Proc. Alg. and Geom. (1995) 103–116

Jacobian Conjecture

T. T. Moh

Department of Mathematics
Purdue University
W. Lafayette, IN 47907, U. S. A.

Abstract

This is a survey talk on the Jacobian Conjecture. The main results of T. T. Moh, S. S. Abhyankar, S. S. S. Wang, H. Bass, E. Connell, D. Wright, A. Sathaye and others are presented. Possible research lines are mentioned.

1 Introduction

The Jacobian Conjecture is a fun problem.

There are only two theorems in Calculus: the fundamental theorem of Calculus and the (local) implicit function theorem. The rest are definitions and rules of computations. The second theorem states that if the Jacobian of a map does not vanish at a point, then the inverse map exists locally. We wish to globalize the above theorem and generalize it to other fields if possible.

It is easy to observe that if the ground field k is of positive characteristic p, then even in one variable case for $\pi(x) = x + x^p$, $\pi'(x) = 1$, and there is no inverse map of π, i.e., there is no polynomial σ such that $\sigma(x) + \sigma(x)^p = x$. We'd better consider only characteristic 0 case. Furthermore, in our possible generalization of (local) implicit function theorem, if we allow differentiable or analytic functions, then the generalization fails when we consider

$$\begin{aligned} f(x,y) &= e^x \\ g(x,y) &= ye^{-x} \end{aligned} \tag{1.1}$$

Note that the Jacobian of the map is 1, and e^x is not invertible. We conclude that we'd better consider only **polynomial maps**.

Another possible way is to require that the Jacobian of the map which is a polynomial vanishes at no point in the space. We have, for instance, the Real Jacobian Conjecture: if the Jacobian of the map of $\mathbf{R}^n \mapsto \mathbf{R}^n$ vanishes at nowhere, then the map is globally invertible. Although there were two proofs of this Conjecture circulated around, last year there was a counter-example by Serguey Pinchuk of Russia [22]. His counter-example is a polynomial map of $\mathbf{R}^2 \mapsto \mathbf{R}^2$ consisting of a pair of polynomials of degrees 40 and 9 with Jacobian sum of squares. It is easily checked.

The true property required is the constancy of the Jacobian of the map, i.e., the change of the volume should be a constant through out.

We shall formulate the Jacobian Conjecture as follows:

Conjecture 1.1 (Jacobian Conjecture). *Let k be a field of characteristic 0 and n a positive integer. Consider a polynomial map $\pi : k^n \mapsto k^n$ which is given as follows,*

$$\begin{aligned} y_1 &= \pi(x_1) = f_1(x_1, \cdots, x_n) \\ y_2 &= \pi(x_2) = f_2(x_1, \cdots, x_n) \\ &\cdots\cdots\cdots \\ y_n &= \pi(x_n) = f_n(x_1, \cdots, x_n) \end{aligned} \tag{1.2}$$

If the Jacobian J of the map defined as usual as

$$J = \det\left(\frac{\partial y_i}{\partial x_j}\right) \tag{1.3}$$

is a non-zero constant, then the map is bijective.

Note that it follows from a result of Ax that for polynomial maps, injection implies bijection. The reasoning of Ax is as follows, (1) for finite fields, and bounded degree, it follows from 'pigeon hole principle' that injection implies bijection. (2) using a theorem of Tarski and Lowenheim Skolem theorem from logic, we conclude that the statement is true in general. The preceding sounds mysterious. Recently, Ming-Chang Kang gives a short mathematical proof of the above statement.

Let us assume from now on that the field k is *algebraically closed* (which is convenient while non-essential) and the Jacobian J is a non-zero constant.

