C(v_j)$.

By construction it follows that the cardinal of this matching is $|Q| + \sum_{j=1}^{S} |C(v_j) - 1| > |P'| + \sum_{j=1}^{S} |C(v_j) - 1| = |P|$, this is a contradiction since $|P|$ is a maximum matching of $\mathcal{G}_x$, therefore $|P'|$ must be a maximum matching of $\mathcal{G}_{x'}$. □

Claim 6.38. There exists a matching of $\mathcal{G}_y$ of cardinality at least $\lceil \frac{S}{2} \rceil$.

Proof. Since P' is a maximum matching of x' and $\{\overline{\mathfrak{v}}_j\}_{j=1}^{S}$ enumerates its unmatched vertices, we can define the sets $U_{x'}, V_{x'}$ and the function $L_{x'} : \widetilde{\mathcal{V}}_{x'} \to 2^y$, and apply Claims 6.35 and 6.36 to them. Note that $U_{x'} \subset U_x$.

Let us define the sets

$$Z_{x'} := \{v \in V_{x'} : |L_{x'}(v)| \geq 2\} \text{ and } W_{x'} := V_{x'} - Z_{x'} = \{v \in V_{x'} : |L_{x'}(v)| = 1\}.$$

Note that for each $v \in W_{x'}$ with $v = \{\overline{\mathfrak{v}}_j\}$ for some j, if $\{Y\} = L_{x'}(v)$, then $\overline{\mathfrak{v}}_j \subset Y$ and $L_x(v_j) = \{Y\} = L_x(\overline{\mathfrak{v}}_j)$. Since $x_B \not\leq y_B$ for all $B \in x+y$, we know that $Y \neq \overline{\mathfrak{v}}_j$, and by Claim 6.35.2. we know that there exists $\{A, B\} = u \in U_{x'}$ such that $Y \cap (A \cup B) \neq \emptyset$. Since $u \in U_{x'}$, we know that $u \notin C(v_i)$, for all $i \in \{1, ..., S\}$, so $L_{x'}(u) \neq \{Y\}$. Therefore, we see that for each $v \in W_{x'}$ there exists $u \in U_{x'}$ such that there is an edge e incident to $\{u, v\}$ and $L_{x'}(u) \neq L_{x'}(v)$. So, we can define a function $\pi_{x'} : W_{x'} \to U_{x'}$ such that there exists an edge incident to $\{v, \pi_{x'}(v)\}$ and $L_{x'}(v) \neq L_{x'}(\pi_{x'}(v))$.

For each $v \in Z_{x'}$, we can define a pair $\{Y_v^0, Y_v^1\} \subset L_{x'}(v)$, by Lemma 6.35.1. We know that if $v \neq v'$ then $\{Y_v^0, Y_v^1\} \cap \{Y_{v'}^0, Y_{v'}^1\} = \emptyset$, therefore the set $M_{Z_{x'}} := \{\{Y_v^0, Y_v^1\}\}_{v \in Z_{x'}}$ is a matching of $\mathcal{G}_y$.

On the other hand, if $u = \pi_{x'}(v)$ for some $v \in W_{x'}$, we can write $\pi_{x'}^{-1}(u) = \{w_i\}_{i=1}^{m}$. Writing $L_{x'}(w_i) = \{Y_{w_i}\}$ for $i = 1, ..., m$, we know, by Corollary 6.36.2, that there exists $X \in u \subset x$ such that for all $i \in \{1, ..., m\}, X \cap Y_{w_i} \neq \emptyset$. By Lemma 6.35.1, we have that $i \neq i'$ implies $Y_{w_i} \neq Y_{w_{i'}}$ and by the construction of $\pi_{x'}$ and Corollary 6.36.1 that $L_{x'}(u) \not\subset \bigcup_{i=1}^{m} L_{x'}(w_i)$. Therefore, we can define a matching of $\mathcal{G}_y$, $M_u :=$

$\{\{Y^0_{u,j}, Y^1_{u,j}\}\}_{j=1}^{\lceil m/2 \rceil}$ where $Y^i_{u,j} := Y_{w_{2j-i}}$ for $i = 0, 1$ and $j = 1, ..., \lfloor m/2 \rfloor$ and, if m is odd, $Y^0_{u,\lceil m/2 \rceil} := Y_{w_m}$ and $Y^1_{u,\lceil m/2 \rceil} \in L_{x'}(u) - \{Y_{w_i}\}_{i=1}^m$.

