MA 598 Sect 528 AANT : Analytic Number Theory : a first course.

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§1. Introduction : generating functions.

We are familiar with the idea of generating functions of the shape
$$A(z) = \sum_{l=0}^{\infty} a_l z^l \qquad (\text{formal power series}).$$
The coefficients $a_l \in \mathbb{C}$ can be extracted from the power series $A(z)$ via Cauchy's formula
$$a_n = \frac{1}{2\pi i} \oint \frac{A(z)}{z^{n+1}}\, dz,$$
provided $A(z)$ converges appropriately and a suitable contour of integration is chosen. Moreover, if $B(z) = \sum_{m=0}^{\infty} b_m z^m$ is a second such power series, then
$$A(z)B(z) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} a_l b_m z^{l+m} = \sum_{n=0}^{\infty} c_n z^n,$$
where
$$c_n = \sum_{\substack{l=0 \\ l+m=n}}^{n} \sum_{m=0}^{n} a_l b_m \qquad (n \geq 0).$$
Thus (formal) power series are appropriate vehicles for discussing additive structure. The associated Fourier analysis begins with the Fourier series obtained by setting $z = e^{2\pi i \alpha}$, which we write as $e(\alpha)$, to obtain the (formal) Fourier series
$$\tilde{A}(\alpha) = \sum_{l=0}^{\infty} a_l\, e(l\alpha).$$

What about multiplicative structure? The generating functions of interest here lie at the core of this course. We now consider the (formal) Dirichlet series

$$\alpha(s) = \sum_{\ell=1}^{\infty} a_\ell \ell^{-s}.$$

It transpires that the coefficients $a_\ell \in \mathbb{C}$ may again be extracted via a relative of Cauchy's formula. Moreover, if $\beta(s) = \sum_{m=1}^{\infty} b_m m^{-s}$ is a second such Dirichlet series, then

$$\alpha(s)\beta(s) = \sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty} a_\ell b_m (\ell m)^{-s} = \sum_{n=1}^{\infty} c_n n^{-s},$$

where

$$c_n = \sum_{\substack{\ell=1 \\ \ell m = n}}^{n} \sum_{m=1}^{n} a_\ell b_m \qquad (n \geqslant 1).$$

This makes it evident that (formal) Dirichlet series are appropriate vehicles for discussing multiplicative structure. The associated Fourier analysis begins with the series obtained by setting $s = it$ $(t \in \mathbb{R})$ to obtain the (formal) series

$$\tilde{\alpha}(it) = \sum_{\ell=1}^{\infty} a_\ell \ell^{-it}.$$

Notice that this series can be written instead as

$$\tilde{\alpha}(it) = \sum_{\ell=1}^{\infty} a_\ell e\left(-\frac{t}{2\pi}\log \ell\right)$$

with frequencies $\log \ell$ $(\ell \in \mathbb{N})$. As $\ell \to \infty$, the gaps between these successive frequencies goes to 0.

Example 1.1. A central example of a Dirichlet series is the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \qquad (1.1)$$

Thus far, the right hand side is merely a formal Dirichlet series. However, when $Re(s) > 1$, we have

$$\sum_{n=1}^{\infty} |n^{-s}| \leqslant \sum_{n=1}^{\infty} n^{-Re(s)} < \infty.$$

By tradition, in this subject we think of s as being of the shape

$$s = \sigma + it,$$

where $\sigma, t \in \mathbb{R}$. Thus, for $\sigma > 1$ we have that the series on the right hand side of (1.1) converges absolutely. The Riemann zeta function $\zeta(s)$ is defined by the series (1.1) for $\sigma > 1$, and by analytic continuation (where possible) for $\sigma \leqslant 1$. This is a matter that will occupy our attention in the next 3 classes.

The multiplicative structure of Dirichlet series has the potential to reveal the properties of multiplicative entities. This idea can be seen already with $\zeta(s)$. Since the series (1.1) converges absolutely for $Re(s) > 1$, we may rearrange terms to see that

$$\sum_{n=1}^{\infty} n^{-s} = \sum_{\substack{n=1 \\ n = p_1^{h_1} \cdots p_r^{h_r} \\ \text{(prime factorisation)}}}^{\infty} \left(p_1^{h_1} \cdots p_r^{h_r}\right)^{-s} = \prod_{\substack{p \\ \text{all distinct primes}}} \left(1 + p^{-s} + p^{-2s} + \cdots\right),$$

Whence

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1}. \qquad (\sigma > 1) \quad \text{(1.2)}$$

This <u>Euler product</u> representation of $\zeta(s)$ encodes critical multiplicative information.

A hint of the hidden content of the relation (1.2) is revealed by a proof of the infinitude of primes. One has

$$\sum_{n \le x} \frac{1}{n} = \log x + O(1)$$

$$\wedge\!\!\wedge$$

$$\prod_{p \le x} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \prod_{p \le x} \left(1 - \frac{1}{p}\right)^{-1}.$$

Thus

$$-\sum_{p \le x} \log\left(1 - \frac{1}{p}\right) \geqslant \log\log x + O\left(\frac{1}{\log x}\right)$$

$$\|$$

$$\sum_{p \le x} \left(\frac{1}{p} + \frac{1}{2p^2} + \cdots\right) = \sum_{p \le x} \frac{1}{p} + O(1).$$

We therefore conclude that $\displaystyle\sum_{p \le x} \frac{1}{p} \geqslant \log\log x + O(1) \longrightarrow \infty$

as $x \to \infty$. In particular, there are infinitely many prime numbers. Using the standard notation

$$\pi(x) = \sum_{p \le x} 1,$$

we see that if $\pi(x) = n \in \mathbb{N}$, then crudely

$$\sum_{m \le n} \frac{1}{m} \geq \sum_{p \le x} \frac{1}{p} \geq \log\log x + O(1),$$

whence $\log n \geq \log\log x + O(1)$. Then $\pi(x) \gg (\log x)$ for large x, meaning that there is a positive number c with the property that

$$\pi(x) \geq c \log x$$

for all large x.

We shall work harder with these ideas, and one of our early objectives will be the proof that

$$\pi(x) \sim \frac{x}{\log x},$$

and more precisely

$$\pi(x) = \mathrm{li}(x) + O\left(x \exp(-c\sqrt{\log x})\right),$$

where

$$\mathrm{li}(x) = \int_2^x \frac{dt}{\log t} = x \sum_{k=1}^{K-1} \frac{(k-1)!}{(\log x)^k} + O_K\left(\frac{x}{(\log x)^K}\right), \qquad (K \in \mathbb{N})$$

with c a suitable positive constant.

<u>Next:</u> Riemann - Stieltjes integration and analytic properties of Dirichlet series

§2. Analytic properties of Dirichlet series.

Recall ~~that~~ we were discussing Dirichlet series of the shape

$$\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}. \qquad (2.1)$$

Just as it makes sense to consider the radius of convergence of a power series $\sum_{n=0}^{\infty} b_n z^n$, so it is desirable to understand the region of the complex plane in which the series $\alpha(s)$ converges.

Definition 2.1. (i) The abscissa of convergence of a Dirichlet series $\alpha(s)$ as in (2.1) is

$$\sigma_c = \inf \Big\{ \sigma \in \mathbb{R} : \sum_{n=1}^{\infty} a_n n^{-s} \text{ converges for all } s \text{ with } \mathrm{Re}(s) > \sigma \Big\};$$

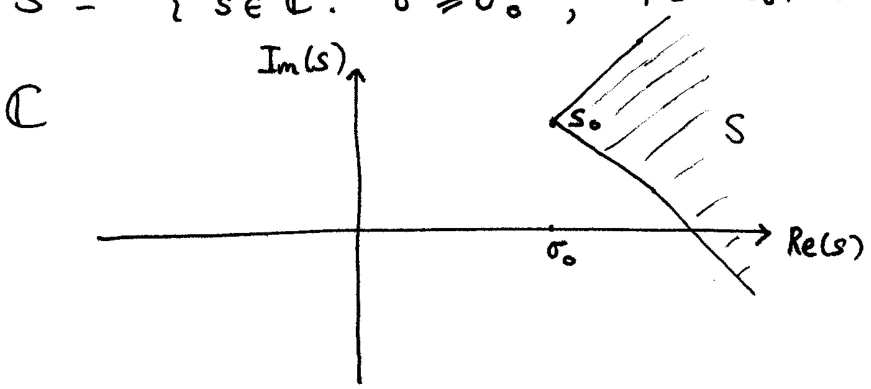
(ii) the line $\sigma = \sigma_c$ is the line of convergence of $\alpha(s)$;

(iii) the half-plane of convergence of $\alpha(s)$ is

$$H_c = \{ s \in \mathbb{C} : \mathrm{Re}(s) > \sigma_c \}.$$

Theorem 2.2. Suppose ~~that~~ $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converges at the point $s = s_0$. Then whenever $H > 0$, the series $\alpha(s)$ is uniformly convergent in the sector

$$S = \{ s \in \mathbb{C} : \sigma \geqslant \sigma_0 , \ |t - t_0| \leqslant H(\sigma - \sigma_0) \}$$



By taking H as large as necessary, it follows ~~that~~ $\alpha(s)$ converges for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \sigma_0$.

Corollary 2.3. The abscissa of convergence of a Dirichlet series $\alpha(s)$ has ~~the~~ property ~~that~~ $\alpha(s)$ converges for all $s \in \mathbb{C}$ with $\sigma > \sigma_c$, and for no $s \in \mathbb{C}$ with $\sigma < \sigma_c$. Further, when $s_0 \in \mathbb{C}$ satisfies $\sigma_0 > \sigma_c$, then there is a neighbourhood of s_0 in which $\alpha(s)$ converges uniformly.

Proof of Theorem 2.2: We'll use properties of Riemann - Stieltjes integration, so we take a diversion. We are interested in defining integrals that accommodate differentials of functions with jump discontinuities. This is all fairly intuitive, and agrees with Riemann integration where applicable.

Define $\int_a^b f(x)\, d\, g(x)$ as a limit of Riemann sums $\sum_n f(\xi_n)\Delta g(x_n)$.

When $a < b$ and $a = x_0 \leq x_1 \leq \dots \leq x_N = b$ is a partition of $[a,b]$, and for each n we have a real number ξ_n ~~satisfying~~ $x_{n-1} \leq \xi_n \leq x_n$, we put

$$S(\underline{x}, \underline{\xi}) = \sum_{n=1}^{N} f(\xi_n)\left(g(x_n) - g(x_{n-1})\right).$$

We say ~~that~~ the Riemann - Stieltjes integral $\int_a^b f(x)\, d\, g(x)$ exists with value I if, for each $\varepsilon > 0$, there exists $\delta > 0$ such ~~that~~

$$|S(\underline{x}, \underline{\xi}) - I| < \varepsilon,$$

whenever

$$\mathrm{mesh}\,\{\underline{x}_\bullet\} := \max_{1\leq n\leq N} (x_n - x_{n-1}) \leq \delta.$$

One can check as exercises the following properties:

(A) The Riemann - Stieltjes integral $\int_a^b f(x)\,dg(x)$ exists whenever f is continuous on $[a,b]$ and g is of bounded variation on $[a,b]$.

$$\left[\text{Thus} \quad \mathrm{Var}_{[a,b]}(g) = \sup_{\underline{x}} \sum_{n=1}^{N} |g(x_n) - g(x_{n-1})| < \infty.\right]$$

(B) If $\int_a^b f\,dg$ exists, then so too does $\int_a^b g\,df$, and

$$\int_a^b g\,df = f(b)g(b) - f(a)g(a) - \int_a^b f\,dg.$$

(C) If g' is continuous on $[a,b]$, then

$$\int_a^b f\,dg = \int_a^b f g'\,dx$$

(D) Suppose that f is continuous on $[0,M]$ and $(a_n)_{n=1}^{\infty}$ is a sequence of complex numbers. Then on writing

$$A(x) = \sum_{1 \le m \le x} a_m,$$

we have

$$\sum_{m=1}^{M} a_m f(m) = \int_0^M f(x)\,d\,A(x).$$

Moreover, if f' is continuous on $[0,M]$, then

$$\sum_{m=1}^{M} a_m f(m) = A(M)f(M) - \int_0^M A(x) f'(x)\,dx.$$

Properties (A), (B), (C) are not completely trivial formal exercises — the readers will profit by convincing themselves of their validity. As for the first of the conclusions in property (D), observe that in the situation described in the definition of

Riemann - Stieltjes integration, with mesh $\{\underline{x}\} \leqslant \delta$, we have

$A(x_n) - A(x_{n-1}) = 0$ whenever x_n, x_{n-1} both lie within an interval between successive integers,

and

$$A(x_n) - A(x_{n-1}) = a_m \quad \text{when} \quad x_{n-1} < m < x_n \text{ with } m \in \mathbb{Z} \cap [0, M].$$

Thus, on noting ~~that~~ the continuity of f implies ~~that~~ we may suppose ~~that~~ $|f(\xi_n) - f(m)| < \varepsilon^2$ in ~~the~~ latter situation, we find ~~that~~

$$\left| S(\underline{x}, \underline{\xi}) - \sum_{m=1}^{M} a_m f(m) \right| < \varepsilon^2 \sum_{m=1}^{M} |a_m| < \varepsilon,$$

say. The desired conclusion follows. □

Back to the proof of Theorem 2.2. We consider ~~the~~ tail of the infinite series defining $\alpha(s)$. Put

$$R(u) = \sum_{n > u} a_n n^{-s_0}.$$

We can apply Riemann - Stieltjes integration to infer what happens at a general argument s in place of s_0. Thus,

$$\sum_{n=M+1}^{N} a_n n^{-s} \overset{(A),(D)}{=} -\int_M^N u^{s_0 - s} \, d\, R(u)$$