Certainly, we have $k[y_1, \ldots, y_n] \subseteq k[x_1, \ldots, x_n]$. We claim that $k(x_1, \cdots, x_n)$ is an *algebraic extension* of $k(y_1, \cdots, y_n)$. Otherwise, let

$t \in k(x_1, \cdots, x_n)$ be transcendental over $k(y_1, \ldots, y_n)$. Then we have

$$\begin{aligned} \frac{\partial x_1}{\partial t} &= \sum_i \frac{\partial x_1}{\partial y_i}\frac{\partial y_i}{\partial t} \\ \frac{\partial x_2}{\partial t} &= \sum_i \frac{\partial x_2}{\partial y_i}\frac{\partial y_i}{\partial t} \\ &\cdots\cdots\cdots \\ \frac{\partial x_n}{\partial t} &= \sum_i \frac{\partial x_n}{\partial y_i}\frac{\partial y_i}{\partial t} \end{aligned} \tag{1.4}$$

It follows from the non-zero constant Jacobian that $\frac{\partial x_j}{\partial y_i}$ exists, and from the transcendental property of t that $\frac{\partial y_i}{\partial t}$ are all zeroes. Therefore $\frac{\partial x_j}{\partial t}$ are all zeroes. We conclude that $\frac{\partial t}{\partial t}$ is zero! A contradiction.

We claim that the map is bijective if and only if $k[x_1, \cdots, x_n] = k[y_1, \cdots, y_n]$.

If the above is an equality, then the map is an automorphism of k^n, and hence an injection and a bijection.

On the other hand if the map is bijective, let the minimal primitive polynomial satisfied by x_j over $k[y_1, \cdots, y_n]$ be as follows,

$$\sum_0^{m_j} h_{ji}(y_1, \cdots, y_n)x_j^i = 0 \tag{1.5}$$

If $m_j > 1$, recall that k is an infinit field, there are points $(a_1, \cdots, a_n)$ such that $h_{jm_j}(a_1, \cdots, a_n) \neq 0$, and there are m_j possible solutions for x_j, then the map is not injective. We conclude that $m_j = 1$ for all j. If h_{j1} is not a constant, then we may select a root $(b_1, \cdots, b_n)$, then there is no point mapping to it. We now have established our claim.

We may reformulate the Jacobian Conjecture as follows,

Conjecture 1.2 (Jacobian Conjecture for n variables). *Given a polynomial map π of the ring $k[x_1, \cdots, x_n]$ of n variables over an algebraically closed field of characteristic 0. Let $\pi(x_i) = y_i$. The Jacobian J of the map is a non-zero constant, if and only if $k[x_1, \cdots, x_n] = k[y_1, \cdots, y_n]$*

The Jacobian Conjecture as stated is unknown except the trivial case of $n = 1$. We shall start with $n = 2$. We will restate

Conjecture 1.3 (Jacobian Conjecture for two variables). *Given two polynomials $f(x, y), g(x, y)$ in two variables over an algebraically closed field* *k of characteristic* 0*, suppose that the following Jacobian condition is satisfied,*

$$J_{x,y}(f(x,y),g(x,y)) = \frac{\partial(f,g)}{\partial(x,y)} \in k^* \tag{1.6}$$

Then we have $k[x,y] = k[f,g]$.

The above conjectures are beautiful and elegant. Are there any truths to them? Could they just be illusions? In 1972 we obtained a verification for polynomial maps of degree less than or equal to 100 for the two variables cases. Later R. C. Heitmann [16] used a different method to get similar results. Could nature (or God) be so mean? It is likely that the Jacobian Conjecture is true for two variables. For three or more variables, there is hardly any evidence for an affirmative answer. In the 70's, while we started learning the computer software 'macsyma', A. Sathaye helped us to use it to compute all degree 3 polynomial maps of three variables with constant Jacobian. The Jacobian Conjecture is true for those cases. However, the computer printout (without any details of computations) is 100 pages long. Since nobody (including us) ever double check the program, we could only say that it is very probably to be true for three variables in the cases of degree three maps.

