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The Jacobian Conjecture And

The Degree of Field Extension

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THE DEGREE OF FIELD EXTENSION

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To my grandmother and grandfather and my parents

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ABSTRACT

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Let k be an algebraically closed field of chacteristic zero. If two polynomials $f(x,y)$ and $g(x,y)$ satisfy the Jacobian condition $f_x g_y - f_y g_x \in k^*$,then the degree of the field extension of $k(x,y)$ over $k(f,g)$, $[k(x,y):k(f,g)]$, is less than or equal to the minimum of $\deg f$ and $\deg g$.

INTRODUCTION

The well-known Jacobian conjecture states "let k be an algebraically closed field of characteristic zero and $f(x,y), g(x,y)$ two polynomials over k. If

$$(1) \qquad f_x g_y - f_y g_x \in k^*$$

then $k[x,y] = k[f,g]$." In other words a polynomial map from $\mathbb{A}^2$ to $\mathbb{A}^2$ is one-one and onto if and only if the Jacobian of the map is a nonzero constant. It was proved by Professor T.T.Moh among numerous other results [3] that this conjecture holds if

$$\max\{\deg f, \deg g\} \leq 100.$$

If two polynomials $f(x,y)$ and $g(x,y)$ satisfy the Jacobian condition (1), then it is easy to see that $f(x,y)$ and $g(x,y)$ are algebraically independent over k. Hence the field $k(x,y)$ is a finite extension of the field $k(f,g)$. In fact it is well-known [1] that if $f(x,y)$ and $g(x,y)$ satisfy the Jacobian condition (1), and $k(x,y) = k(f,g)$, then $k[x,y] = k[f,g]$.

In the present paper, by means of the decomposition of polynomials over a *then the degree of field extension of $k(x,y)$ over $k(f,g)$, $[k(x,y):k(f,g)]$, satisfies*

$$[k(x,y):k(f,g)] \leq \min\{\deg f, \deg g\}.$$

Throughout this paper, the ground field k is algebraically closed and of characteristic zero. The second and third sections are preparations for the proof of the theorem, where we do not assume that the polynomials satisfy the Jacobian condition.

In Chapter 1, we will give the definition and basic properties of π-approximate roots of a polynomial over the Puiseux field, which was introduced by Professor T.T.Moh [3]. We will also give a sufficient condition for a polynomial $g(x,y)$ satisfying the condition that the field $k(g)$ is relatively algebraically closed in $k(x,y)$, by the mean of π-approximate roots.

In Chapter 2, we will give a general result which states that if $f(x,y)$ and $g(x,y)$ are algebraically independent over k and monic in y, satisfying

(i) $k(g)$ is relatively algebraically closed in $k(x,y)$,

(ii) $k(x,f,g) = k(x,y)$,

then there exists a constant $c \in k$ such that $g(x,y) - c$ is irreducible and such that

$$[k(x,y):k(f,g)] = [k(\bar{x},\bar{y}):k(f(\bar{x},\bar{y}))]$$

where $\bar{x}$ and $\bar{y}$ are the images of x and y under the quotient map are satisfied by $f(x,y)$ and $g(x,y)$. Hence, with the same notations as above and by a result in algebraic function theory, we have

$$[k(x,y):k(f,g)] = -\sum_{v(f(\bar{x},\bar{y}))<0} v(f(\bar{x},\bar{y}))$$

where the summation is taken over all valuations v on $k(\bar{x},\bar{y})/k$ such that $v(f(\bar{x},\bar{y})) < 0$. Now the key point of the proof of our theorem is that under the Jacobian condition (1) we can prove

$$\begin{aligned} -\sum_{v(f(\bar{x},\bar{y}))<0} v(f(\bar{x},\bar{y})) &\leq -\sum_{v(\bar{x})<0} v(\bar{x}) \\ &= \deg_y g. \end{aligned}$$

Clearly the theorem then follows.

CHAPTER 1

π-APPROXIMATE ROOTS OF A POLYNOMIAL

Following Professor Moh, for a polynomial $g(x,y)$ over k, we put $x = t^{-1}$ and regard g as a polynomial in the variable y over the Puiseux field $k \ll t \gg$. If g is monic in y, then $g(t^{-1},y)$ has the following decomposition over $k \ll t \gg$.

(2)
$$g(t^{-1},y) = \prod_{i=1}^{n}(y - \tau_i)$$

with $\tau_i \in k \ll t \gg$.