By Claim 6.35 and Claim 6.36.1, we know that for all u, u' distinct in the image of $\pi_{x'}$ the respective sets of matched vertices in M_u and $M_{u'}$ are pairwise disjoint (if $\{A, B\} \in M_u$ and $\{B, C\} \in M_{u'}$ then $\{A, B\} \cap \{C, D\} = \emptyset$) and pairwise disjoint from the matched vertices in $M_{Z_{x'}}$, thus these matchings may be combined to form a matching $M = M_{Z_{x'}} \cup (\bigcup_{u \in Im(\pi_{x'})} M_u)$ of $\mathcal{G}_y$. We now calculate

$$\begin{aligned} |M| &= |M_{Z_{x'}}| + \sum_{u \in \pi_{x'}(W_{x'})} |M_u| \\ &= |Z_{x'}| + \sum_{u \in \pi_{x'}(W_{x'})} \left\lceil \frac{|\pi_{x'}^{-1}(u)|}{2} \right\rceil \\ &\geq \left\lceil \frac{|Z_{x'}|}{2} \right\rceil + \left\lceil \frac{\sum_{u \in \pi_{x'}(W_{x'})} |\pi_{x'}^{-1}(u)|}{2} \right\rceil \\ &\geq \left\lceil \frac{|Z_{x'}| + |W_{x'}|}{2} \right\rceil = \left\lceil \frac{|V_{x'}|}{2} \right\rceil = \left\lceil \frac{S}{2} \right\rceil \end{aligned}$$

which establishes the claim. □

Now, besides the maximum matching of $\mathcal{G}_x$ previously fixed, we fix a maximum matching $\{\{A'_i, B'_i\}\}_{i=1}^M$ of the vertices of $\mathcal{G}_y$ and let $\{\{C'_j\}\}_{j=1}^R$ enumerate the unmatched vertices. Using the maximum matching of $\mathcal{G}_x$ we define a singular partition $z^* \in \Gamma(x)$ with main block $B_{z^*} := \{a_i, b_i\}_{i=1}^N \cup \{c_j\}_{j=1}^S$ where $a_i \in A_i, b_i \in B_i$ and $\{a_i, b_i\} \subseteq Y \in y$ for $i = 1, ..., N$ and $c_j \in C_j$ for $j = 1, ..., S$. Similarly, we can define a singular partition $w_* \in \Gamma(y)$ using the maximum matching of $\mathcal{G}_y$. It is clear by its construction that $\max_{w \in \Gamma(y)} |B_{z^*} \cap B_w| \leq N + S$ and $\max_{z \in \Gamma(x)} |B_z \cap B_{w^*}| \leq M + R$, therefore

$$\min\{\gamma(x, y), \gamma(y, x)\} \leq N + S, M + R$$

Since $\#x = 2N + S$ and $\#y = 2M + R$ we note that

$$4 \min\{\gamma(x, y), \gamma(y, x)\} \leq 2(M + R) + 2(N + S) = \#x + \#y + R + S$$

To finish the proof, observe that by Claim 6.38 we know that $S \leq 2M$. Thus, we get $R + S \leq R + 2M \leq \#y + \#(x + y)$ and conclude that

$$4 \min\{\gamma(x, y), \gamma(y, x)\} \leq \#x + 2\#y + \#(x + y).$$ □

6.2. **Connections with continuous partition lattices.** Björner and Lovász [8] introduced an inductive limit-type construction on the class of pseudomodular lattices in order to produce examples of continuous limits of lattices. We refer the reader to [7] for a broad survey of this topic. In [6] Björner constructed in this fashion a continuous limit of partition lattice which we will denote by Π_∞. This was realized as a sublattice of a lattice of measurable partitions in subsequent work of Haiman [14].

We now describe the construction of Π_∞. We will use Π_n to denote the finite partition lattice of the set $\{0, 1, ..., n\}$. As metric lattices Π_n and P_{n+1} are isomorphic. If $kn = m$ then a lattice homomorphism

$$\phi_n^m : \Pi_n \to \Pi_m$$

is defined as follows. For a partition $\pi = \{B_0, \dots, B_p\} \in \Pi_n$ such that $0 \in B_0$, $\phi_n^m(\pi) := \{C_0\} \cup \{C_{ij}\}_{1 \le i \le p, 0 \le j \le k-1}$ where $C_{ij} := \{kb - j | b \in B_i\}$ and $C_0 := \{0, 1, \dots, m\} - \bigcup_{i,j} C_{ij}$.