We'd better clarify the concept of the degrees of a polynomial map. Let us consider the following,

$$\begin{aligned} \pi(x) &= x \\ \pi(y) &= y + x^2 \end{aligned} \tag{1.7}$$

The degree of π with respect to variables x, y is 2 (or the pair (1, 2)). Let us consider a different pair of generators u, v with $u = y, v = x + y^n$. Then we have

$$\begin{aligned} \pi(u) &= y + x^2 = u + (v - u^n)^2 \\ \pi(v) &= x + (y + x^2)^n = (v - u^n) + (u + (v - u^n)^2)^n \end{aligned} \tag{1.8}$$

Therefore, the degree of π with respect to the variables u, v is $2n^2$. If we allow all possible pairs of variables, then we may define the degree of a polynomial map π as the minimal degree with respect to all pairs of variables in the domain and all pairs of variables in the range. Then this degree is a property of the map. The Jacobian Conjecture states in a different way that if the Jacobian is a constant, then the (minimal) degree is 1.

2 A Brief History

The history of the Jacobian conjecture is well-known, over a hundred papers has been published on it. Originally the conjecture was formulated by Keller

[1] as a problem associated with the "Ganze Cremona -Transformationen" in 1939. The article [2] of Bass, Connell and Wright is indispensable reading. Besides history, the authors presented many equivalent forms of the conjecture and discussed many lines of research. Personally, in the late 60's, we were giving a talk about other materials in a seminar at Purdue University while Professor O. Zariski was in the audience. We use the Jacobian Conjecture as a proved theorem in our presentation. At once Professor Zariski pointed out it is unknown. Shafarevich used Jacobian Conjecture for n variables as a known fact in one of his paper of 1967, and later made a similar mistake in his talk in 1970. We are in good company. Since the kernel of the problem is about the Jacobian of a transformation, we decided to call it the Jacobian Conjecture. It becomes the 'official name' of this problem.

This problem had been attacked from all angles. Several wrong proofs were published. The victims include B. Segre [6], Gröbner, W. Engel et al [15]. B. Segre had the distinction of published three faulty proofs. One faulty proof was reviewed by a great French Mathematician to be 'correct'. The other contributions of mathematicians to our enlightenments will be reported in the later part of this article.

The approaches used by analysts and geometers are beyond the scope of this report. Let us consider the essential algebraic approaches.

3 K-theoretic Approach

Using a generalization of '*mean value theorem*', S. S. S. Wang [5] was able to prove

Theorem 3.1. *Let k be a field of characteristic $\neq 2$, and* $\max\{\deg \pi(x_i)\} \leq 2$. *Then π is invertible.*

Proof. It suffices to prove that π is injective. Suppose the contrary. After translations we may assume that $(0, \cdots, 0)$ and $(a_1, \cdots, a_n)$ are sent to $(0, \cdots, 0)$. Let

$$x_i = a_i t \text{ for } i = 1, \cdots, n \tag{3.1}$$

Then we have

$$f_i(x_1, \cdots, x_n) = f_{i1}(a_1, \cdots, x_n)t + f_{i2}(a_1, \cdots, a_n)t^2 \tag{3.2}$$

The above equations vanish at $t = 0, 1$, and they are parabolic, therefore their derivatives vanish at $t = 1/2$. We have

$$\frac{df_i}{dt} = \sum \frac{\partial f_i}{\partial x_i} a_i \tag{3.3}$$

Since the Jacobian is non-zero, then the above system of linear equations in a_i with $t = 1/2$ will imply $a_i = 0$ for all i. A contradiction. □

Let us compare the above theorem with the following theorem of D. Wright [15]:

Theorem 3.2. *If the Jacobian Conjecture for m variables is true for all positive integer m with the further restriction that* $\deg \pi(x_i) \le 3$*, then the Jacobian Conjecture for n variables is true for a given number n without the restriction on the degree of the map.*

Proof. Let

$$d = \max\{\deg \pi(x_i)\} \tag{3.4}$$

If $d \le 3$, then there is nothing to be proved. Suppose that $d > 3$. Let the number of terms of degree d in the set $\{f_i(x_1, \cdots, x_n)\}$ be s. We shall make induction on s. Note that if s drops to zero, then d has to drop.