DEFINITION 1. ([3]) Let π be a symbol, and let $\sigma = \sum_{j<\delta} a_j t^j + \pi t^\delta \in k[\pi] \ll t \gg$ with $a_j \in k$. Then σ is said to be a π-approximate root (abbr. π-root) of a polymonial $g(y)$ over $k \ll t \gg$ if in the expression in $k[\pi] \ll t \gg$

$$g(\sigma) = f_\sigma(\pi)t^\lambda + \text{higher terms in} \quad t$$

we have $1 \leq \deg_\pi g_\sigma(\pi) < \infty$. The multiplicity of σ as a π-root of $g(y)$ is defined to be $\deg_\pi f_\sigma(\pi)$ as a polynomial in π.

for some $c \in k$ and some rational number ι. In this case we call σ a π-root associated to the root τ.

On the other hand, for every π-root σ of $g(y)$, there exists a root τ of the equation $g(y) = 0$ such that σ is associated to τ.

(II) Let τ be a root of the equation $g(y) = 0$ and let σ be a π-root of $g(y)$ associated to τ. Then for every $\tau^* \in k \ll t \gg$, we have

$$ord_t(\tau - \tau^*) \geq ord_t(\sigma - \tau^*),$$

and therefore

$$ord_t f(\tau) \geq ord_t f(\sigma)$$

for every polynomial $f(y)$ over $k \ll t \gg$.

LEMMA 1.. *Let $g(x, y)$ be a polynomial monic in y. If $g(t^{-1}, y)$ has a π-root σ of multiplicity one such that*

$$ord_t g(t^{-1}, \sigma) = 0,$$

then the field $k(g)$ is relatively algebraically closed in $k(x, y)$.

Proof.: Let $\overline{k(g)}$ be the algebraic closure of $k(g)$ in $k(x, y)$. We want to prove $\overline{k(g)} = k(g)$. Let R denote the integral closure of $k[g]$ in $\overline{k(g)}$. Since every element in R is integral over $k[x, y]$ which is integrally closed, we have $R \subset k[x, y]$. Since $\overline{k(g)}$ is the fraction field of R, it suffices to show that $R = k[g]$. Let h be a non-zero polynomial in R. Then h satisfies an equation

Write

$$h(t^{-1},\sigma) = h_\sigma(\pi) + \text{higher terms in } \quad t$$

. Since $\deg g_\sigma(\pi) = 1$,there exists a polynomial H in one variable such that $h_\sigma(\pi) = H(g_\sigma(\pi))$, and therefore

$$(4) \qquad ord_t(h(t^{-1},\sigma) - H(g(t^{-1},\sigma)) > 0.$$

Since (3) holds for every non-zero polynomial in R and $h - H(g) \in R$, by (4) we have

$$h - H(g) = 0.$$

It follows that

$$R = k[g].$$

Lemma 1 is proved. ∎

CHAPTER 2

DEGREE OF FIELD EXTENSION

The main result of this seection is the following

LEMMA 2. . *Let $f(x, y)$ and $g(x, y)$ be two polynomials over k such that*

(i) *$f(x, y)$ and $g(x, y)$ are algebraically independent over k and monic in y,*

(ii) *the field $k(g)$ is relatively algebraically closed in $k(x, y)$,and*

(iii) *$k(x, f, g) = k(x, y)$.*

Then there exists a constant $c \in k$ such that

$$[k(x, y) : k(f, g)] = [k(\bar{x}, \bar{y}) : k(f(\bar{x}, \bar{y}))] \tag{5}$$

where $\bar{x}$ and $\bar{y}$ are the images of x and y under the quotient map

$$k[x, y] \longrightarrow k[x, y]/(g(x, y) - c).$$

Proof: . Let

Let $n = \deg_y g$. It follows from the above discussion that the polynomial $F(X,Y,Z)$ can be taken as the form

$$F(X,Y,Z) = Y^n + F_1(X,Z)Y^{n-1} + \cdots + F_n(X,Z)$$

with

$$\max_{0<i\leq n} \deg_X F_i(X,Z) = [k(x,y) : k(f,g)].$$

By condition (iii), there exists a non-zero polynomial $G(X,Y,Z)$ such that

$$yG(x,f,g) \in k[x,f,g].$$

It is obvious that for all but finitely many constants $c \in k$

(a) $$G(x, f(x,y), g(x,y)) \not\equiv 0 \pmod{g(x,y) - c}$$

and

(b) $$\max_{0<i\leq n} \deg_X F_i(X,c) = [k(x,y) : k(f,g)].$$

On the other hand, by condition (ii) and the second Bertini theorem (see [4]), for all but finitely many constants $c \in k$ we have

(c) $g(x,y) - c$ is irreducible.