[The -ve sign here: notice ~~that~~ as u increases in $R(u)$, we are removing terms]

$$\overset{(B)}{=} -u^{s_0-s} R(u) \Big|_M^N + \int_M^N R(u) \, d u^{s_0 - s}$$

$$\overset{(C)}{=} M^{s_0 - s} R(M) - N^{s_0 - s} R(N) + (s_0 - s) \int_M^N R(u) u^{s_0 - s - 1} du. \tag{2.2}$$

Since $\alpha(s)$ converges at $s = s_0$, we may suppose that M is chosen with $|R(u)| \leq \varepsilon$ for all $u \geq M$. Then, whenever $\sigma > \sigma_0$, we deduce that

$$\left| \sum_{n=M+1}^{N} a_n n^{-s} \right| \leq 2\varepsilon M^{\sigma_0 - \sigma} + |s_0 - s| \int_M^\infty |R(u)| \, u^{\sigma_0 - \sigma - 1} \, du$$

$$\leq 2\varepsilon + \varepsilon |s_0 - s| \int_M^\infty u^{\sigma_0 - \sigma - 1} \, du$$

$$\leq \left(2 + \frac{|s - s_0|}{\sigma - \sigma_0} \right) \varepsilon .$$

When $|t - t_0| \leq H(\sigma - \sigma_0)$, we have

$$|s - s_0| \leq \sigma - \sigma_0 + |t - t_0| \leq (H+1)(\sigma - \sigma_0),$$

whence $$\left| \sum_{n=M+1}^{N} a_n n^{-s} \right| \leq (H+3)\varepsilon . \quad \text{(so "tail" of } \alpha(s) \text{ is small)}$$

We conclude that $\alpha(s)$ converges uniformly in S. //

It follows from Theorem 2.2 that $\alpha(s)$ converges in the half-plane $\mathrm{Re}(s) > \sigma_0$. The first conclusion of Corollary 2.3 follows at once.

Two easy consequences of Corollary 1.3:

- since $\underset{\overset{\shortparallel}{e^{-s \log n}}}{n^{-s}}$ $(n \in \mathbb{N})$ is an analytic function of s, and $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is locally uniformly convergent for $\sigma > \sigma_c$, it follows that $\alpha(s)$ is analytic for $\sigma > \sigma_c$.

- Likewise, the differentiated series
$$\alpha'(s) = -\sum_{n=1}^{\infty} a_n (\log n) n^{-s}$$
is locally uniformly convergent for $\sigma > \sigma_c$.

How do we compute σ_c from the sequence $(a_n)_{n=1}^{\infty}$?

Theorem 2.4. Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and write $A(x) = \sum_{n \leq x} a_n$.

(i) When $\sigma_c \geq 0$, one has
$$\sigma_c = \limsup_{x \to \infty} \frac{\log |A(x)|}{\log x};$$

(ii) When $\sigma_c < 0$, the function $A(x)$ is bounded;

(iii) When $\sigma > \max\{\sigma_c, 0\}$, one has
$$\sum_{n=1}^{\infty} a_n n^{-s} = s \int_1^{\infty} \frac{A(x)}{x^{s+1}} \, dx. \qquad (2.3)$$

Proof. Put
$$\sigma^* = \limsup_{x \to \infty} \frac{\log |A(x)|}{\log x}.$$
If $\sigma_0 > \sigma^*$, then $A(x) \ll x^{\sigma_0}$ (with an implicit constant depending on $\underline{a}$ and σ_0). Then whenever $\sigma > \sigma_0$, we see that
$$\int_1^{\infty} \left| \frac{A(x)}{x^{s+1}} \right| dx \ll \int_1^{\infty} \frac{x^{\sigma_0}}{x^{\sigma+1}} \, dx < \infty.$$
Meanwhile, by Riemann-Stieltjes integration, one has
$$\sum_{n=1}^{N} a_n n^{-s} = \int_{1^-}^{N} x^{-s} \, dA(x) = A(x) x^{-s} \Big|_{1^-}^{N} - \int_{1^-}^{N} A(x) \, dx^{-s}$$
$$= A(N) N^{-s} + s \int_1^N A(x) x^{-s-1} \, dx.$$
The second term on the right hand side here is absolutely convergent

as $N \to \infty$ whenever $\sigma > \sigma_0$, and likewise the first satisfies

$$|A(N)N^{-s}| \ll N^{\sigma_0-\sigma} \to 0 \quad \text{as } N \to \infty.$$

Thus, whenever $\sigma > \sigma^*$, one finds that

$$\sum_{n=1}^{\infty} a_n n^{-s} = s \int_1^{\infty} A(x) x^{-s-1}\, dx. \qquad (2.4)$$

We now divide into cases.

(a) Suppose that $\sigma_c < 0$. Then Corollary 2.3 (with $s=0$) shows that $A(x)$ converges as $x \to \infty$, whence $\sigma^* \le 0$. We may then conclude that $A(x)$ is bounded and (2.3) holds for $\sigma > 0$. This proves (ii)

(b) Suppose that $\sigma_c \geq 0$. Then Corollary 2.3 shows that $\alpha(s)$ diverges when $\sigma < \sigma_c$, whence $\sigma^* \geq \sigma_c$. To show that $\sigma^* \le \sigma_c$, consider $\sigma_0 > \sigma_c$ and observe that (2.2) delivers the estimate

$$A(N) = -R(N)N^{\sigma_0} + \sigma_0 \int_0^N R(u) u^{\sigma_0 - 1}\, du \qquad (M=0 \ \& \ s=0),$$

in which $R(u) = \sum_{n>u} a_n n^{-s_0}$. This series is bounded (because $\alpha(s_0)$ converges), and hence $A(N) \ll_{a,\sigma_0} N^{\sigma_0}$. But then $\sigma^* \le \sigma_0$.

Since this relation holds whenever $\sigma_0 > \sigma_c$, we must have $\sigma^* \le \sigma_c$. Thus we have $\sigma_c = \sigma^*$, and this proves (i), and also (in view of (2.4)) it proves (iii). //

Definition 2.5. The absissa of absolute convergence of a Dirichlet series $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ is defined to be

$$\sigma_a = \inf \{ \sigma \in \mathbb{R} : \sum_{n=1}^{\infty} |a_n| n^{-\sigma} < \infty \}.$$

[This is the absissa of convergence of $\sum_{n=1}^{\infty} |a_n| n^{-s}$]

Theorem 2.6. For any Dirichlet series $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, One has $\sigma_c \le \sigma_a \le \sigma_c + 1$.

Proof. Problem set 1.

Theorem 2.7. Suppose ~~that~~ $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ has abscissa of convergence σ_c. Suppose also ~~that~~ $\delta > 0$ and $\sigma \geq \sigma_c + \delta$. Then for each $\varepsilon > 0$ with $\varepsilon < \delta$, one has

$$\alpha(s) \ll_{\underline{a}, \delta, \varepsilon} (|t| + 4)^{1-\delta+\varepsilon}.$$

Proof. Put $s_0 = \sigma_c + \varepsilon$ and note ~~that~~ by (2.2) one has

$$\alpha(s) = \sum_{n=1}^{M} a_n n^{-s} + R(M) M^{\sigma_c + \varepsilon - s} + (\sigma_c + \varepsilon - s) \int_M^{\infty} R(u) u^{\sigma_c + \varepsilon - s - 1}\, du.$$

Since $\alpha(\sigma_c + \varepsilon)$ converges, we have $a_n \ll n^{\sigma_c + \varepsilon}$, whence $R(u) \ll 1$. Then

$$\alpha(s) \ll \sum_{n=1}^{M} n^{-\delta+\varepsilon} + M^{-\delta+\varepsilon} + \frac{|\sigma_c + \varepsilon - s|}{\sigma - \sigma_c - \varepsilon} M^{\sigma_c + \varepsilon - \sigma}$$

$$\ll M^{1-\delta+\varepsilon}.$$

Thus, on putting $M = [|t|+4]$ we obtain ~~the~~ desired conclusion. //

Finally, we have uniqueness of Dirichlet series expansions.

Theorem 2.8. Suppose ~~that~~ for all s with $\sigma > \sigma_0$, one has

$$\sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} b_n n^{-s}.$$

Then $a_n = b_n$ for all $n \in \mathbb{N}$.

Proof. Put $c_n = a_n - b_n$ and consider $\sum_{n=1}^{\infty} c_n n^{-s}$. We prove by induction ~~that~~ $c_n = 0$ for all n. We may suppose ~~that~~ $\sigma > \sigma_0$ and ~~that~~ $\sum_{n=1}^{\infty} c_n n^{-\sigma} = 0$. Thus

$$c_1 = -\sum_{n=2}^{\infty} c_n n^{-\sigma}.$$

The right hand side here is absolutely convergent for $\sigma > \sigma_0 + 1$, and by dominated convergence one has

$$|c_1| \leqslant \sum_{n=2}^{\infty} |c_n| n^{-\sigma} \to 0 \quad \text{as } \sigma \to \infty.$$

Then $c_1 = 0$, establishing the base of the induction. If we have $c_n = 0$ for $n < N$ and $N \geq 2$, meanwhile, then

$$|c_N| \leqslant \sum_{n > N} |c_n| (N/n)^{\sigma} \to 0 \quad \text{as } \sigma \to \infty,$$

in like manner. Thus $c_N = 0$ and the induction is complete. //

§3. The Riemann zeta function : basic properties.

Observe first that the multiplicative property of (formal) Dirichlet series asserted in our first class is justified in regions of absolute convergence.

Theorem 3.1. Let

$$\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s} \quad \text{and} \quad \beta(s) = \sum_{n=1}^{\infty} b_n n^{-s}$$

be Dirichlet series, and put

$$\gamma(s) = \sum_{n=1}^{\infty} c_n n^{-s}, \quad \text{where} \quad c_n = \sum_{lm=n} a_l b_m.$$

Then provided that $\alpha(s)$ and $\beta(s)$ are both absolutely convergent at $s \in \mathbb{C}$, so too is $\gamma(s)$, and $\gamma(s) = \alpha(s)\beta(s)$.

Proof. The absolute convergence of $\alpha(s)$ and $\beta(s)$ implies that we may rearrange terms to see ~~that~~

$$\alpha(s)\beta(s) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} a_l b_m (lm)^{-s} = \sum_{n=1}^{\infty} c_n n^{-s} = \gamma(s),$$

with

$$\sum_{n=1}^{\infty} |c_n n^{-s}| \leq \sum_{n=1}^{\infty} \sum_{lm=n} |a_l b_m| (lm)^{-\sigma}$$

$$= \left(\sum_{l=1}^{\infty} |a_l| \, l^{-\sigma} \right) \left(\sum_{m=1}^{\infty} |b_m| m^{-\sigma} \right) < \infty. //$$

Motivated by this multiplicative relation, we return to an earlier observation concerning $\zeta(s)$.

Recall that an <u>arithmetic function</u> $f: \mathbb{N} \to \mathbb{C}$ is called <u>multiplicative</u> if $f(1) = 1$ and, whenever $(m,n) = 1$ then
$f(mn) = f(m) f(n)$.

Notice the distinction here between multiplicative functions f and <u>totally multiplicative functions</u> $g: \mathbb{N} \to \mathbb{C}$ for which
$f(mn) = f(m) f(n)$ for all $m, n \in \mathbb{N}$. $\boxed{f(n) = \prod_{p^h \| n} f(p^h).}$

<u>Examples of multiplicative functions:</u>

(i) $f(n) = \mathbb{1}(n) = \{ 1, \quad n \in \mathbb{N}$.

(ii) (Möbius function) $\mu(n) = \begin{cases} (-1)^{\omega(n)}, & n \text{ squarefree} \\ 0, & n \text{ not squarefree} \end{cases}$, where $\omega(n) = \sum_{p|n} 1$.

(iii) (Liouville lambda function) $\lambda(n) = \{ (-1)^{\Omega(n)}$, where $\Omega(n) = \sum_{p^h \| n} h$.

(iv) $\tau(n) = \sum_{d|n} 1$. (divisor function)

<u>Theorem 3.2.</u> Let f be multiplicative and write
$$\varphi(s) = \sum_{n=1}^{\infty} f(n) n^{-s}.$$

Then whenever $\varphi(s)$ is absolutely convergent, one has the <u>Euler product</u> decomposition
$$\varphi(s) = \prod_p \sum_{h=0}^{\infty} \frac{f(p^h)}{p^{hs}} = \prod_p \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots \right)$$

Suppose that $\phi(s)$ is absolutely convergent at $s \in \mathbb{C}$.