Let $c\prod x_i^{r_i}$ be a term in one of $f_i(x_1, \cdots, x_n)$, say $i = 1$. Since the total degree is at least 4, we may separate it into two terms, say $c\prod x_i^{p_i}$ and $\prod x_i^{q_i}$, of degrees at least 2 each. Let us introduce two more variables x_{n+1}, x_{n+2}, extend the map π as $\pi(x_{n+1}) = x_{n+1} + c\prod x_i^{p_i}$ and $\pi(x_{n+2}) = x_{n+2} + \prod x_i^{q_i}$. It is easy to see the Jacobian of the extended map stays the same. Let us consider the automorphism σ defined as $\sigma(x_1) = x_1 - x_{n+1}x_{n+2}$ and $\sigma(x_i) = x_i$ for $i > 1$. Let us consider the composition $\sigma\pi$. The we have $\sigma\pi(x_1) = f_1 - (x_{n+1} + c\prod x_i^{p_i})(x_{n+2} + \prod x_i^{q_i})$. Therefore the particular term $c\prod x_i^{r_i}$ disappears from $\sigma\pi(x_1)$, and there is a drop of s. By induction, we conslude that $\sigma\pi$ is bijective. It follows that π is bijective. We are done. □

If the Jacobian Conjecture is true for n variables, then the above theorem is a possible step in proving it. It is clear from the above proof that we are trading the number of coefficients with the number of variables. Furthermore, it is unlikely that the degrees can be further reduced to 2. There seems to be an abyss between degrees 2 and 3.

By more linear transformations we may assume the degree 3 polynomials are all 'pure' in the sense that the quadratic terms are missing, and linear part yield identity map. Along this road, it was proved that we may assume the map $\pi = I + H$ where I is the identity map and the Jacobian matrix of H is nilpotent. These line of attack was carried on by H. Bass, E. Connell, D. Wright [2] and a group of Polish Mathematicians. There are many interesting results, for instance, if the square of the nilpotent matrix is zero, then the Jacobian conjecture for n variables is true (c. f. Bass [2]).

4 Formal Inverse Approach

We may consider $k[x_1, \cdots, x_n] \subset k[[x_1, \cdots, x_n]]$ the formal power series ring. The Jacobian criteria of formal power series ring implies that

$k[[x_1, \cdots, x_n]] = k[[y_1, \cdots, y_n]]$ (this is another way to see $y_1, \cdots, y_n$ are algebraically independent). Therefore x_i are formal power series in $y_1, \cdots, y_n$. The Jacobian conjecture simply states that these formal power series are polynomials, i.e., all but finitely many coefficients vanish. This is a beautiful and mysterious phenomenon that finitely vanishing, i.e., in the computation of the Jacobian, all finitely many but one equations vanish, implies the vanishing of infinitely many equations. To carry out this line of attack, we need an estimate of the degree of the inverse map, and a formula to compute the inverse power series map. We have the following theorem conjectured by S.S.S.Wang, and proved by Ofer Gabber,

Theorem 4.1. *Let us fix a set of variables $x_1, \cdots, x_n$. If π is an automorphism, then we have* $\deg \pi^{-1} \le (\deg \pi)^{n-1}$.

The next thing we need is the formal inverse. There are two inverse formula due to Gurjar and Abhyankar based on a formula of Goursat;

Gurjar's formula:

$$x_i = \sum \frac{1}{\prod r_j! \prod s_j!} \frac{\partial^{r+s}}{(\partial x_1)^{r_1} \cdots (\partial x_n)^{r_n} (\partial y_1)^{s_1} \cdots (\partial y_n)^{s_n}} \cdot (x_i J \prod (x_i - y_i)^{r_i}) \prod y_j^{s_j} \tag{4.1}$$

Abhyankar's formula:

$$x_i = \sum \frac{1}{\prod r_j!} \frac{\partial^r}{(\partial x_1)^{r_1} \cdots (\partial x_n)^{r_n}} (x_i J \prod (x_i - y_i)^{r_i}) \tag{4.2}$$

It becomes a complicated game played by Bass, Connell and Wright [2].