Notice that $k[x,y]/(g(x,y)-c)$ is an integral domain and $k(\bar{x},\bar{y})$ is its fraction field. We claim that in the present case (5) holds. It follows from (a) that $G(\bar{x},f(\bar{x},\bar{y}),c)\neq 0$, and therefore $\bar{y}\in k(\bar{x},f(\bar{x},\bar{y}))$. Hence

$$(6)\qquad [k(\bar{x},f(\bar{x},\bar{y})):k(\bar{x})]=n=\deg_Y F(X,Y,c).$$

Clearly we have

$$(7)\qquad F(\bar{x},f(\bar{x},\bar{y}),c)=0.$$

If the polynomial $F(X,Y,c)$ is reducible, then $f(\bar{x},\bar{y})$ satisfies an equation over $k(\bar{x})$ whose degree is less than n. This contradicts (6). Hence $F(X,Y,c)$ is irreducible and (7) is a defining equation of $\bar{x}$ and $f(\bar{x},\bar{y})$ over k. By (b), it follows that

$$[k(\bar{x},\bar{y}):k(f(\bar{x},\bar{y}))]=[k(\bar{x},f(\bar{x},\bar{y})):k(f(\bar{x},\bar{y}))]=\deg_X F(X,Y,c)=[k(x,y):k(f,g)],$$

as claimed. ∎

To end this chapter we state some basic terminologies and results in algebraic function theory. Let K denote a function field over k, that is, K is a finitely generated field of transcendence degree one over k. A valuation on K/k is a map $v: K\to \mathbb{Z}$ such that

$$v(\alpha\beta)=v(\alpha)+v(\beta)\qquad \text{for all}\quad \alpha,\beta\in K,$$

(II) Let $\alpha \in K, \alpha \notin k$. Then K is a finite extension of $k(\alpha)$, and

$$[K : k(\alpha)] = - \sum_{v(\alpha)<0} v(\alpha)$$

where the sum is taken over all valuations v on K/k satisfying $v(\alpha) < 0$.

(III) In the case that K is generated by $\bar{x}$ and $\bar{y}$, let

$$g(\bar{x}, \bar{y}) = 0$$

be a defining equation of $\bar{x}$ and $\bar{y}$ over k. Then for every valuation v on K/k satisfying $v(\bar{x}) < 0$, there exists a root τ of the equation $g(t^{-1}, y) = 0$ in $k \ll t \gg$ and a positive integer a such that

$$v(h(\bar{x}, \bar{y})) = a\{ord_t(h(t^{-1}, \tau))\}$$

for every polynomial $h(x, y)$.

CHAPTER 3

PROOF OF THEOREM

From now on, we will assume that $f(x,y)$ and $g(x,y)$ are two polynomials monic in y satisfying

(i) $$f_x g_y - f_y g_x \in k^*,$$

(ii) $$\deg_y g(x,y) \leq \deg_y f(x,y).$$

Let $n = \deg_y g(x,y)$. To prove the theorem, it suffices to show that under the above conditions

$$[k(x,y) : k(f,g)] \leq n.$$

For notational simplicity we will write $f(y) = f(t^{-1},y)$, $g(y) = g(t^{-1},y)$. Let $\sigma = \sum_{j<\delta} a_j t^j + \pi t^\delta$ be a π-root of $g(y)$ such that $ord_t g(\sigma) = 0$. Put $\lambda = ord_t f(\sigma)$ and write

$$g(\sigma) = g_\sigma(\pi) + \text{higher terms in} \quad t,$$

$$f(\sigma) = f_\sigma(\pi) t^\lambda + \text{higher terms in} \quad t$$

Proof: . (i). It is obvious that for every $\tau \in k \ll t \gg$

$$ord_t(\sigma - \tau) \leq \delta.$$

Hence, in view of (2), we have

(8) $$0 = ord_t g(\sigma) \leq n\delta.$$

It is easy to see that the Jacobian of $f(\sigma)$ and $g(\sigma)$ with respect to t and π is of the form

(9) $$J_{t,\pi}(f(\sigma), g(\sigma)) = \lambda f_\sigma(\pi) g'_\sigma(\pi) t^{\lambda-1} + \text{higher terms in} \quad t.$$

On the other hand, by the chain rule and the Jacobian condition we get

(10) $$J_{t,\pi}(f(\sigma), g(\sigma)) = J_{x,y}(f,g) J_{t,\pi}(t^{-1}, \sigma) = c_1 t^{-2+\delta}$$

for some $c_1 \in k^*$. To prove $\lambda \geq -1$, we may assume $\lambda \neq 0$. Comparing the right-hand sides of (9) and (10) we get

(11) $$\lambda = -1 + \delta.$$

By (8) and (11) we get (i).