<u>Proof.</u> Each factor in the Euler product is absolutely convergent, since

$$\sum_{h=0}^{\infty} \left| \frac{f(p^h)}{p^{hs}} \right| \leq \sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\sigma}} < \infty .$$

We may therefore rearrange terms in any finite product of such Euler factors. Put $\mathcal{N}(x) = \{ n \in \mathbb{N} : p|n \Rightarrow p \leq x \}$. Then

$$\prod_{p \leq x} \sum_{h=0}^{\infty} \frac{f(p^h)}{p^{hs}} = \sum_{n \in \mathcal{N}(x)} \frac{f(n)}{n^s} ,$$

whence

$$\left| \sum_{n=1}^{\infty} f(n) n^{-s} - \prod_{p \leq x} \sum_{h=0}^{\infty} \frac{f(p^h)}{p^{hs}} \right| \leq \sum_{n \notin \mathcal{N}(x)} \frac{|f(n)|}{n^{\sigma}}$$

$$\leq \sum_{n > x} \frac{|f(n)|}{n^{\sigma}} .$$

But given any $\varepsilon > 0$, we can take x large enough ~~that~~ the right hand side here is smaller than ε (since $\phi(s)$ is absolutely convergent). Thus we see ~~that~~

$$\lim_{x \to \infty} \prod_{p \leq x} \sum_{h=0}^{\infty} \frac{f(p^h)}{p^{hs}} = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \phi(s). \quad //$$

<u>Corollary 3.3.</u> When $\sigma > 1$, one has

(i) $$\sum_{n=1}^{\infty} n^{-s} = \zeta(s) = \prod_p (1 - p^{-s})^{-1} ;$$

(ii) $$\sum_{n=1}^{\infty} \mu(n) n^{-s} = 1/\zeta(s) = \prod_p (1 - p^{-s}) ;$$

(iii) $$\sum_{n=1}^{\infty} \lambda(n) n^{-s} = \frac{\zeta(2s)}{\zeta(s)} = \prod_p (1 + p^{-s})^{-1} ;$$

(iv) $\sum_{n=1}^{\infty} \tau(n) n^{-s} = \zeta(s)^2 = \prod_p (1-p^{-s})^{-2}$.

Proof. When $\sigma > 1$, the series in (i), (ii), (iii) are absolutely convergent, since $\sum_{n=1}^{\infty} n^{-\sigma} < \infty$. The series in (iv) is therefore absolutely convergent by virtue of Theorem 3.1.

The coefficients of the Dirichlet series (i) - (iv) are all multiplicative, so in cases (i) and (ii) we obtain the Euler products

$$\zeta(s) = \prod_p \sum_{h=0}^{\infty} \frac{\mathbb{1}(p^h)}{p^{hs}} = \prod_p (1+p^{-s}+p^{-2s}+\dots) = \prod_p (1-p^{-s})^{-1}, \quad \square$$

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \sum_{h=0}^{\infty} \frac{\mu(p^h)}{p^{hs}} = \prod_p (1-p^{-s}).$$

Thus $\zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = 1 \rightarrow \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$. $\square$

In case (iii) we obtain similarly

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p \sum_{h=0}^{\infty} \frac{\lambda(p^h)}{p^{hs}} = \prod_p (1-p^{-s}+p^{-2s}-\dots) = \prod_p (1+p^{-s})^{-1}.$$

Thus

$$\zeta(s) \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p (1-p^{-2s})^{-1} = \zeta(2s). \quad \square$$

Finally, one has

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} = \prod_p \sum_{h=0}^{\infty} \frac{\tau(p^h)}{p^{hs}} = \prod_p (1+2p^{-s}+3p^{-2s}+\dots)$$

$$= \prod_p (1-p^{-s})^{-2} = \zeta(s)^2. \quad \square$$ //

<u>Connection with Möbius inversion formula</u>: Suppose f, g arithmetic functions

Recall that if $g(n) = \sum_{d|n} f(d)$, then $\rightleftarrows$ $f(n) = \sum_{d|n} \mu(d) g(n/d)$.

Corresponding (formal) analogue in Dirichlet series:

$$\zeta(s) \sum_{n=1}^{\infty} f(n) n^{-s} = \sum_{n=1}^{\infty} \Big(\sum_{d|n} f(d) \Big) n^{-s} = \sum_{n=1}^{\infty} g(n) n^{-s}$$

$$\sum_{n=1}^{\infty} f(n) n^{-s} = \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} g(n) n^{-s}.$$

Define the <u>von Mangoldt function</u> $\Lambda(n)$:

$$\Lambda(n) = \begin{cases} \log p, & \text{when } n = p^h, \ h \geq 1, \ p \text{ prime}, \\ 0, & \text{otherwise}. \end{cases}$$

<u>Corollary 3.4.</u> When $\sigma > 1$, one has

$$\log \zeta(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s}$$

and

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s} \qquad \text{(logarithmic derivative of } \zeta(s)\text{)}.$$

<u>Proof.</u> Using the Euler product for $\zeta(s)$, one has

$$\log \zeta(s) = -\sum_{p} \log(1 - p^{-s}) = \sum_{p} \sum_{k=1}^{\infty} k^{-1} p^{-ks} = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s},$$

Thus, by differentiating,

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} \cdot \log n \cdot n^{-s} = -\sum_{n=1}^{\infty} \Lambda(n) n^{-s}. \quad //$$

Notice ~~that~~

$$\sum_{n=1}^{\infty} (\log n) n^{-s} = -\zeta'(s) = \zeta(s) \Big(-\frac{\zeta'}{\zeta}(s) \Big) = \sum_{n=1}^{\infty} \Big(\sum_{d|n} \Lambda(n/d) \Big) n^{-s},$$

whence $\sum_{d|n} \Lambda(d) = \log n$.

Analytic continuation of the Riemann zeta function.

Thus far we have defined $\zeta(s)$ as an analytic function for $\sigma > 1$ by means of the series $\sum_{n=1}^{\infty} n^{-s}$. It is natural to attempt to analytically continue $\zeta(s)$ into the plane $\sigma \leq 1$.

Theorem 3.5. Suppose that $s \in \mathbb{C}$ satisfies $s \neq 1$, $\sigma > 0$, and that $x > 0$. Then one has

$$\zeta(s) = \sum_{n \leq x} n^{-s} + \frac{x^{1-s}}{s-1} + \frac{\{x\}}{x^s} - s \int_x^{\infty} \frac{\{u\}}{u^{s+1}}\, du.$$

[Note: $\{\theta\} = \theta - \lfloor \theta \rfloor$, so the last integral here is absolutely convergent for $\sigma > 0$]

Proof: Noting that for $\sigma > 1$ one has

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{1 \leq n \leq x} n^{-s} + \sum_{n > x} n^{-s}, \qquad (3.1)$$

we may apply Riemann-Stieltjes integration to rewrite the final sum as

$$\sum_{n>x} n^{-s} = \int_{x^+}^{\infty} u^{-s}\, d\underbrace{\lfloor u \rfloor}_{\sum_{n \leq u} 1} = \int_{x^+}^{\infty} u^{-s}\, du - \int_{x^+}^{\infty} u^{-s}\, d\{u\}$$

$$= \frac{x^{1-s}}{s-1} + \{x\} x^{-s} + \int_x^{\infty} \{u\}\, du^{-s}. \qquad (3.2)$$

The formula claimed in the theorem follows for $\sigma > 1$ and $s \neq 1$ by combining (3.1) and (3.2).

Since the integral $\int_x^{\infty} \frac{\{u\}}{u^{s+1}}\, du$ is convergent for $\sigma > 0$, and uniformly for $\sigma > \delta > 0$, and the integrand is an analytic function of s, it follows ~~that~~ the integral is analytic for $\sigma > 0$. Thus, by the uniqueness of analytic continuation, the claimed formula for $\zeta(s)$ holds for $\sigma > 0$. //

<u>Corollary 3.6</u> The Riemann zeta function has a simple pole at $s = 1$ with residue 1, but is otherwise analytic for $\sigma > 0$.

<u>Proof.</u> Apply Theorem 3.5 with $x = 1$ to obtain

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{s+1}}\,du \qquad (\sigma > 0)$$

$$= \frac{s}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{s+1}}\,du \qquad (\sigma > 0). \quad // \qquad (3.3)$$

Can integrate by parts to analytically continue to $\sigma > -1, \sigma > -2, \ldots$ (use Euler - Maclaurin summation formula).

<u>Corollary 3.7.</u> Let C_0 denote Euler's constant (often called γ), defined by

$$C_0 = \lim_{x\to\infty}\Big(\sum_{1\le n\le x}\frac{1}{n} - \log x\Big).$$

Then the Laurent expansion of $\zeta(s)$ about $s=1$ is given by

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^\infty a_k (s-1)^k,$$

in which $a_k = C_0$.

<u>Proof.</u> We first contemplate Euler's constant C_0. One has

$$\sum_{1\le n\le x}\frac{1}{n} = \int_{1^-}^x u^{-1}\,d\lfloor u\rfloor = \int_{1^-}^x u^{-1}\,du - \int_{1^-}^x u^{-1}\,d\{u\}$$

$$= \log x + 1 - \{x\}/x - \int_1^x \{u\}u^{-2}\,du$$

Then

$$\sum_{1 \leq n \leq x} \frac{1}{n} = \log x + c_0 + O\left(\frac{1}{x}\right), \qquad (3.4)$$

where

$$c_0 = 1 - \int_1^\infty \{u\} u^{-2} \, du .$$

By comparing the definition of C_0 with (3.4), we see ~~that~~ $c_0 = C_0$.

Also, by comparison with (3.3), we find that

$$\zeta(s) = \frac{1}{s-1} + \underbrace{\left(1 - \int_1^\infty \frac{\{u\}}{u^2} \, du\right)}_{C_0} + O(|s-1|)$$

as $s \to 1$.

This confirms the claim concerning the Laurent series for $\zeta(s)$. //

§4. Arithmetic functions – mean values and elementary estimates.

Given an arithmetic function $f : \mathbb{N} \to \mathbb{C}$, we are interested in this section in obtaining statistical information — average value, max/min value, variance, etc. This is, of course, a topic of great importance in its own right. However, if we are to understand the convergence of a Dirichlet series

$$\sum_{n=1}^\infty f(n) n^{-s},$$

then the order of growth of the partial sums $\sum_{n \leq x} f(n)$ plays a critical role in determining the abscissa of convergence σ_c. This is important for the discussion of the last two sections.

When arithmetic functions f and g are connected via the relation

$$f(n) = \sum_{d | n} g(d), \qquad (4.1)$$

then one has a particularly simple approach to computing the

mean value
$$\frac{1}{N}\sum_{n=1}^{N} f(n).$$

Thus, one has
$$\sum_{n=1}^{N} f(n) = \sum_{n=1}^{N} \sum_{d|n} g(d) = \sum_{1\le d\le N} g(d) \sum_{\substack{1\le n\le N\\ d|n}} 1$$
$$= \sum_{1\le d\le N} g(d) \left\lfloor \frac{N}{d} \right\rfloor$$

Since $\left\lfloor \frac{N}{d} \right\rfloor = \frac{N}{d} + O(1)$, we deduce ~~that~~
$$\frac{1}{N}\sum_{n=1}^{N} f(n) = \sum_{1\le d\le N} \frac{g(d)}{d} + O\Big(\sum_{1\le d\le N} |g(d)| \Big).$$

Oftentimes one finds ~~that~~ the first sum converges, or at least is relatively well-behaved, and the second is also well-controlled.

Note ~~that~~ given $f: \mathbb{N} \to \mathbb{C}$, one may obtain a function g satisfying (4·1) via Möbius inversion:
$$f(n) = \sum_{d|n} g(d) \iff g(n) = \sum_{d|n} \mu(d) f(n/d).$$

Example 4·1 Let $\sigma(n) = \sum_{d|n} d$. Then one has
$$\sum_{n\le x} \frac{\sigma(n)}{n} = \sum_{n\le x} \sum_{d|n} \frac{d}{n} = \sum_{n\le x} \sum_{d|n} \frac{1}{d} \qquad \left(\begin{matrix}\text{duality of divisors}\\ d \longleftrightarrow n/d\end{matrix} \right)$$
$$= \sum_{1\le d\le x} \frac{1}{d} \sum_{1\le m\le x/d} 1 = \sum_{1\le d\le x} \frac{1}{d} \left\lfloor \frac{x}{d} \right\rfloor$$

$$= \sum_{1 \le d \le x} \frac{x}{d^2} + O\left(\sum_{1 \le d \le x} \frac{1}{d} \right)$$

$$= x \left(\sum_{d=1}^{\infty} \frac{1}{d^2} + O\left(\sum_{d > x} \frac{1}{d^2} \right) \right) + O(\log x)$$

$$= x \zeta(2) + O(\log x).$$

So $\displaystyle \frac{1}{x} \sum_{n \le x} \frac{\sigma(n)}{n} = \frac{\pi^2}{6} + O\left(\frac{\log x}{x} \right)$. $\square$

Example 4·2. Compute the average value of $\frac{n}{\varphi(n)}$, where $\varphi(n)$ denotes Euler's totient.

[Thus $\displaystyle \varphi(n) = \prod_{p^h \| n} (p^h - p^{h-1}) = n \prod_{p \mid n} (1 - 1/p)$.]

We first seek a function $g : \mathbb{N} \to \mathbb{C}$ with

$$\sum_{d \mid n} g(d) = n / \varphi(n).$$

But by Möbius inversion, one has

$$g(n) = \sum_{d \mid n} \mu(d) \frac{n/d}{\varphi(n/d)}.$$

This looks complicated, but since $\mu(n)$ and $n/\varphi(n)$ are both multiplicative functions, it suffices to consider prime power values of n (Ex. if f is multiplicative, so ~~too~~ is $\sum_{d \mid n} f(d)$; if a, b multiplicative, then so ~~too~~ is $\sum_{d \mid n} a(d) b(n/d)$).