5 Field and Ring Extensions

As we point out before that $k(x_1, \cdots, x_n)$ is an algebraic extension of $k(y_1, \cdots, y_n)$. If $k(x_1, \cdots, x_n)$ is a Galois extension of $k(y_1, \cdots, y_n)$, then L. A. Campbell [14] and later S. Abhyankar [13], proved that the Jacobian Conjecture is true. Let D be the field degree $[k(x_1, \cdots, x_n) : k(y_1, \cdots, y_n)]$. If $D = 1$, then certainly we have Galois extension case, and the Jacobian Conjecture is true. Nevertheless, let us present the following proof due to S. Abhyankar,

Theorem 5.1. *If $D = 1$, then the Jacobian Conjecture is true.*

Proof. Let P be a prime ideal of $A = k[y_1, \cdots, y_n]$ of height 1. Then $P = aA$ for some non-unit a. This element a remains a non-unit in $B = k[x_1, \cdots, x_n]$. So there exists a prime Q in B of height 1 with $Q \supset aB$. Hence $Q \cap A \supset P$.

The Jacobian being non-zero constant implies that height $(Q \cap A) =$ height $Q = 1$. Hence $Q \cap A \supset P$ and $A_P \subset B_Q$. Since A_P and B_Q are valuation rings with the same quotient field, it follows that $A_P = B_Q \supset B$. Hence we have

$$A = \cap\{A_P : \text{height} P = 1\} \supset B, A = B \tag{5.1}$$

□

It is meaningful to study the relation between the two rings $k[x_1, \cdots, x_n]$ and $k[y_1, \cdots, y_n]$. For instance, we have the following theorem,

Theorem 5.2. *If* $\mathbf{C}[x_1, \cdots, x_n]$ *is an integral extension of* $\mathbf{C}[y_1, \cdots, y_n]$, *then the Jacobian Conjecture is true.*

Proof. The constancy of Jacobian implies that the mapping is a local homeomorphism. Integral extension implies that under the mapping, $\mathbf{C}^n$ is a covering of $\mathbf{C}^n$. We know that $\mathbf{C}^n$ is simply connected. Therefore the covering degree is 1. Hence the map is a bijection. □

An algebraic proof of the above theorem shows that the complex field $\mathbf{C}$ can be replaced by any field.

Let us consider the two variable case. In general we may wish to bound the field degree. A result of our formal student Zhang [7] shows that the field degree is bounded by gcd $(\deg f, \deg g)$.

6 Analysis of Singularities at ∞

We shall only consider the case of two variables. Let $f(x,y), g(x,y) \in k[x,y]$ such that the map π defined by $\pi(x) = f(x,y)$ and $\pi(y) = g(x,y)$ is with non-zero constant Jacobian. Since $x, f(x,y), g(x,y)$ are algebraically dependent, let $F(x,f,g) = 0$ be the defining equation. We [3] wish to study the two curves $f = 0, g = 0$ over the field k and the curve $F(x,f,g) = 0$ over the field $k(x)$, especially the singularities of them at infinity.

We observe that $k(x) \subset \cup_i k((x^{-1/i})) = K$. As classically known, the field K is algebraically closed. We may find the roots in K, and establish the expanding technique at ∞. Let us consider the following equation

$$f(x,y) = 0 \tag{6.1}$$

in variable y over the field K. Therefore it splits completely. Let $v_{ij} = \text{ord}_{x^{-1}}(y_i - y_j)$ where y_i, y_j are roots of the above equation. Then the set $\{v_{ij}\}$ gives us the *configuration* [11] of the roots and the geometric properties of the singularities at ∞. We shall study the influence of the constant Jacobian on this set.