Lemma 4. . *The field $k(g)$ is relatively algebraically closed in $k(x,y)$.*

Proof: . By Lemma 1, it suffices to show that $g(y)$ has a π-root σ with multiplicity one such that

(12) $$ord_t g(\sigma) = 0.$$

First we claim that $g(y)$ has a π-root σ satisfying (12) and

(13) $$ord_t f(\sigma) < 0.$$

On the contrary let us assume that (13) is false. Then for every π-root σ of $g(y)$ satisfying (12) we have

$$ord_t f(\sigma) \geq 0.$$

It follows from Section 2,(II) that for every root τ of the equation $g(y) = 0$ we have

(14) $$ord_t f(\tau) \geq 0.$$

Let g_1 be an irreducible factor of g and let K be the fraction field of the quotient ring $k[x,y]/(g_1(x,y))$. As we did in Section 3, we let $\bar{x}$ and $\bar{y}$ be the images of x and y under the quotient map $k[x,y] \longrightarrow k[x,y]/(g_1(x,y))$. For every valuation v on K/k, if $v(\bar{x}) \geq 0$, then we have $v(f(\bar{x},\bar{y})) \geq 0$. If $v(\bar{x}) < 0$, then by (14) and Section 3,(III) we also get $v(f(\bar{x},\bar{y})) \geq 0$. It therefore follows from Section 3,(I) that

$$f(\bar{x},\bar{y}) \in k.$$

Thus, there exists a constant $c \in k$ such that

(15) $$f(x,y) \equiv c \pmod{g_1(x,y)}.$$

REMARK. .The decomposition of $f(y)$ and $g(y)$ over $k \ll t \gg$ was explicitly investigated by Professor Moh [3] where much deeper results were obtained. In fact, the existance of a π-root σ of $g(y)$ with multiplicity one and satisfying (12) can also be deduced from [3], Section 5, and the result in Lemma 3 is contained in [3], Section 4.

LEMMA 5. . *The polynomials $f(x,y)$ and $g(x,y)$ satisfy*

$$k(x,f,g) = k(x,y).$$

Proof: . See [3], Section 2. ∎

It is well-known that the Jacobian condition implies that $f(x,y)$ and $g(x,y)$ are algebraically independent over k. Hence, by Lemma 4 and 5, $f(x,y)$ and $g(x,y)$ satisfy all conditions in Lemma 2. It now follows from Lemma 2 that there exists a constant $c \in k$ such that $g(x,y) - c$ is irreducible and such that

$$[k(x,y) : k(f,g)] = [k(\bar{x},\bar{y}) : k(f(\bar{x},\bar{y}))] \tag{16}$$

where $\bar{x}$ and $\bar{y}$ are images of x and y respectively under the quotient map $k[x,y] \longrightarrow k[x,y]/(g(x,y) - c)$. We will simply write $\bar{f} = f(\bar{x},\bar{y})$, and, without loss of generality, assume $c = 0$. By Section 3,(II) we get

$$[k(\bar{x},\bar{y}) : k(\bar{f})] = -\sum v(\bar{f}) \tag{17}$$

$k(\bar{x},\bar{y})/k$ satisfying $v(\bar{f}) < 0$. By Chapter 3,(III), there exists a root τ of the equation $g(y) = 0$ and a positive integer a such that

$$(18) \qquad v(\bar{f}) = a\{ord_t f(\tau)\}, \quad v(\bar{x}) = a\{ord_t(t^{-1})\} = -a.$$

By Section 2, (I), there exists a π-root σ associated to τ such that $ord_t g(\sigma) = 0$. It follows from Lemma 3 that

$$ord_t f(\sigma) \geq -1.$$

Since σ is associated to τ, by (18) and Section 2, (II) we get

$$v(\bar{f}) \geq -a = v(\bar{x}).$$

From the above discussion we conclude that

$$(19) \qquad \sum_{v(\bar{f})<0} v(\bar{f}) \geq \sum_{v(\bar{f})<0} v(\bar{x}) \geq \sum_{v(\bar{x})<0} v(\bar{x}).$$

By Section 3, (II) we get

$$(20) \qquad \sum_{v(\bar{x})<0} v(\bar{x}) = -[k(\bar{x},\bar{y}) : k(\bar{x})] = -\deg_y g.$$

By (16), (17), (19) and (20), we complete the proof of the theorem. ∎

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BIBLIOGRAPHY

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VITA

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