But

(24)

$$g(1) = 1$$

$$g(p) = \sum_{d \mid p} \mu(d) \frac{p/d}{\phi(p/d)} = \frac{p}{p-1} - 1 = \frac{1}{p-1}$$

$$g(p^h) = \sum_{\ell=0}^{h} \mu(p^\ell) \frac{p^h/p^\ell}{\phi(p^h/p^\ell)} = \frac{p^h}{p^h(1-1/p)} - \frac{p^{h-1}}{p^h(1-1/p)} = 0 \quad (h \geq 2).$$

Thus $$g(n) = \prod_{p^h \| n} g(p^h) = \frac{\mu(n)^2}{\varphi(n)}.$$

We next deduce ~~that~~

$$\sum_{1 \leq n \leq x} \frac{n}{\varphi(n)} = \sum_{1 \leq n \leq x} \sum_{d \mid n} g(d)$$

$$= \sum_{1 \leq d \leq x} \frac{\mu(d)^2}{\varphi(d)} \sum_{1 \leq m \leq x/d} 1$$

$$= x \sum_{1 \leq d \leq x} \frac{\mu(d)^2}{d\varphi(d)} + O\Big(\sum_{1 \leq d \leq x} \frac{1}{\varphi(d)}\Big)$$

$$= x \sum_{d=1}^{\infty} \frac{\mu(d)^2}{d\,\varphi(d)} + O(x^\varepsilon),$$

using $\varphi(d) \gg_\varepsilon d^{1-\varepsilon}$, any $\varepsilon > 0$.

The infinite sum here is absolutely convergent with Euler product

$$\prod_p \Big(1 + \frac{\mu(p)^2}{p(p-1)} + \frac{\mu(p^2)^2}{p^2 \cdot p(p-1)} + \cdots\Big) = \prod_p \Big(1 + \frac{1}{p(p-1)}\Big).$$

$$= \prod_p \Big(\frac{p(p-1)}{p^2-p+1}\Big)^{-1} = \prod_p \Big(\frac{(p^3-1)}{(p^6-1)}\, p(p-1)(p+1)\Big)^{-1}$$

$$= \prod_p (1-p^{-6})^{+1} \cdot \prod_p (1-p^{-3})^{-1} \cdot \prod_p (1-p^{-2})^{-1}$$

$$= \frac{\zeta(2)\zeta(3)}{\zeta(6)} .$$

Thus we conclude that

$$\frac{1}{x} \sum_{1 \le n \le x} \frac{n}{\phi(n)} = \frac{\zeta(2)\zeta(3)}{\zeta(6)} + O(x^{\varepsilon-1}). \qquad \square$$

Variants of the basic idea:

(a) Powered convolutions of shape $f(n) = \sum_{d^k \mid n} g(d) \quad (k > 1)$.

Example. (exercise) Show that $\mu(n)^2 = \sum_{d^2 \mid n} \mu(d)$, and hence deduce that

$$\sum_{\substack{1 \le n \le x \\ n \text{ squarefree}}} 1 = \frac{6}{\pi^2} x + O(x^{1/2}).$$

(b) Beyond Möbius inversion I: rewrite the convolution sum in the shape

$$f(n) = \sum_{d \mid n} a(d)\, b(n/d),$$

for suitable arithmetic functions a, b. Then

$$\sum_{1 \le n \le x} f(n) = \sum_{1 \le d \le x} a(d) \underbrace{\sum_{1 \le m \le x/d} b(m)}_{= B(x/d)\text{, say}}$$

— need a good estimate here.

(c) <u>Beyond Möbius inversion, II.</u> The "bilinear" structures can be used to improve error terms — idea is ~~that~~ if $n = kl$, then either $k \le \sqrt{n}$ or $l \le \sqrt{n}$.

More generally, suppose ~~that~~ $1 \le y \le x$,

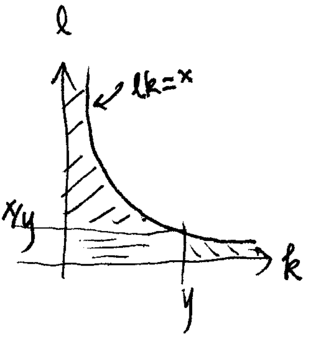
$$f(n) = \sum_{d \mid n} a(d)\, b(n/d) = \sum_{kl=n} a(k) b(l),$$

and write $A(z) = \sum_{1 \le n \le z} a(n)$ and $B(z) = \sum_{1 \le n \le z} b(n)$.

Then

$$\sum_{1 \le n \le x} f(n) = \sum_{1 \le kl \le x} a(k)\, b(l)$$

$$= \sum_{1 \le k \le y} \sum_{1 \le l \le x/k} a(k) b(l) + \sum_{1 \le l \le x/y} \sum_{1 \le k \le x/l} a(k) b(l)$$



$$- \sum_{1 \le k \le y} \sum_{1 \le l \le x/y} a(k) b(l)$$

"overcounting". included in both previous terms

$$= \sum_{1 \le k \le y} a(k) B(x/k) + \sum_{1 \le l \le x/y} b(l) A(x/l) - A(y) B(x/y)$$

<u>Example.</u> Observe ~~that~~ naively, we have

$$\sum_{1 \le n \le x} \tau(n) = \sum_{1 \le d \le x} \sum_{1 \le m \le x/d} 1 = \sum_{1 \le d \le x} \left\lfloor \frac{x}{d} \right\rfloor = x \sum_{1 \le d \le x} \frac{1}{d} + O(x)$$

$$\sum_{1 \le n \le x} \sum_{d \mid n} 1 \qquad = x \log x + O(x).$$

Meanwhile, using the decomposition described in (c), we have

$$\sum_{1\le n\le x} \tau(n) = \sum_{1\le kl\le x} 1 = \sum_{1\le k\le y} \lfloor \tfrac{x}{k} \rfloor + \sum_{1\le l\le x/y} \lfloor \tfrac{x}{l} \rfloor - \lfloor y \rfloor \lfloor x/y \rfloor$$

$$= x \sum_{1\le k\le y} \frac{1}{k} + x \sum_{1\le l\le x/y} \frac{1}{l} - y\cdot\frac{x}{y} + O(y + x/y)$$

$$= x\left(\log y + C_0 + O\left(\tfrac{1}{y}\right)\right) + x\left(\log\left(\tfrac{x}{y}\right) + C_0 + O\left(\tfrac{y}{x}\right)\right) - x + O(y + x/y)$$

$$= x\log x + (2C_0 - 1)x + O(y + x/y).$$

The error term here is $O(\sqrt{x})$ if we take $y = \sqrt{x}$. //

§5. Prime number theorems: the work of Chebyshev and Mertens.

Some notation:

$$\pi(x) = \sum_{p\le x} 1, \qquad \theta(x) = \sum_{p\le x} \log p, \qquad \psi(x) = \sum_{n\le x} \Lambda(n).$$

Theorem 5.1. When $x \ge 2$, one has $\psi(x) \asymp x$.

Proof. We apply a common strategy in analytic number theory, that of majorising the characteristic function of a set. Since

$$\sum_{d|m} \mu(d) = \begin{cases} 1, & \text{when } m = 1, \\ 0, & \text{otherwise}, \end{cases}$$

it is apparent that

$$\psi(x) = \sum_{1\le n\le x} \Lambda(n) = \sum_{1\le n\le x} \sum_{1\le m\le x/n} \left(\sum_{d|m} \mu(d)\right) \Lambda(n)$$

(28) It transpires that the Möbius function is difficult to handle, but we may majorise the characteristic function

$$1 = \sum_{1 \le m \le y} \sum_{d \mid m} \mu(d)$$

by substituting for $\mu(d)$ an arithmetic function having appropriate properties. A particularly simple approach is to take

$$a(d) = \begin{cases} 1 & , \text{ when } d=1, \\ -2 & , \text{ when } d=2, \\ 0 & , \text{ when } d \geqslant 3. \end{cases}$$

We then have

$$\sum_{1 \le m \le y} \sum_{d \mid m} a(d) = \sum_{1 \le d \le y} a(d) \sum_{1 \le k \le y/d} 1 = \sum_{1 \le d \le y} a(d) \lfloor y/d \rfloor$$

$$= \lfloor y \rfloor - 2 \lfloor y/2 \rfloor \in \{0, 1\}.$$

We deduce that

$$\sum_{1 \le n \le x} \Lambda(n) \left(\left\lfloor \frac{x}{n} \right\rfloor - 2 \left\lfloor \frac{x}{2n} \right\rfloor \right) \le \psi(x).$$

Since for $x \geqslant n > x/2$, the parenthetic expression is equal to 1, it follows that

$$\psi(x) - \psi(x/2) \le \sum_{1 \le n \le x} \Lambda(n) \sum_{1 \le m \le x/n} \sum_{d \mid m} a(d) \le \psi(x)$$

$$\| $$

$$\sum_{1 \le kdn \le x} \Lambda(n)\, a(d) = \sum_{1 \le \ell d \le x} a(d) \sum_{n \mid \ell} \Lambda(n)$$

$$\| $$

$$\sum_{1 \le \ell d \le x} a(d) \log \ell = \sum_{1 \le d \le x} a(d) \left(\sum_{1 \le \ell \le x/d} \log \ell \right)$$

(29) By Riemann-Stieltjes integration, one finds that

$$\sum_{1 \le l \le y} \log l = \int_{1^-}^{y} \log u \, d\lfloor u \rfloor = y \log y - y + O(\log y).$$

Thus

$$\sum_{1 \le d \le x} a(d) \sum_{1 \le l \le x/d} \log l = \left(x \log x - x + O(\log x) \right) - 2\left(\frac{x}{2} \log\left(\frac{x}{2}\right) - \frac{x}{2} + O(\log x) \right)$$

$$= x(\log 2) + O(\log x).$$

Consequently, we find that

$$\psi(x) \geq x(\log 2) + O(\log x)$$

and

$$\psi(x) - \psi(x/2) \leq x(\log 2) + O(\log x).$$

By summing over dyadic intervals, we conclude that

$$\psi(x) \leq (2 \log 2) x + O((\log x)^2),$$

and thus

$$(\log 2) x + O(\log x) \leq \psi(x) \leq (2 \log 2) x + O((\log x)^2).$$

There are more efficient choices for the sequence $(a(d))_{d \in \mathbb{N}}$. Thus, one may take $a(1) = a(30) = 1$ and $a(2) = a(3) = a(5) = -1$, with $a(d) = 0$ otherwise. This gives

$$(0.9212)x + O(\log x) \leq \psi(x) \leq (1.1056)x + O((\log x)^2).$$

[Even the weak version of the Prime Number Theorem embodied in Corollary 5.2 yields precise information concerning various sums and products involving $1/p$.]

<u>Corollary 5.2.</u> When $x \geqslant 2$, one has

$$\theta(x) = \psi(x) + O(x^{1/2}) \quad \text{and} \quad \pi(x) = \frac{\psi(x)}{\log x} + O\left(\frac{x}{(\log x)^2}\right).$$

Moreover, one has $\theta(x) \asymp x$ and $\pi(x) \asymp x/\log x$.

<u>Proof.</u> By the definitions of $\psi(x)$ and $\theta(x)$, it follows ~~that~~

$$\psi(x) = \sum_{p^k \leqslant x} \log p = \sum_{k=1}^{\infty} \theta(x^{1/k}),$$

But for each $y \in \mathbb{R}_{\geqslant 2}$, one has

$$\theta(y) \leqslant \psi(y) \asymp y,$$

and ~~thus~~ we deduce ~~that~~

$$\psi(x) - \theta(x) = \sum_{k \geqslant 2} \theta(x^{1/k}) \ll x^{\frac{1}{2}} + x^{1/3} \log x \ll x^{\frac{1}{2}}.$$

This establishes our first claim concerning $\theta(x)$ and $\psi(x)$. Since $\psi(x) \asymp x$, moreover, we see also that $\theta(x) = \psi(x) + O(x^{1/2}) \asymp x$.

We apply Riemann - Stieltjes integration to obtain the result on $\pi(x)$. Thus, we have

$$\pi(x) = \int_{2^-}^{x} (\log u)^{-1}\, d\theta(u) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(u)}{u(\log u)^2}\, du$$

$$= \frac{\theta(x)}{\log x} + O\left(\int_2^x \frac{du}{(\log u)^2}\right) \asymp \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right).$$

Hence $\pi(x) \asymp x/\log x$. Also, since $\theta(x) - \psi(x) = O(x^{1/2})$, it follows also that $\pi(x) = \dfrac{\psi(x)}{\log x} + O\left(\dfrac{x}{(\log x)^2}\right)$. //

Theorem 5.3. When $x \geq 2$, one has

(a) $$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1) \quad \text{and} \quad \sum_{p \leq x} \frac{\log p}{p} = \log x + O(1);$$

(b) $$\int_1^x \frac{\psi(u)}{u^2}\,du = \log x + O(1);$$

(c) $$\sum_{p \leq x} \frac{1}{p} = \log\log x + C_0 - \sum_p \sum_{k=2}^{\infty} \frac{1}{kp^k} + O\left(\frac{1}{\log x}\right),$$

(d) $$\prod_{p \leq x} (1 - 1/p)^{-1} = e^{C_0} \log x + o(1)$$ (Mertens' Theorem).