Another way to study the configuration of roots is the following. Let τ be an indeterminate. We shall consider the following substitution $\sigma : y \mapsto \alpha + \tau x^{-\delta}$ where $\alpha \in K$ and $\delta \in \mathbf{Q}$. Let us consider

$$\sigma(f(x,y)) = f(x,\sigma(y)) = f_\sigma(\tau)x^\lambda + \cdots \tag{6.2}$$

where $\deg f_\sigma(\tau)$ gives us the number of roots y_i of $f(x,y)$ such that $\operatorname{ord}_{x^{-1}}(\alpha - y_i) \geq \delta$. Let α be one of the roots, and let δ go to ∞. We shall mark down $\deg f_\sigma(\tau)$ as it drops. This way we get a sequence of numbers $\{v_i m/d_i : i = s, \cdots, 1\}$ along a root α, where d_i will be defined below as integers. It turns out v_i is an integer. Note the reversing order of the sequence.

It is not hard to see that by letting α go through all roots, the preceding equation will provide us all informations about the configurations of roots. Thus we may be able to detect the influence of the constant Jacobian by changing variables from x, y to x, τ. That computation was carried out in our previous work [3].

Note that $F(x,f,g)$ defines a genus zero curve with one place at ∞. We build a machine, the tool of *approximate roots*, to handle it in one of our previous work. Mixing the configuration of roots and approximate roots, we were able to detect the implications on the singularities at ∞. First the curve $F(x,f,g) = 0$ or its *parametric* form $(f(x,y), g(x,y))$ with x as constant and y as variable produces a sequence of numbers, i.e., the characteristic data, as follows; let

$$\begin{aligned}
\deg_{x,y} f(x,y) = \deg_y f(x,y) = m, \qquad \deg_{x,y} g(x,y) = \deg_y g(x,y) = n \\
g(x,y) = \eta^{-n}, \qquad \eta \in k[x]((y^{-1})) \\
f(x,y) = \eta^{-m} + \sum_{j>-m} f_i(x)\eta^j \in k[x]((\eta)) \\
d_1 = \deg_{x,y} g(x,y) = \deg_y\ g(x,y) = n \\
m_1 = \min\{i : f_i \neq 0, d_1 \nmid i\} \\
\cdots\cdots \\
d_{j+1} = \gcd\{n, m_1, \cdots, m_j\} \\
m_{j+1} = \min\{i : f_i \neq 0, d_{j+1} \nmid i\}
\end{aligned} \tag{6.3}$$

Note that the above sequences are independent of all roots of $f(x,y), g(x,y)$. The characteristic data control the configurations of roots of $f(x,y)$ and $g(x,y)$ under the assumption that the Jacobian is a constant as follows.

If for a root α, the associated sequence $\{v_i m/d_i\}$ has the following property for all i,

$$v_{i+1}d_i/d_{i+1} \geq v_i > d_i/(n - m_i) \tag{6.4}$$

Then the root is called a *major* root. Otherwise it is called a *minor* root. For a major root α, the value of the place such that $\deg f_\sigma(\tau)$ drops is given by

$$\delta_i = 1 - \frac{(n-m_i)\prod_{j=i+1}^{s}[v_j(n-m_j)-d_j]}{(n-m_s-1)\prod_{j=i+1}^{s}[v_j(n-m_{j-1})-d_j]} \tag{6.5}$$

From the above formula, we conclude that

Proposition 6.1. *If $m_s \neq n-2$, then the Jacobian Conjecture is true. If $m_s = n-2$, then there are two points at ∞. Moreover, if $s=2$, i.e., there are two characteristic pairs, then δ_2 has too large a denominator to be possible (consider conjugations of the field K).*

The above proposition was discovered independently by S. S. Abhyankar. The important phenomena are the existence of major roots, the values of possible δ_i, and the non-splitting of the minor roots until $\operatorname{ord} f = \operatorname{ord} g = 0$. We used the above analysis in a computer program for assigned values of the characteristic pairs to check if the denominators are too large along a major root. It enables us to verify all maps with degrees up to 100.

Heitmann studies $dx/\frac{\partial f}{\partial y}$ on the family of curves $f+\lambda$ to deduce similar results later.

Recently Hai-chau Chang and Lih-chung Wang obtain similar results using group actions.