We then define $\Pi_{(\infty)}$ as the direct limit of $(\Pi_k, \phi_1^k$, which is a lattice with a rank function $|\cdot| : \Pi_{(\infty)} \to [0, 1]$ that only takes rational values. We then define Π_∞ as the metric completion of $\Pi_{(\infty)}$ and note that $\Pi_{(\infty)}$ is a sublattice of Π_∞. Observe that for each k the construction induces a lattice embedding $\phi_k : \Pi_k \to \Pi_\infty$.

Question 6.39. Is Π_∞ a model of T_{FPL}?

We do not have a clear answer to the question. We note that it suffices to show that for every $\mathfrak{L}$-formula $\psi(\bar{x})$ we have that $\sup_{\bar{a}_n} |\psi^{\Pi_\infty}(\phi_n(\bar{a}_n)) - \psi^{\Pi_n}(\bar{a}_n)| \to 0$ as $n \to \infty$ where $\bar{a}_n$ ranges over all tuples in Π_n. This would be trivially satisfied if all embeddings $\phi_n : \Pi_k \to \Pi_\infty$ are elementary. However, this is not the case.

Remark 6.40. For all positive integers $n, k > 1$, $\phi_n^{kn}(\Pi_n)$ is not an elementary substructure of Π_{kn}. Moreover, no $\phi_n : \Pi_n \to \Pi_\infty$ is an elementary embedding.

Proof. Let's consider the formula $\psi(x) = \sup_y \varphi(x, y)$ where $\varphi(x, y)$ is as in equation line (18). Picking a modular element $a \in \Pi_n, a \neq 0$, it is clear that $\psi^{\Pi_n}(a) = 0$. By Corollary 6.6 we know that a is a singular partition, so there is a main block A with cardinality at least 2. By the definition of ϕ_n^{kn} we know that $\{A_j\}_{0 \le \le k-1} \subset \phi_n^{kn}(a)$ where $A_j = \{kb - j | b \in A\}$. Since $\phi_n^{kn}(a)$ has k blocks of cardinal ≥ 2, it is not a singular partition, thus by Corollary 6.6 we conclude that $\psi^{\Pi_{kn}}(\phi_n^{kn}(a)) \neq 0$. ☐

Despite knowing that the embeddings are not elementary, we can still ask whether the embeddings allow us to approximate formulas of T_{FPL} uniformly. In other we are asking if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \ge N$, all $\mathfrak{L}$-formula $\psi(\bar{x})$ and all $\bar{a} \in \Pi_n$, $|\psi^{\Pi_n}(\bar{a}) - \psi^{P_{Bjorn}}(\varphi_n(\bar{a}))| < \epsilon$.

The answer turns out to still be no. Let us pick $\epsilon = \frac{1}{3}$, for each $n \in \mathbb{N}$ we can define the $\mathfrak{L}$-formula with n free variables, $\psi_n(x_1, ..., x_n) := \sup_y \min\{d(x_1, y), ..., d(x_n, y)\}$. It is clear that for all $n \in \mathbb{N}$, if we denote $|\Pi_n| = N$ and $\Pi_n = \{a_i\}_{i=1}^N$, then $(\psi_N(a_1, ..., a_N))^{\Pi_n} = 0$. However, in Π_{2n} we can define the partition $z := \{\{0\}\} \cup \{\{2i-1, 2i\}\}_{i=1}^n$. Note that for any partition $x = \{B_0, ..., B_p\} \in \Pi_n$ we know that $\varphi_n^{2n}(x) = \{C_0\} \cup \{C_{ij}\}_{1 \le i \le p, j=1,0}$ where $C_{ij} = \{2b - j | b \in B_i\}$ and $C_0 = \{0, 1, ..., 2n\} - \bigcup_{i,j} C_{ij}$, so for each $m \in \{1, ..., n\}$ if $m \in B_i \in x$ for some $i \in \{1, ..., p\}$ then $2m \in C_{i0}$ and $2m - 1 \in C_{i1}$. This implies that for all $x \in \Pi_n$, $\varphi_n^{2n}(x) + z = \{C_0\} \cup \{C_{i0} \cup C_{i1}\}_{i=1}^p$, and that $\#(\varphi_n^{2n}(x) + z) = \#x$. Now, if we compute the distance of $\varphi_n^{2n}(x)$ and z we get that

$$d(\varphi_n^{2n}(x), z) = \frac{\#\varphi_n^{2n}(x) + \#z - 2\#(\varphi_n^{2n}(x) + z)}{2n} = \frac{2\#x - 1 + n + 1 - 2\#x}{2n} = \frac{n}{2n} = \frac{1}{2}$$