Proof: (a) We have $\log n = \sum_{d|n} \Lambda(d)$, and hence

$$\sum_{1 \leq n \leq x} \log n = \sum_{1 \leq d \leq x} \Lambda(d) \sum_{1 \leq m \leq x/d} 1 = x \sum_{1 \leq d \leq x} \frac{\Lambda(d)}{d} + O(\psi(x)).$$

$$\parallel$$

$$x \log x + O(x)$$

Thus $$\sum_{1 \leq d \leq x} \frac{\Lambda(d)}{d} = \frac{x \log x + O(x)}{x} = \log x + O(1).$$

Moreover, we have

$$\sum_{1 \leq d \leq x} \frac{\Lambda(d)}{d} = \sum_{p \leq x} \frac{\log p}{p} + \sum_{k=2}^{\infty} \sum_{p^k \leq x} \frac{\log p}{p^k}$$

$$= \sum_{p \leq x} \frac{\log p}{p} + \sum_p \frac{\log p}{p(p-1)} = \sum_{p \leq x} \frac{\log p}{p} + O(1).$$

$\square$

(b) We have

$$\sum_{1 \leq d \leq x} \frac{\Lambda(d)}{d} = \int_{2^-}^x u^{-1}\,d\psi(u) = \frac{\psi(u)}{u}\Big|_{2^-}^x + \int_2^x \frac{\psi(u)}{u^2}\,du$$

$$= \int_1^x \frac{\psi(u)}{u^2}\,du + O(1).$$

The proof of parts (c) and (d) in the precise form stated is a little complicated.

(c) Define the functions $L(x)$ and $R(x)$ by putting

$$L(x) = \sum_{p \le x} \frac{\log p}{p} \quad \text{and} \quad R(x) = L(x) - \log x,$$

so that according to part (a), we have $R(x) \ll 1$. Then by R-S integration,

$$\sum_{p \le x} \frac{1}{p} = \int_{2^-}^{x} (\log u)^{-1}\, dL(u) = \int_{2^-}^{x} \frac{d(\log u)}{\log u} + \int_{2^-}^{x} \frac{dR(u)}{\log u}$$

$$= \log\log u \Big|_{2^-}^{x} + \frac{R(u)}{\log u}\Big|_{2^-}^{x} + \int_{2^-}^{x} \frac{R(u)}{u(\log u)^2}\,du$$

absolutely convergent as $x \to \infty$.

$$= \log\log x - \log\log 2 + \frac{R(x)}{\log x} + 1 + \int_2^{\infty} \frac{R(u)}{u(\log u)^2}\,du$$

$R(2^-) = -\log 2$

$$- \int_x^{\infty} \frac{R(u)}{u(\log u)^2}\,du$$

Write $\quad b = 1 - \log\log 2 + \int_2^{\infty} \frac{R(u)}{u(\log u)^2}\,du,\quad$ and recall that

$R(n) \ll 1$. Then we deduce ~~that~~

$$\sum_{p \le x} \frac{1}{p} = \log\log x + b + O\left(\frac{1}{\log x}\right).$$

Note ~~that~~ we have yet to confirm ~~that~~ $b = C_0 - \sum_p \sum_{k=2}^{\infty} \frac{1}{kp^k}$, a task to which we shall shortly return.

(d). Note ~~that~~

$$\log\left(\prod_{p \le x} (1 - \tfrac{1}{p})^{-1}\right) = -\sum_{p \le x} \log(1 - \tfrac{1}{p})$$

$$= \sum_{p \le x} \frac{1}{p} + \sum_{p \le x} \sum_{k=2}^{\infty} \frac{1}{kp^k}.$$

Making use of our conclusion in part (c), we deduce ~~that~~

$$\log\left(\prod_{p \le x} (1 - \tfrac{1}{p})^{-1}\right) = \log\log x + c + \underbrace{\sum_{p > x} \sum_{k=2}^{\infty} \frac{1}{kp^k}}_{\le \sum_{n > x} \frac{1}{n^2} = O\left(\frac{1}{x}\right)} + O\left(\frac{1}{\log x}\right),$$

where

$$c = b + \sum_p \sum_{k=2}^{\infty} \frac{1}{kp^k}.$$

Hence

$$\prod_{p \le x} (1 - \tfrac{1}{p})^{-1} = (e^c \log x)\left(1 + O\left(\frac{1}{\log x}\right)\right) = e^c \log x + O(1).$$

Here, we have yet to confirm ~~that~~ $c = C_0$. This we confirm by comparing $\sum_{n \le \log x} \frac{1}{n}$ with $\sum_{p \le x} \frac{1}{p}$.

We begin by observing ~~that~~

$$\sum_{1<n\leq x} \frac{\Lambda(n)}{n\log n} = \sum_{p\leq x}\sum_{k=1}^{\infty} \frac{\Lambda(p^k)}{p^k\log(p^k)} - \sum_{\substack{p\leq x\\ p^k>x}} \frac{\Lambda(p^k)}{p^k\log(p^k)}$$

$$= -\sum_{p\leq x} \log\left(1-\frac{1}{p}\right) + O\left(\frac{1}{\log x}\sum_{p\leq x}\sum_{k\geq 2}\frac{\log p}{p^k}\right)$$

$$= \log\left(\prod_{p\leq x}\left(1-\frac{1}{p}\right)^{-1}\right) + O\left(\frac{1}{\log(2x)}\right).$$

Hence we deduce ~~that~~

$$\sum_{1<n\leq x}\frac{\Lambda(n)}{n\log n} = \log\log x + c + O(1/\log(2x))$$

$$= \sum_{n\leq \log x}\frac{1}{n} + (c-C_0) + O(1/\log(2x)) \quad -(5.1)$$

(using (3.4)).

We next ~~introduce the~~ auxiliary quantities, for $\delta>0$,

$$I_1(\delta) = \delta\int_{1^+}^{\infty} x^{-1-\delta}\sum_{1<n\leq x}\frac{\Lambda(n)}{n\log n}\,dx,$$

$$I_2(\delta) = \delta\int_{1^+}^{\infty} x^{-1-\delta}\sum_{1\leq n\leq \log x}\frac{1}{n}\,dx,$$

$$I_3(\delta) = \delta\int_{1^+}^{\infty} x^{-1-\delta}\,dx = 1.$$

Then it follows from (5.1) ~~that~~

$$I_1(\delta) = I_2(\delta) + c - C_0 + I_4(\delta),$$

where

$$I_4(\delta) \ll \delta \int_{1^+}^{\infty} \frac{x^{-1-\delta}}{\log(2x)}\,dx \ll \delta + \delta \int_2^{e^{1/\delta}} \frac{dx}{x\log x} + \delta^2 \int_{e^{1/\delta}}^{\infty} x^{-1-\delta}\,dx$$

$$\ll \delta \log(1/\delta).$$

But by Theorem 2.4 (iii)

$$I_1(\delta) = \sum_{n=1}^{\infty} a_n n^{-\delta}, \quad \text{where} \quad a_n = \Lambda(n)/(n\log n),$$

whence $I_1(\delta) = \log \zeta(1+\delta) = \log\left(\frac{1}{\delta}\right) + O(\delta)$ as $\delta \to 0+$

(using $\zeta(s) = \frac{1}{s-1} + C_0 + O(|s-1|)$). Also,

$$I_2(\delta) = \delta \sum_{n=1}^{\infty} \frac{1}{n} \int_{e^n}^{\infty} x^{-1-\delta}\,dx = \sum_{n=1}^{\infty} \frac{1}{n} e^{-\delta n}$$

$$= -\log(1-e^{-\delta}) = -\log(\delta + O(\delta^2)) = \log\left(\frac{1}{\delta}\right) + O(\delta)$$

as $\delta \to 0+$.

We therefore conclude that as $\delta \to 0+$, one has

$$\log\left(\frac{1}{\delta}\right) = \log\left(\frac{1}{\delta}\right) + c - C_0 + O\left(\underbrace{\delta \log\left(\frac{1}{\delta}\right)}_{\to 0}\right),$$

whence $c = C_0$.

We may now conclude that

$$\prod_{p \le x} \left(1 - \frac{1}{p}\right)^{-1} = e^{C_0} \log x + O(1),$$

and that $b = C_0 - \sum_p \sum_{k=2}^{\infty} \frac{1}{kp^k}$. //

Corollary 5.4. One has

$$1 \geq \liminf_{x\to\infty} \frac{\pi(x)}{x/\log x} \quad \text{and} \quad \limsup_{x\to\infty} \frac{\pi(x)}{x/\log x} \geq 1.$$

Proof. By Corollary 5.2 we may focus on establishing that

$$\liminf_{x\to\infty} \frac{\psi(x)}{x} \leq 1 \quad \text{and} \quad \limsup_{x\to\infty} \frac{\psi(x)}{x} \geq 1.$$

If, however, one were to have $\limsup_{x\to\infty} \frac{\psi(x)}{x} = a$, then given $\varepsilon > 0$ one must have $\psi(x) \leq (a+\varepsilon)x$ for all large enough x, say for $x \geq x_0$. But then

$$\int_1^x \psi(u)u^{-2}\,du \leq \int_1^{x_0} \psi(u)u^{-2}\,du + (a+\varepsilon)\int_{x_0}^x u^{-1}\,du$$
$$\leq (a+\varepsilon)\log x + O_\varepsilon(1),$$

whence $\displaystyle\int_1^x \frac{\psi(u)}{u^2}\,du \leq (a+o(1))\log x$.

But we have $\displaystyle\int_1^x \frac{\psi(u)}{u^2}\,du = \log x + O(1)$, so necessarily $a \geq 1$.

The argument adapts easily to confirm that $\liminf_{u\to\infty} \frac{\psi(u)}{u} \leq 1$. //

§6. Bounds for arithmetic functions.

The relatively crude version of the Prime Number Theorem ~~that~~ we have established thus far permits fairly precise bounds to be derived for arithmetic functions of interest.

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Theorem 6.1 When $n \geq 3$, one has

$$\varphi(n) \geq \frac{n}{\log\log n}\left(e^{-c_0} + O\left(\frac{1}{\log\log n}\right)\right)$$

and

$$1 \leq \omega(n) \leq \frac{\log n}{\log\log n}\left(1 + O\left(\frac{1}{\log\log n}\right)\right).$$

Proof. Denote the k-th largest prime by p_k, ~~so that~~ $p_1 = 2$, $p_2 = 3, \ldots$, and put $m = \omega(n)$. Then once

$$\frac{\varphi(n)}{n} = \prod_{p \mid n}(1 - 1/p) \geq \prod_{i=1}^{m}(1 - 1/p_i),$$

we find from Mertens' ~~theorem~~ ~~that~~

$$\frac{\varphi(n)}{n} \geq \frac{e^{-c_0}}{\log p_m} + O(1).$$

However, we have $p_1 \cdots p_m \leq n$, whence

$$\underbrace{\sum_{p \leq p_m} \log p}_{\substack{\| \\ \theta(p_m) \asymp p_m + O(1)}} \leq \log n \Rightarrow \log p_m \leq \log\log n + O(1).$$

Hence $$\frac{\varphi(n)}{n} \geq \frac{e^{-c_0}}{\log\log n}\left(1 + O\left(\frac{1}{\log\log n}\right)\right).$$ □

Likewise, we have $\omega(n) \leq m$, where m is the largest natural number with $\underbrace{p_1 \cdots p_m}_{\substack{\| \\ n^*,\ \text{say}}} \leq n$. Then we have

$$\underbrace{\sum_{p \leq p_m} \log p}_{\substack{\| \\ \theta(p_m) \asymp p_m + O(1)}} = \log n^* \Rightarrow \log p_m = \log\log n^* + O(1).$$

Also,

$$m = \pi(p_m) = \frac{\theta(p_m)}{\log p_m} + O\left(p_m / (\log p_m)^2 \right)$$

$$= \frac{\log n^*}{\log\log n^* + O(1)} + O\left(\frac{\log n^*}{(\log\log n^*)^2} \right)$$

$$\leq \frac{\log n}{\log\log n} \left(1 + O\left(\frac{1}{\log\log n} \right) \right). \quad \square$$

Even ~~though~~ $\omega(n)$ may be as large as $\log n / \log\log n$, it is typically much smaller.

<u>Theorem 6.2.</u> (Turán) For $x \geq 3$, one has

$$\sum_{n \leq x} (\omega(n) - \log\log n)^2 \ll x \log\log x . \quad \text{——— (6.1)}$$

Thus, for 'almost all' integers n, one has $\omega(n) = \log\log n + O((\log\log n)^{\frac{1}{2}+\varepsilon})$.