7 Resultant

Let us assume that

$$\deg_{x,y} f(x,y) = \deg_y f(x,y) = m, \qquad \deg_{x,y} g(x,y) = \deg_y g(x,y) = n \tag{7.1}$$

Note that the above can be achieved by a general linear transformation. Let us consider the resultant of $f(x,y)+p, g(x,y)+q$ with respect to y. Then we get a polynomial

$$\Phi(p,q,x) = \text{Resultant}(f+p, g+q, y) = \phi(p,q)x^t + \cdots + \text{lower terms in x} \tag{7.2}$$

If $\phi(p,q)$ is a non-zero constant, then x is integral over $k[f,g]$. Since y is integral over $k[x,f]$, then $k[x,y]$ is integral over $k[f,g]$. if $t=1$, then

$x \in k(f,g)$. Both cases are simple and have been discussed before. Let us use the materials of our analysis of singularities at ∞. Then we have

$$\begin{aligned} f(x,y) &= \prod_{\text{major}} (y-u_i) \prod_{\text{minor}} (y-u_i) \\ g(x,y) &= \prod_{\text{major}} (y-v_i) \prod_{\text{minor}} (y-v_i) \end{aligned} \tag{7.3}$$

Note that all major roots split before $\text{ord}(f)$ and $\text{ord}(g)$ reach 0, and all minor roots will not split completely before $\text{ord}(f)$ and $\text{ord}(g)$ reach 0. Due to the arbitrary numbers p, q, we have

$$\begin{aligned} t &= -\text{ord}_{x^{-1}}(\Phi(p,q,x)) = -\text{ord}_{x^{-1}}(\text{Resultant}(f+p, g+q, y)) \\ &= -\text{ord}_{x^{-1}}(\prod_{\text{major}} (u_j - v_i)) \end{aligned} \tag{7.4}$$

and let $\sigma(y) = \alpha + \tau x^{\delta}$ be a typical minor root such that $\text{ord}\,\sigma(f(x,y)) = 0$, then we have

$$\phi(p,q) = \prod_{\text{minor}} \text{Resultant}(f(x,\sigma)+p, g(x,\sigma)+q, \tau) \tag{7.5}$$

Therefore the most important informations about the resultant are provided by our analysis of singularities at ∞.

8 Pencil of Curves

We may study the pencil of curves $f(x,y)+\lambda$. If the Jacobian Conjecture is true, then $f(x,y)$ serves as a variable. We shall expect that $f(x,y)+\lambda$ is indistinguishable from $x+\lambda$. For instance, $f(x,y)+\lambda$ should be irreducible for all λ. But this is unknown.

Since last year, we had constantly discussed the Jacobian problem with A. Sathaye. The following is the base of our discussions. Let us assume that

$$\deg_{x,y} f(x,y) = \deg_y f(x,y) = m, \qquad \deg_{x,y} g(x,y) = \deg_y g(x,y) = n \tag{8.1}$$

We shall study the pencil $f + ag + b$ over the field k and pay attention to the pencil at ∞. The essential tool is the 'Zeuthen-Segre-Jung' formulas.

We shall use the 'Zeuthen-Segre-Jung' formula as follows. In general, we have

(1)Let f, g be arbitrary polynomials. First, Consider the projective plane P^2. For any projective curve F, we consider the following:

(A) F' shall denote the **reduced** part of F. At any point P, we define:

$$J(F,P) = J(F',P) = \langle F'_u, F'_v, P\rangle \tag{8.2}$$

Here, the last notation describes the local intersection multiplicity of partial derivatives with respect to local parameters u, v at the point P. Since F' has no multiple factors, the intersection multiplicity is a finite nonnegative integer.

(B) $J(F)$ shall denote the sum of $J(F,P)$ over all points of F.

(C) We define:

$$r(F) = r(F') = 2p_a(F') - J(F) \tag{8.3}$$

here P_a stands for the arithmetic genus.

An easy reformulation for irreducible F' gives that

$$r(F) = 2p_g(F') + \sum(\nu(F',P) - 1) \tag{8.4}$$

Here $\nu(F',P)$ denotes the number of branches to F' at P.