for all $x \in \Pi_n$. Therefore, $\min\{d(\varphi_n^{2n}(a_1), z), ..., d(\varphi_n^{2n}(a_N), z)\} = \frac{1}{2}$ and it follows that $\psi_N^{\Pi_\infty}(\varphi_n(a_1, ..., a_N)) \geq \psi_N^{\Pi_{2n}}(\varphi_n^{2n}(a_1, ..., a_N)) \geq \frac{1}{2} > 0 = \psi_N^{\Pi_n}(a_1, ..., a_N)$. We have proved that for all $n \in \mathbb{N}$ there exists a formula $\psi_{|\Pi_n|}(x_1, ..., x_{|\Pi_n|})$ and elements $a_1, ..., a_{|\Pi_n|} \in \Pi_n$ such that $|\psi_{|\Pi_n|}^{\Pi_n}(\bar{a}) - \psi_{|\Pi_n|}^{\Pi_\infty}(\varphi_n(\bar{a}))| = \psi_{|\Pi_n|}^{\Pi_\infty}(\varphi_n(\bar{a})) - \psi_{|\Pi_n|}^{\Pi_n}(\bar{a}) > \epsilon = \frac{1}{3}$, so there is no uniform elementary approximation of the $\mathfrak{L}$-formulas through the embeddings.

Towards settling Question 6.39, it turns out that we can, in fact, control the behavior for a restricted set of formulas. We say that an L-formula is in *prenex form* if it is of the form $Q_{x_1}^1 Q_{x_2}^2 \cdots Q_{x_n}^n \psi(\bar{x}, \bar{y})$ where ψ is a quantifier-free formula and each Q_i is either sup or inf. Every formula is equivalent to one in prenex form. We call a prenex formula $Q_{x_1}^1 Q_{x_2}^2 \cdots Q_{x_n}^n \psi(\bar{x}, \bar{y})$ $\forall\exists$ if there is $1 \leq i \leq n$ so that $Q_j = \sup$ for all $j \leq i$ and $Q_j = \inf$ for all $j > i$.

Proposition 6.41. *If $\sigma \in T_{FPL}$ is an $\forall\exists$ sentence, then $\Pi_\infty \models \sigma$.*

Proof. First we assume that σ is universal, that is $\sigma = \sup_{x_1,\ldots,x_j} \psi(x_1, \ldots, x_j)$, with ψ quantifier. Since $\sigma \in T_{FPL}$, we get that

$$0 = \psi^{\Pi_n}(a_1, \ldots, a_j) = \psi^{\Pi_\infty}(\phi_n(a_1), \ldots, \phi_n(a_j))$$

for every $a_1, \ldots, a_j \in \Pi_n$ for every n, thus $\Pi_\infty \models \sigma$, by uniform continuity of ψ and density of $\Pi_{(\infty)}$ in Π_∞.

If σ is existential, that is, $\sigma = \inf_{x_1,\ldots,x_j} \psi(x_1, \ldots, x_j)$, then Since ψ is quantifier free and

$$(\inf_{x_1,\ldots,x_j} \psi(x_1, \ldots, x_j))^{\Pi_n} = 0$$

for every n, we can pick some tuple $\bar{a}_n$ in P_n so that $\psi^{P_n}(a_1, \ldots, a_j) = 0$. Therefore $\psi^{\Pi_\infty}(\phi_n(a_1), \ldots, (a_j)) = 0$ and $\Pi_\infty \models \sigma$.

For the general case, consider $\sigma = \sup_{x_1,\ldots,x_k} \inf_{x_{k+1},\ldots,x_j} \psi(x_1, \ldots, x_j)$. We have that for every n,

$$(\sup_{x_1,\ldots,x_k} \inf_{x_{k+1},\ldots,x_j} \psi(x_1, \ldots, x_j))^{\Pi_n} = 0$$

Since each Π_n is a finite partition lattice and ψ is quantifier free, for every $a_1, \ldots, a_k \in \Pi_n$, exists a $b_{k+1}, \ldots, b_j \in \Pi_n$ such that

$$0 = \psi^{\Pi_n}(a_1, \ldots, a_k, b_{k+1}, \ldots, b_j) = \psi^{\Pi_\infty}(\phi_n(a_1), \ldots, \phi_n(a_k), \phi_n(b_{k+1}), \ldots, \phi_n(b_j)) = 0.$$