<u>Proof:</u> We first show ~~that~~ (6.1) holds with $\log\log n$ replaced by $\log\log x$. Indeed, one has

$$\sum_{n \leq x} (\omega(n) - \log\log x)^2 = S_2 - 2(\log\log x) S_1 + \lfloor x \rfloor (\log\log x)^2, \text{ — (6.2)}$$

where

$$S_j = \sum_{n \leq x} \omega(n)^j \qquad (j = 1, 2).$$

But

$$S_1 = \sum_{n \leq x} \omega(n) = \sum_{n \leq x} \sum_{p \mid n} 1 = \sum_{p \leq x} \left\lfloor \frac{x}{p} \right\rfloor$$

$$= x \sum_{p \leq x} \frac{1}{p} + O\left(\sum_{p \leq x} 1 \right) = x(\log\log x + b) + O(x/\log x).$$

Also,

$$S_2 = \sum_{n \le x} \Big(\sum_{p_1 | n} 1 \Big) \Big(\sum_{p_2 | n} 1 \Big) = \sum_{p_1 \le x} \sum_{p_2 \le x} \sum_{\substack{1 \le n \le x \\ p_1 | n \,\&\, p_2 | n}} 1 .$$

The contribution to S_2 from the terms with $p_1 = p_2$ is

$$\sum_{p \le x} \Big\lfloor \frac{x}{p} \Big\rfloor = x \log\log x + O(x).$$

Meanwhile, the corresponding contribution with $p_1 \ne p_2$ is

$$\sum_{p_1 \le x} \sum_{\substack{p_2 \le x \\ p_2 \ne p_1}} \Big\lfloor \frac{x}{p_1 p_2} \Big\rfloor \le x \Big(\sum_{p \le x} \frac{1}{p} \Big)^2 = x (\log\log x)^2 + O(x \log\log x).$$

Hence

$$S_2 \le x (\log\log x)^2 + O(x \log\log x),$$

By substituting into (6.2), we conclude ~~that~~

$$\sum_{n \le x} (\omega(n) - \log\log x)^2 \le x (\log\log x)^2 + O(x \log\log x) - 2(\log\log x) x \log\log x$$
$$+ x(\log\log x)^2 + O(x \log\log x)$$
$$\ll x \log\log x .$$

Finally, we have

$$\sum_{3 \le n \le x} (\log\log n - \log\log x)^2 \le \int_{3^-}^{x} (\log\log u - \log\log x)^2 \, d\lfloor u \rfloor$$

$$= \lfloor u \rfloor (\log\log u - \log\log x)^2 \Big|_{3^-}^{x} - 2 \int_{3^-}^{x} \lfloor u \rfloor \frac{(\log\log u - \log\log x)}{u \log u} \, du$$

$$= -2 \int_{3^-}^{x} \frac{\log\log u}{\log u} \, du + 2 \int_{3^-}^{x} \frac{\log\log x}{\log u} \, du + O\Big(\int_{3^-}^{x} \frac{|\log\log u|}{u \log u} \, du \Big)$$

$$\ll x \log\log x$$

Then it follows by the triangle inequality ~~that~~

$$\Big(\sum_{n \le x} (\omega(n) - \log\log n)^2 \Big)^{\frac{1}{2}} \ll (x \log\log x)^{1/2} + (x \log\log x)^{\frac{1}{2}} \ll (x \log\log x)^{\frac{1}{2}}.$$

§7. Additive and multiplicative characters.

Our goal in the next several lectures will be the proof of Dirichlet's theorem:

Theorem 7.1 (Dirichlet, 1837) Suppose that $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $(a,q)=1$. Then there are infinitely many primes p with $p \equiv a \pmod q$.

The detection of the congruence class a modulo q requires a consideration of Fourier analysis associated with the additive group $\mathbb{Z}/q\mathbb{Z}$ and multiplicative group $(\mathbb{Z}/q\mathbb{Z})^\times$. This we discuss in the present section.

Additive characters.

First we recall the theory associated with the additive group of residues $\mathbb{Z}/q\mathbb{Z}$. Write $e(\theta) := e^{2\pi i \theta}$, and, when $a \in \mathbb{Z}$ and $q \in \mathbb{N}$, put $e_q(a) = e(a/q)$.

Orthogonality (Ex.): One has

$$q^{-1}\sum_{a=1}^{q} e(ah/q) = \begin{cases} 1, & \text{when } h \equiv 0 \pmod q, \\ 0, & \text{when } h \not\equiv 0 \pmod q. \end{cases}$$

[When $q \nmid h$, have $\sum_{a=1}^{q} e(ah/q) = e(h/q)\,(\underbrace{e(h)-1}_{=0})/(e(h/q)-1)$]

This permits us to define Fourier expansions via the finite Fourier transform of a function $f : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$:

$$\hat{f}(k) := q^{-1}\sum_{n=1}^{q} f(n)\,e(-kn/q). \qquad (7.1)$$

We claim that given a function $f : \mathbb{Z} \to \mathbb{C}$ periodic with period q, the formula (7.1) delivers the expansion

$$f(n) = \sum_{k=1}^{q} \hat{f}(k) e(kn/q). \qquad (7.2)$$

Proof of claim: By orthogonality,

$$\sum_{k=1}^{q} \hat{f}(k) e(kn/q) = q^{-1} \sum_{k=1}^{q} \left(\sum_{m=1}^{q} f(m) e(-km/q) \right) e(kn/q)$$

$$= \sum_{m=1}^{q} \underbrace{\left(q^{-1} \sum_{k=1}^{q} e(k(n-m)/q) \right)}_{\begin{cases} =0 & \text{when } n \not\equiv m \pmod q \\ =1 & \text{when } n \equiv m \pmod q \end{cases}} f(m)$$

$$= f(n). \qquad \square$$

Analogue of Parseval / Plancharel: One has (Ex.)

$$\sum_{k=1}^{q} |\hat{f}(k)|^2 = q^{-1} \sum_{n=1}^{q} |f(n)|^2. \qquad (7.3).$$

Notice that the characters $e(kn/q)$ are indeed additive for $k \in \mathbb{Z}$, in the sense that $e(k_1 n/q) e(k_2 n/q) = e((k_1+k_2)n/q)$.

Multiplicative characters.

We now turn to consider the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^\times$, and more generally functions $f: \mathbb{Z} \to \mathbb{C}$ having the property of being periodic with period q, and being multiplicative.

Definition 7.1. A Dirichlet character is a totally multiplicative function $\chi : \mathbb{Z} \to \mathbb{C}$ having the properties:

(i) $\chi(n) = 0$ if and only if $(n,q) > 1$; and

(ii) χ is periodic with period q.

Theorem 7.2. Suppose that $f : \mathbb{N} \to \mathbb{C}$ is multiplicative, and satisfies the conditions:

(i) $f(n) = 0$ whenever $(n,q) > 1$; and

(ii) f is periodic with period q.

Then f is a Dirichlet character modulo q.

Proof. It suffices to show that f is totally multiplicative with the property that $f(n) \neq 0$ whenever $(n,q) = 1$.

First, when $(mn, q) > 1$, then either $(m,q) > 1$ or $(n,q) > 1$, and hence $f(m) f(n) = 0 = f(mn)$. Suppose then that $(mn, q) = 1$, when $(m,q) = (n,q) = 1$. There exists $k \in \mathbb{Z}$ having the property that $n + kq \equiv 1 \pmod{m}$, so that $(m, n+kq) = 1$. Then the multiplicative property of f ensures that

$$\begin{array}{ccccc} & f(m(n+kq)) & = & f(m) f(n+kq) & \\ \text{(periodicity)} & \| & & \| & \text{(periodicity)} \\ & f(mn) & & f(m) f(n) & \end{array}$$

Then f is totally multiplicative.

In order to check that $f(n) \neq 0$ whenever $(n,q) = 1$, observe that $f(n)^{\phi(q)} = f(n^{\phi(q)}) = f(1) = 1$, whence $f(n)$ is a $\phi(q)$-th root of unity and hence non-zero. //

Notice that we can always think of a Dirichlet character χ modulo q as being a map from $(\mathbb{Z}/q\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$. Since $\chi(mn) = \chi(m)\chi(n)$ for all m and n, moreover, one sees that $\chi : (\mathbb{Z}/q\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ is a group homomorphism.

- The identity element is the principal character χ_0 defined by taking $\chi_0(n) = 1$ whenever $(n,q) = 1$.
- The inverse of a character χ is the conjugate $\overline{\chi}$ defined by taking $\overline{\chi}(n) = \overline{\chi(n)}$ for $(n,q) = 1$.

In order to see the latter, one observes that the non-zero values of a character are roots of unity. Indeed, as in the conclusion of the proof of Theorem 7.2, we see ~~that~~ when χ is a character modulo q, then $\chi(n)^{\varphi(q)} = 1$ whenever $(n,q) = 1$.

In order to understand the structure of the group of characters, it is useful to start with some basic examples.

<u>Example 7.3.</u> Let q be a prime number, say p, and consider the character χ modulo p.

Note that $(\mathbb{Z}/p\mathbb{Z})^\times$ is a cyclic group of order $\varphi(p) = p-1$ having a generator g (a primitive root). Thus, given an integer n with $(n,p) = 1$, there is an integer $u = \mathrm{ind}_g n$ having the property that $n \equiv g^u \pmod p$. We then have

$$\chi(n) = \chi(g^u) = \chi(g)^u.$$

So the character χ is determined by its value at a single primitive root g modulo p. This value, moreover, is a $(p-1)$-th root of unity.

It follows that there are precisely $(p-1)$ characters modulo p, namely

$$\chi_k : (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times,$$

defined by taking

$$\chi_k(n) = e\left(\frac{k \operatorname{ind}_g(n)}{p-1}\right) \qquad (0 \leq k \leq p-2).$$

Since the values $\chi_k(g) = e\left(\frac{k}{p-1}\right)$ are distinct for $0 \leq k \leq p-2$, these Dirichlet characters are all distinct. Moreover, the character $\chi_0(\cdot) = 1$ is indeed the principal character.

Notice that the group of characters $\{\chi_0, \ldots, \chi_{p-2}\} = \{\chi_1^k : 0 \leq k \leq p-2\} \cong C_{p-1}$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^\times$.

Example 7.4. Let q be an odd prime power, say p^h, or else be 2 or 4, and consider the character χ modulo q.

In this situation $(\mathbb{Z}/p^h\mathbb{Z})^\times$ (either p odd, or $p=2$ and $h = 1, 2$) is again cyclic of order $\varphi(p^h)$, having a generator g (a primitive root). Indeed, if g is a primitive root modulo p^2, then it is primitive modulo p^h for $h \geqslant 3$, when p is odd. In such circumstances, a discussion paralleling that of Example 7.3 shows that there are precisely $\varphi(p^h) = p^h - p^{h-1}$ characters modulo p^h, namely

$$\chi_k : (\mathbb{Z}/p^h\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times,$$

defined by taking

$$\chi_k(n) = e\left(k \operatorname{ind}_g(n) / \varphi(p^h)\right) \qquad (0 \leq k \leq \varphi(p^h)-1).$$

Also, writing $\hat{G}$ for the group of characters modulo p^h, one sees that
$$\hat{G} = \langle \chi_1 \rangle \cong C_{\varphi(p^h)} \cong (\mathbb{Z}/p^h\mathbb{Z})^\times .$$

<u>Example 7.5.</u> Let $q = 2^h$ with $h \geq 3$, and consider the character χ modulo 2^h.

In this case $(\mathbb{Z}/2^h\mathbb{Z})^\times$ is generated by the elements -1 and 5, and is isomorphic to $C_2 \times C_{2^{h-2}}$. Given an integer n with $(n, 2^h)=1$, there is a pair of integers $\underset{\operatorname{ind}^*_{-1} n}{\underset{\|}{v}}$ and $\underset{\operatorname{ind}^*_{5} n}{\underset{\|}{u}}$ having the property that
$$n \equiv (-1)^v 5^u \pmod{2^h}.$$

We then have
$$\chi(n) = \chi((-1)^v 5^u) = \chi(-1)^v \chi(5)^u .$$

So the character χ is determined by its values at -1 and 5. Moreover, one has $\chi(5)^{2^{h-2}} = 1$ and $\chi(-1)^2 = 1$.

It follows that there are precisely $2^{h-1} = \varphi(2^h)$ characters modulo 2^h, namely
$$\chi_{l,k} : (\mathbb{Z}/2^h\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times,$$

defined by taking
$$\chi_{l,k}(n) = e\left(\frac{l \operatorname{ind}^*_{-1}(n)}{2} + \frac{k \operatorname{ind}^*_5(n)}{2^{h-2}} \right)$$

In this case, the group of characters

$$\{ \chi_{\ell,k} : \ell \in \{0,1\},\ 0 \le k \le 2^{h-2}-1 \} = \langle \chi_{0,1}, \chi_{1,0} \rangle$$
$$\cong C_2 \times C_{2^{h-2}} \cong (\mathbb{Z}/2^h\mathbb{Z})^\times .$$

~

Example 7.6: Let $q \in \mathbb{N}$ with $q > 1$, and consider the character χ modulo q.

By the Chinese Remainder Theorem, one finds that

$$(\mathbb{Z}/q\mathbb{Z})^\times \cong \bigotimes_{p^\alpha \| q} (\mathbb{Z}/p^\alpha\mathbb{Z})^\times ,$$

and so $(\mathbb{Z}/q\mathbb{Z})^\times$ is generated by suitable representatives of primitive roots g_p, for $p^\alpha \| q$, with -1 and 5 if $8 | q$. It follows that there are precisely $\varphi(q)$ characters modulo q, namely

$$\chi_{\underline{k}} : (\mathbb{Z}/q\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$$

defined by taking

$$\chi_{\underline{k}}(n) = e\left(\sum_{p^\alpha \| q} \frac{k_{p^\alpha} \mathrm{ind}_{g_p}(n)}{\varphi(p^\alpha)} \right), \quad \text{when } 8 \nmid q,$$

with $0 \le k_{p^\alpha} \le \varphi(p^\alpha) - 1$,

and

$$\chi_{\underline{\ell},\underline{k}}(n) = e\left(\frac{\ell_0 \mathrm{ind}^*_{-1}(n)}{2} + \frac{\ell_1 \mathrm{ind}^*_5(n)}{2^{h-2}} + \sum_{\substack{p^\alpha \| q \\ p \text{ odd}}} \frac{k_{p^\alpha} \mathrm{ind}_{g_p}(n)}{\varphi(p^\alpha)} \right),$$

when $2^h \| q$ with $h \ge 3$, wherein $\ell_0 \in \{0,1\}$, $0 \le \ell_1 \le 2^{h-2}-1$, and $0 \le k_{p^\alpha} \le \varphi(p^\alpha) - 1$ (p odd).