(D) Consider a noncomposite pencil $F_\lambda = F + \lambda G$. The Zeuthen-Segre-Jung formula states that:

$$r(F) + r(G) + T = \sum_{\lambda\neq 0} r(H) - r(F_\lambda) \tag{8.5}$$

Here H stands for a generic member of the pencil and T is *one less* than the number of base points. Note that the points are determined over the whole projective plane and we only count their number without any attached multiplicities.

(2) We apply the above 'Zeuthen-Segre-Jung' formula to a pair of polynomials satisfying the Jacobian Condition. Let

$$\Phi(p,q,x) = \text{ Resultant}(f+p, g+q, Y) = \phi(p,q)x^t + \cdots + \text{lower terms in X} \tag{8.6}$$

Then $\phi(p,q)$ is completely determined by the numerical *minor roots* in our previous work. This is one of the connections between our old approach and the new one jointly with A. Sathaye. There are many consequences of 'Zeuthen-Segre-Jung' formula in this particular case. For instance, we have

$$T = \sum_i d_i(1 + \sum_{\phi_i} \mu_{i,P} - \nu_{i,P}) \tag{8.7}$$

where $\phi(p,q) = \prod_i \phi_i(p,q)^{d_i}$, and where $\nu_{i,\infty}$ is the number of branches of C_i at infinity and $\mu_{i,\infty}$ is the multiplicity of the corresponding point.

All numbers above can be computed given the the analysis of singularities at ∞ in [3]. Some new restrictions are discovered by using 'Zeuthen-Segre-Jung' formula.

References

[1] O. H. Keller, *Monatshefte Math. Phys.* **47** (1939) 299.

[2] H. Bass, E. Connell, and D. Wright, *Bull. Amer. Math. Soc. (N. S.)* **7** (1982) 310.

[3] T. T. Moh, *J. Reine Angew. Math.* **340** (1983) 140.

[4] W. Van der Kulk, *Nieuw Arch. Wisk.* **I** (1953) 33.

[5] S. S. S. Wang, *J. Alg.* **65** (1980) 453.

[6] B. Segre, *"Forme differenziali e loro integrali Vol(2)"*, Roma 1956.

[7] Y. T. Zhang, *"On a bound of degrees for Jacobian maps"*, thesis, Purdue University 1991.

[8] T. T. Moh, *J. Alg.* **65** (1980) 301.

[9] T. T. Moh, *Proc. Amer. Math. Soc.* **44** (1974) 22.

[10] S. Abhyankar and T. T. Moh, *J. Reine Angew. Math.* **260** (1973) 47.

[11] T. T. Moh, *J. Math. Soc. Japan* **34** (1982) 637.

[12] S. Abhyankar and T. T. Moh, *J. Reine Angew. Math.* **261** (1973) 29.

[13] S. Abhyankar, *"On Expansion Techniques in Algebraic Geometry"*, TATA Institute Bombay 1977.

[14] L. A. Campbell, *Math. Ann.* **205** (1973) 243.

[15] W. Engel, *Math. Ann.* **130** (1955) 11.

[16] R. Heitmann, *J. Pure Appl. Alg.* **60** (1990) 35.

[17] David Wright, *Illinois J. Math.* **25** (1981) 423.

[18] H. W. E. Jung, *J. Reine Angew. Math.* **184** (1942) 161.

[19] H. Bass, E. Connell, and D. Wright, *"Polynomial automorphisms and the Jacobian Conjecture"*, preprint.

[20] M. Nagata, *"On the automorphism group of K[x,y]"*, in *"Lectures in Mathematics"*, Tokyo 1972.

[21] G. Angermuller, *Math. Z.* **153** (1977) 267.

[22] S. Pinchuk, "*A Counterexample to the Real Jacobian Conjecture*", manuscript 1994.

[23] M. Razar, *Israel J. Math.* **32** (1979) 97.

[24] A. Sathaye, *Amer. J. Math.* **99** (1977) 1105.