Therefore $(\inf_{x_{k+1},\ldots,x_j} \psi(\phi_\infty(a_1), \ldots, \phi_\infty(a_k), x_{k+1}, \ldots, x_j)^{\Pi_\infty} = 0$ for every $a_1, \ldots, a_k \in \Pi_{(\infty)}$. By uniform continuity of $\inf_{x_{k+1},\ldots,x_j} \psi(x_1, \ldots, x_k, x_{k+1}, \ldots, x_j)$, we conclude that

$$(\inf_{x_{k+1},\ldots,x_j} \psi(a_1, \ldots, a_k, x_{k+1}, \ldots, x_j)^{\Pi_\infty} = 0$$

for every $a_1, \ldots, a_k \in \Pi_\infty$, so $\Pi_\infty \models \sigma$. □

A natural question at this point is whether every formula in T_{FPL} is equivalent to an $\forall\exists$ formula. Again, the answer is not totally clear. An interesting test case would be the sentence $\inf_x \sup_y \varphi(x, y) \in T_{FPL}$. This sentence is also true in Π_∞ by the results of Björner [6].

Perhaps the most fundamental open question is the following.

Question 6.42. How many models of T_{FPL} exist, up to isomorphism, of density character $\aleph_0$?

In an attempt to answer this question, one can try to formulate invariants which could be shown to hold for certain constructions.

Definition 6.43. Let $\mathcal{L}$ be a metric lattice. We say that a net (a_n) in L is *almost modular* if $\varphi(x, a_n) \to 0$ for all $x \in L$, where φ is as in equation line (18). We say that an almost modular net (a_n) is *trivial* if there exists a net of modular elements (b_n) in L so that $d(a_n, b_n) \to 0$. We say that a metric lattice has *property* Γ if there exists a non-trivial almost modular net.

The terminology "property Γ" comes from the theory of von Neumann algebras, specifically from the seminal work of Murray and von Neumann [26], where it is defined as the algebra having an almost central sequence which is bounded away from sequences of central elements.

Question 6.44. Does Π_∞ have property Γ?

Question 6.45. Do all models of T_{FPL} have property Γ?

Proposition 6.46. *The following are equivalent:*

(1) *No model of T_{FPL} has property Γ.*
(2) *For every $\epsilon > 0$ there exists $k \in \mathbb{N}$ and $\delta > 0$ so that for all P_n, a finite partition lattice, there exists $x_1, \ldots, x_k$ in P_n so that $\sum_{i=1}^{k} \varphi(x_i, y) < \delta$ implies that $d(y, \Sigma_n) < \epsilon$.*

Proof. The proof of "(2) implies (1)" is straightforward.

For the proof of "(1) implies (2)," suppose that the second statement is false. That is, there is some $\epsilon > 0$ so that for all $k, l \in \mathbb{N}$ we have that there is $n_l \in \mathbb{N}$ so that for any choice of $x_1, \ldots, x_k \in P_{n_l}$ there is $y_l \in P_{n_l}$ with $d(y_l, \Sigma_{n_l}) \geq \epsilon$ and $\sum_{i=1}^{K} \varphi(y_l, x_i) \leq \frac{1}{k}\epsilon$. Fix a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and consider the ultraproduct lattice $\mathcal{P} = \prod_{l \in \mathcal{U}} P_{n_l}$. For any finite subset $F = \{x_1, \ldots, x_k\}$ of $\mathcal{P}$ choose representing sequences $x_i = (x_{l,i})_{l \in \mathbb{N}}$ and $y_l \in P_{n_l}$ so that $d(y_l, \Sigma_{n_l}) \geq \epsilon$ and $\sum_{i=1}^{K} \varphi(y_l, x_{l,i}) \leq \frac{1}{|F|}\epsilon$. Setting y_F to be the class of (y_l) in $\mathcal{P}$, we have that (y_F) is a non-trivial almost modular net. $\square$

By Proposition 6.46 an affirmative answer to the following question would imply that no model of T_{FPL} has property Γ.

Question 6.47. Do there exist constants C, K so that for all P_n, a finite partition lattice, there exists $x_1, \ldots, x_K$ in P_n so that $d(y, \Sigma_n) \leq C \sum_{i=1}^{K} \varphi(x_i, y)$?

The following definition is inspired by the work of Krivine and Maurey [17].

Definition 6.48. We say that a metric lattice $\mathcal{L}$ is *stable* if $\lim_i \lim_j |x_i+y_j| = \lim_j \lim_i |x_i+y_j|$ for all sequences (x_i) and (y_j) in L for which both limits exist.

Question 6.49. Is there some "combinatorial" description of stable metric lattices?

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