Notice that $\{\chi_{\underline{\ell},\underline{k}}\} \cong (\mathbb{Z}/q\mathbb{Z})^\times$ in this case also.

Theorem 7.7. The multiplicative group $(\mathbb{Z}/q\mathbb{Z})^\times$ of reduced residue classes (mod q) has $\varphi(q)$ Dirichlet characters. If χ is such a character, then

$$\sum_{\substack{n=1 \\ (n,q)=1}}^{q} \chi(n) = \begin{cases} \varphi(q), & \text{when } \chi = \chi_0, \\ 0, & \text{when } \chi \neq \chi_0. \end{cases}$$

Moreover, when $(n,q)=1$, one has

$$\sum_{\chi \in X(q)} \chi(n) = \begin{cases} \varphi(q), & \text{when } n \equiv 1 \pmod q, \\ 0, & \text{otherwise}. \end{cases}$$

Here, we write $X(q)$ for the group of $\varphi(q)$ Dirichlet characters modulo q.

Proof. When $\chi = \chi_0$, the first assertion is plain. Suppose then that $\chi \neq \chi_0$. There exists an integer m with $(m,q)=1$ and $\chi(m) \neq 1$. But then, by noting that multiplication by m permutes the reduced residues modulo q, we deduce that

$$\sum_{\substack{n=1 \\ (n,q)=1}}^{q} \chi(n) = \sum_{\substack{n=1 \\ (n,q)=1}}^{q} \chi(mn) = \chi(m) \sum_{\substack{n=1 \\ (n,q)=1}}^{q} \chi(n),$$

whence

$$\underbrace{(\chi(m)-1)}_{\text{non-zero}} \sum_{\substack{n=1 \\ (n,q)=1}}^{q} \chi(n) = 0 \Rightarrow \sum_{\substack{n=1 \\ (n,q)=1}}^{q} \chi(n) = 0.$$

$\square$

For the second assertion, when $n \equiv 1 \pmod q$ one has

$$\sum_{\chi \in X(q)} \chi(n) = \sum_{\chi \in X(q)} \chi(1) = \mathrm{card}(X(q)) = \varphi(q).$$

Also, when $(n,q)=1$ and $n \not\equiv 1 \pmod q$, there is a character χ_1 modulo q with $\chi_1(n) \neq 1$ (by our characterisation of Dirichlet characters). Thus

$$\sum_{\chi \in X(q)} \chi(n) = \sum_{\chi \in X(q)} (\chi_1 \chi)(n) = \chi_1(n) \sum_{\chi \in X(q)} \chi(n)$$

use group structure of $X(q)$

$$\Rightarrow \left(\chi_1(n) - 1\right) \sum_{\chi \in X(q)} \chi(n) = 0 \quad \Rightarrow \quad \sum_{\chi \in X(q)} \chi(n) = 0. \quad \square \; //$$

Another consequence of our characterisation of characters.

<u>Theorem 7.8.</u> (i) If χ_i is a character modulo q_i for $i=1,2$, then $\chi_3 : \mathbb{Z} \to \mathbb{C}$ is a character modulo q, where $q = [q_1, q_2]$ and $\chi_3(n) = \chi_1(n)\chi_2(n)$ for $n \in \mathbb{Z}$.

(ii) If $q = q_1 q_2$ with $(q_1, q_2) = 1$, and χ is a character modulo q, ~~then~~ there exist unique characters $\chi_i \pmod{q_i}$ $(i=1,2)$ such that $\chi(n) = \chi_1(n)\chi_2(n)$ for all n.

<u>Exercise:</u> Analogues of Parseval / Plancherel :

(i) $$\frac{1}{\phi(q)} \sum_{\chi \in X(q)} \left| \sum_{n=1}^{q} c_n \chi(n) \right|^2 = \sum_{\substack{n=1 \\ (n,q)=1}}^{q} |c_n|^2 \qquad (c_n \in \mathbb{C})$$

(ii) $$\frac{1}{\phi(q)} \sum_{n=1}^{q} \left| \sum_{\chi \in X(q)} c_\chi \chi(n) \right|^2 = \sum_{\chi \in X(q)} |c_\chi|^2 \qquad (c_\chi \in \mathbb{C}).$$

§8. Dirichlet L-functions and Dirichlet's theorem on primes.

The Dirichlet L-function $L(s, \chi)$ associated with the character χ modulo q is defined for $\sigma > 1$ by the Dirichlet series

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}. \quad \text{———} \quad (8\cdot 1)$$

This series is, of course, very much analogous to the series defining $\zeta(s)$, and we can develop the associated theory, and consequences for prime number sums, in a manner analogous to that employed above.

First we consider questions of convergence in $(8\cdot1)$, and the associated Euler product. Plainly, the series $(8\cdot1)$ is absolutely convergent for $\sigma > 1$, since

$$\sum_{n=1}^{\infty} |\chi(n) n^{-s}| = \sum_{n=1}^{\infty} n^{-\sigma} < \infty.$$

When $\chi \neq \chi_0$, we have

$$\sum_{1 \le n \le kq} \chi(n) = k \sum_{n=1}^{q} \chi(n) = 0,$$

when

$$\Big| \sum_{1 \le n \le x} \chi(n) \Big| \le q.$$

Thus Theorem 2·4 implies that the series $(8\cdot1)$ converges for $\sigma > 0$.

The absolute convergence of $(8\cdot1)$ for $\sigma > 1$ ensures that one has the Euler product

$$L(s,\chi) = \prod_p (1 - \chi(p) p^{-s})^{-1} \qquad (\sigma > 1). \quad \text{———} \quad (8\cdot2)$$

Hence, in particular, one sees that

$$L(s,\chi_0) = \sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} n^{-s} = \zeta(s)\prod_{p|q}(1-p^{-s}). \qquad (\sigma>1). \quad \text{— (8.3)}$$

This relation shows that in fact $L(s,\chi_0)$ is analytic for $\sigma>0$ except for a simple pole at $s=1$.

As in our previous discussion for $\zeta(s)$, information for prime sums may be inferred by examining the logarithmic derivative of $L(s,\chi)$. We summarise such deliberations in the next theorem.

<u>Theorem 8.1.</u> If $\chi \neq \chi_0$, then $L(s,\chi)$ is analytic for $\sigma>0$. Meanwhile, when $\sigma>0$ the function $L(s,\chi_0)$ is analytic except for a simple pole at $s=1$ with residue $\varphi(q)/q$. In either case, when $\sigma>1$, one has

$$\log L(s,\chi) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n}\chi(n)n^{-s}$$

and

$$-\frac{L'}{L}(s,\chi) = \sum_{n=1}^{\infty} \Lambda(n)\chi(n)n^{-s}.$$

<u>Proof.</u> The only claim to check concerns the residue of $L(s,\chi_0)$ at $s=1$. For this we have

$$\lim_{s\to 1}\left((s-1)\zeta(s)\prod_{p|q}(1-p^{-s})\right) = \prod_{p|q}(1-p^{-1}) = \varphi(q)/q. \qquad //$$

So far as Dirichlet's theorem on primes $p\equiv a \pmod q$ is concerned, we are interested in the sum

$$\sum_{\substack{1\le n\le x \\ n\equiv a \pmod q}} \Lambda(n) \quad \text{as an analogue of } \psi(x) = \sum_{1\le n\le x}\Lambda(n).$$

However, the congruence class $n \equiv a \pmod q$ is easily singled out via the identity (valid whenever $(q,a)=1$)

$$\frac{1}{\varphi(q)} \sum_{\chi \in X(q)} \bar{\chi}(a)\chi(n) = \begin{cases} 1, & \text{when } n \equiv a \pmod q, \\ 0, & \text{when } n \not\equiv a \pmod q. \end{cases}$$

$$\left(\sum_{\chi \in X(q)} \chi(a^{-1}n) = \begin{cases} 1, & \text{when } a^{-1}n \equiv 1 \pmod q, \\ 0, & \text{when } a^{-1}n \not\equiv 1 \pmod q. \end{cases} \right)$$

Thus we have

$$\sum_{\substack{n=1 \\ n \equiv a \pmod q}}^{\infty} \Lambda(n) n^{-s} = \frac{1}{\varphi(q)} \sum_{n=1}^{\infty} \Lambda(n) n^{-s} \sum_{\chi \in X(q)} \bar{\chi}(a)\chi(n)$$

$$= \frac{1}{\varphi(q)} \sum_{\chi \in X(q)} \bar{\chi}(a) \cdot \left(-\frac{L'}{L}(s,\chi) \right) \qquad (\sigma > 1). \tag{8.4}$$

We can now discern a strategy for proving ~~that~~ there are infinitely many primes p with $p \equiv a \pmod q$ when $(q,a)=1$. It is tempting to believe that when $\chi \neq \chi_0$, the Dirichlet series

$$-\frac{L'}{L}(s,\chi) = \sum_{n=1}^{\infty} \frac{\Lambda(n)\chi(n)}{n^s}$$

converges for $\sigma > 0$. Certainly, for $\sigma > 1$, the series is analytic (since the same is true of $L(s,\chi)$ and hence also $\log L(s,\chi)$) provided that $L(s,\chi) \neq 0$. Then provided that $L(1,\chi) \neq 0$, one has

$$\lim_{s \to 1+} \frac{L'}{L}(s,\chi) = \frac{L'}{L}(1,\chi) < \infty.$$

Moreover, the function $L(s,\chi_0)$ has a simple pole at $s=1$, and hence $-\frac{L'}{L}(s,\chi_0)$ also has a simple pole at $s=1$ with residue 1. Indeed, by examining the Laurent series for $L(s,\chi_0)$

about $s = 1$, namely

$$L(s,\chi_0) = \frac{\varphi(q)/q}{s-1} + a_0 + \sum_{i\geq 1} a_i (s-1)^i,$$

we see ~~that~~

$$\log L(s,\chi_0) = \log\left(\frac{1}{s-1}\right) + \log(\varphi(q)/q) + O(|s-1|),$$

whence

$$-\frac{L'(s,\chi_0)}{L} = \frac{1}{s-1} + b_0 + \sum_{i\geq 1} b_i (s-1)^i.$$

Hence (8.4) yields the formula

$$\sum_{\substack{n=1 \\ n\equiv a \pmod q}}^{\infty} \Lambda(n) n^{-s} = \frac{1}{\varphi(q)}(s-1)^{-1} + c_0 + \sum_{i\geq 1} c_i (s-1)^i,$$

as $s \to 1+$.

It remains to prove ~~that~~ $L(1,\chi) \neq 0$ for $\chi \neq \chi_0$.

<u>Theorem 8.2.</u> (Dirichlet). When χ is a character modulo q with $\chi \neq \chi_0$, one has $L(1,\chi) \neq 0$.

<u>Proof.</u> The argument required to establish ~~that~~ $L(1,\chi)\neq 0$ $(\chi \neq \chi_0)$ divides naturally into two halves, and involves some lemmata the proof of which we will provide en passant. The division of cases is natural in hindsight, and a common theme in the subject — namely ~~that~~ of classifying χ as real or complex.

<u>Case 1: χ complex.</u> A character is called <u>real</u> if all of its values are real, and is otherwise called <u>complex</u>. Note ~~that~~ if χ is a complex character, then the character $\bar{\chi}$

is a distinct complex character. Thus the complex characters come in conjugate pairs that share many of the same properties.

Observe first that, in view of Theorem 8.1, one has

$$\log\Big(\prod_{\chi\in X(q)} L(s,\chi)\Big) = \sum_{\chi\in X(q)} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n}\chi(n)n^{-s}, \qquad (\sigma>1)$$

whence

$$\prod_{\chi\in X(q)} L(s,\chi) = \exp\Big(\varphi(q)\sum_{\substack{n=2\\ n\equiv 1 \pmod q}}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s}\Big).$$

By taking $s=\sigma>1$, ~~therefore,~~ we find that the last sum is a non-negative real number, whence

$$\prod_{\chi\in X(q)} L(\sigma,\chi) \geqslant 1. \qquad (8.5)$$

Consider the Laurent expansions of $L(s,\chi)$ about $s=1$ for $\chi\in X(q)$. The function $L(s,\chi_0)$ is analytic for $\sigma>0$ except for a simple pole at $s=1$ with residue $\varphi(q)/q$, so

$$L(s,\chi_0) = \frac{\varphi(q)/q}{(s-1)} + c_0 + c_1(s-1)+\cdots,$$

for suitable $c_0, c_1, \ldots \in \mathbb{C}$. Meanwhile, when $\chi\neq\chi_0$ the function $L(s,\chi)$ is analytic for $\sigma>0$, whence

$$L(s,\chi) = c_0(\chi) + c_1(\chi)(s-1)+\cdots,$$

for suitable $c_i(\chi)\in\mathbb{C}$. Then we see via (8.5) ~~that~~ one can have $L(1,\chi)=0$ for at most one character $\chi\neq\chi_0$, for otherwise the product on the right hand side of (8.5) would have a zero at $s=1$. Moreover, any zero of $L(s,\chi)$ must be simple.

Suppose ~~that~~ χ is a <u>complex</u> character for which $L(1,\chi)=0$.

Then $\bar{\chi}$ is a character with $\bar{\chi} \neq \chi$ and $L(1, \bar{\chi}) = 0$, contradicting our observation that there is at most one character χ with $\chi \neq \chi_0$ for which $L(1, \chi) = 0$. This assertion that $L(1, \bar{\chi}) = 0$ appears obvious enough, but requires some justification. One can see this by brute-force by examining the partial sums of $L(\sigma, \bar{\chi})$ as $\sigma \to 1+$, which shows that

$$\lim_{\sigma \to 1+} L(\sigma, \bar{\chi}) = \lim_{\sigma \to 1+} \overline{L(\sigma, \chi)} = \overline{\lim_{\sigma \to 1+} L(\sigma, \chi)} = 0.$$

A higher brow approach would be to apply the Schwarz reflection principle to show that $\overline{L(s, \chi)} = L(\bar{s}, \bar{\chi})$ (hint: consider $L(s, \chi) \pm L(s, \bar{\chi})$).

It remains to show that $L(1, \chi) \neq 0$ for real characters χ.

Case 2: χ real and $\chi \neq \chi_0$. If χ is a real character, then $\chi^2 = \chi_0$. A character is called quadratic if it has order 2 in the character group, whence $\chi^2 = \chi_0$ but $\chi \neq \chi_0$. Such characters are tricky to handle owing to their solitary life-style. Rather than deal with χ directly, we derive an auxiliary quantity exhibiting more amenable positivity. This is motivated by the observation that since when $L(1, \chi) = 0$,

$$L(s, \chi) = c_1(\chi)(s-1) + c_2(\chi)(s-1)^2 + \cdots$$

and

$$\zeta(s) = \frac{1}{s-1} + c_0 + \cdots,$$

We have

$$\zeta(s) L(s, \chi) = c_1(\chi) + b_1(s-1) + b_2(s-1)^2 + \cdots, \quad (8.6)$$

for suitable $b_i \in \mathbb{C}$, wherein $c_1(\chi) \neq 0$ (we don't need to know this, but else we contradict (8.5)).

One has

$$\zeta(s)L(s,\chi) = \sum_{n=1}^{\infty} r(n)n^{-s} \qquad (\sigma>1),$$

where

$$r(n) = \sum_{d\mid n}\chi(d).$$

The function $r(n)$ is multiplicative, and one has

$$r(p^h) = \begin{cases} 1, & \text{when } p\mid q, \\ h+1, & \text{when } \chi(p)=+1, \\ 1, & \text{when } \chi(p)=-1 \text{ and } 2\mid h, \\ 0, & \text{when } \chi(p)=-1 \text{ and } 2\nmid h. \end{cases}$$

[Note here ~~that~~ when $\chi(p)=-1$, one has

$$r(p^{2l}) = \sum_{m=0}^{2l}\chi(p)^m = (1-1)+\cdots+(1-1)+1 = +1,$$

$m=0 \quad 1 \quad \ldots \quad 2l-2 \quad 2l-1 \quad 2l$

and

$$r(p^{2l-1}) = \sum_{m=0}^{2l-1}\chi(p)^m = (1-1)+\cdots+(1-1) = 0. \;]$$

Thus we see ~~that~~ $r(n)\geq 0$ for all $n\in\mathbb{N}$, and moreover ~~that~~ $r(n^2)\geq 1$ for all $n\in\mathbb{N}$.

Let σ_c denote the absissa of convergence of the Dirichlet series $\sum_{n=1}^{\infty} r(n)n^{-s}$. Then since

$$\sum_{n=1}^{\infty} r(n)n^{-1/2} \geq \sum_{n=1}^{\infty} r(n^2)n^{-1} \geq \sum_{n=1}^{\infty} n^{-1} = +\infty,$$

it is apparent ~~that~~ $1\geq\sigma_c\geq\frac{1}{2}$. However, when $L(1,\chi)=0$, the function $\zeta(s)L(s,\chi)$ is analytic for $\sigma>0$, including the point $s = 1$ in view of (8.6). This analyticity at $s = \sigma_c$ is untenable (i.e. leads to a contradiction), in view of the following lemma of Landau — and the desired conclusion $L(1, \chi) \neq 0$ follows at once. //

Lemma 8.3 (Landau) Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series whose abscissa of convergence σ_c is finite. Then provided that $a_n \geq 0$ for all $n \in \mathbb{N}$, the point $s = \sigma_c$ is a singularity of the function $\alpha(s)$.

Proof. By replacing a_n by $a_n n^{-\sigma_c}$, we may suppose ~~that~~ the abscissa of convergence of $\alpha(s)$ is $\sigma_c = 0$. If $\alpha(s)$ were not analytic at $s = 0$, then it would possess a singularity there, and the proof would be complete. We may ~~therefore~~ suppose to the contrary ~~that~~ $\alpha(s)$ is analytic at $s = 0$, and hence also analytic in the domain

$$\mathcal{D} = \{ s \in \mathbb{C} : \sigma > 0 \quad \text{or} \quad |s| < \delta \},$$

provided that δ is sufficiently small. Our strategy now is to show that the series $\sum_{n=1}^{\infty} a_n n^{-s}$ converges to $\alpha(s)$ for a real number $\sigma < 0$, and this we achieve by considering a power series expansion for $\alpha(s)$ about $s = 1$.

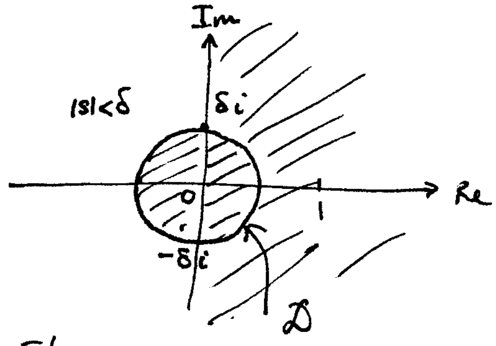
Since $s = 1$ is well inside the half-plane of convergence of $\alpha(s)$, we may apply a Taylor expansion to obtain

$$\alpha(s) = \sum_{k=0}^{\infty} c_k (s-1)^k, \qquad (8.7)$$

where

$$c_k = \frac{\alpha^{(k)}(1)}{k!} = \frac{1}{k!} \sum_{n=1}^{\infty} a_n (-\log n)^k n^{-1}.$$

The radius of convergence of (8.7) is the distance of 1 to the nearest singularity of $\alpha(s)$.



Since $\alpha(s)$ is analytic in $\mathcal{D}$, and the nearest points not in $\mathcal{D}$ are $\pm i\delta$, we see that the radius of convergence of the series (8.7) is at least $\sqrt{1+\delta^2} = 1+\delta'$, say. Thus, when $|s-1| < 1+\delta'$, one has

$$\alpha(s) = \sum_{k=0}^{\infty} \frac{(1-s)^k}{k!} \sum_{n=1}^{\infty} a_n (\log n)^k n^{-1}.$$

If $s = \sigma < 1$, then all the terms are non-negative, and we may rearrange the summation for $-\delta' < \sigma < 1$ to deduce that

$$\alpha(\sigma) = \sum_{n=1}^{\infty} a_n n^{-1} \sum_{k=0}^{\infty} \frac{(1-\sigma)^k (\log n)^k}{k!}$$

$$= \sum_{n=1}^{\infty} a_n n^{-1} \exp\big((1-\sigma)\log n\big) = \sum_{n=1}^{\infty} a_n n^{-\sigma}.$$

Thus $\alpha(s)$ converges at $s = -\delta'/2$, contrary to our assumption that $\alpha(s)$ has abscissa of convergence $\sigma_c = 0$. We are therefore forced to conclude that $\alpha(s)$ is not analytic at $s=0$, but instead has a singularity at $s=0$. //

As an immediate corollary of Theorem 8.2, we deduce Dirichlet's ~~theorem~~.

Corollary 8.4. (Dirichlet). If $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ ~~satisfy~~ $(a,q)=1$, ~~then~~ one has

$$\sum_{p \equiv a \pmod q} \frac{\log p}{p} = +\infty.$$

In particular, ~~there~~ are infinitely many primes p with $p \equiv a \pmod q$.

Proof. Recall from (8.4) ~~that~~ when $(a,q)=1$ and $\sigma > 1$, one has

$$\sum_{\substack{n=1 \\ n \equiv a \pmod q}}^{\infty} \Lambda(n) n^{-s} = -\frac{1}{\phi(q)} \sum_{\chi \in X(q)} \bar{\chi}(a) \frac{L'}{L}(s,\chi).$$

$\uparrow$

analytic for $\sigma > 0$ except at zeros of $L(s,\chi)$, when $\chi \neq \chi_0$

$$= \frac{1}{\phi(q)(s-1)} + O_q(1) \quad \text{as } s \to 1+.$$

Then

$$\lim_{s \to 1+} \sum_{\substack{n=1 \\ n \equiv a \pmod q}}^{\infty} \Lambda(n) n^{-s} = +\infty,$$

and indeed

$$\sum_{\substack{n=1 \\ n \equiv a \pmod q}}^{\infty} \frac{\Lambda(n)}{n} = +\infty.$$

Moreover, ~~the~~ ~~contribution~~ to ~~this~~ sum from prime powers is

$$\sum_{\substack{p^k \equiv a \pmod q \\ k \geq 2}} \frac{\log p}{p^k} \leq \sum_p \log p \sum_{k=2}^{\infty} p^{-k} = \sum_p \frac{\log p}{p(p-1)} < \infty,$$

whence

$$\sum_{p \equiv a \pmod q}^{\infty} \frac{\log p}{p} = +\infty. \quad //$$

We record some consequences of this circle of ideas in the next theorem.

Theorem 8.5. When $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a,q)=1$, and $x \geq 2$, one has:

(a) $$\sum_{\substack{n \leq x \\ n \equiv a \pmod q}} \frac{\Lambda(n)}{n} = \frac{1}{\varphi(q)} \log x + O_q(1) \quad \text{and} \quad \sum_{\substack{p \leq x \\ p \equiv a \pmod q}} \frac{\log p}{p} = \frac{1}{\varphi(q)} \log x + O_q(1);$$

(b) $$\sum_{\substack{p \leq x \\ p \equiv a \pmod q}} \frac{1}{p} = \frac{1}{\varphi(q)} \log\log x + b(q,a) + O_q\left(\frac{1}{\log x}\right),$$

where
$$b(q,a) = \frac{1}{\varphi(q)}\left(C_0 + \sum_{p \mid q} \log\left(1 - \frac{1}{p}\right) + \sum_{\chi \neq \chi_0} \bar{\chi}(a) \operatorname{Log} L(1,\chi)\right) - \sum_{\substack{p^k \equiv a \pmod q \\ k > 1}} \frac{1}{kp^k};$$

(c) $$\prod_{\substack{p \leq x \\ p \equiv a \pmod q}} \left(1 - \frac{1}{p}\right)^{-1} = c(q,a)(\log x)^{1/\varphi(q)}\left(1 + O_q\left(\frac{1}{\log x}\right)\right),$$

where
$$c(q,a) = \left(e^{C_0} \frac{\varphi(q)}{q} \prod_{\chi \neq \chi_0} \left(L(1,\chi)^{\bar{\chi}(a)} \prod_p \left(1 - \frac{1}{p}\right)^{-\chi(p)}\left(1 - \frac{\chi(p)}{p}\right)\right)\right)^{\frac{1}{\varphi(q)}}.$$

Proof. We'll leave this as an exercise for the (energetic!) enthusiast, as a consequence of the following lemma. //

Lemma 8.6. Suppose that χ is a non-principal Dirichlet character. Then for $x \geq 2$, one has

(a) $$\sum_{n\le x} \frac{\chi(n)\Lambda(n)}{n} \ll_\chi 1 \quad \text{and} \quad \sum_{p\le x} \frac{\chi(p)\log p}{p} \ll_\chi 1 ;$$

(b) $$\sum_{p\le x} \frac{\chi(p)}{p} = b(\chi) + O_\chi\left(\frac{1}{\log x}\right);$$

where $$b(\chi) = \log L(1,\chi) - \sum_{p^k\,(k>1)} \frac{\chi(p^k)}{kp^k};$$

(c) $$\prod_{p\le x} \left(1-\frac{\chi(p)}{p}\right)^{-1} = L(1,\chi) + O_\chi\left(\frac{1}{\log x}\right).$$

Note: One can apply these notations, and the orthogonality of characters to detect the reduced residue class a modulo q, to obtain the conclusions of Theorem 8.5. The missing character is χ_0, and $\chi_0(n) = \begin{cases} 1, & \text{when } (q,n)=1, \\ 0, & \text{when } (q,n)>1. \end{cases}$ Thus, expressions such as
$$\sum_{n\le x} \frac{\chi_0(n)\Lambda(n)}{n}$$
are easily estimated by means of corresponding estimates for all the primes. Thus, for example,
$$\sum_{n\le x} \frac{\chi_0(n)\Lambda(n)}{n} = \sum_{n\le x} \frac{\Lambda(n)}{n} - \sum_{\substack{p^k\le x\\ p|q}} \frac{\log p}{p^k} = \sum_{n\le x}\frac{\Lambda(n)}{n} + O_q(1),$$
whence
$$\sum_{n\le x} \frac{\chi_0(n)\Lambda(n)}{n} = \log x + O_q(1).$$

Proof of Lemma 8.6. We establish part (a) using ideas from the corresponding proof of Theorem 5.3. One has
$$\sum_{n\le x} \frac{\chi(n)\log n}{n} = \sum_{n\le x} \frac{\chi(n)}{n}\sum_{d|n}\Lambda(d) = \sum_{d\le x} \frac{\Lambda(d)\chi(d)}{d} \sum_{m\le x/d} \frac{\chi(m)}{m}.$$

The last sum is fairly easily estimated. One has

$$\sum_{m \le y} \frac{\chi(m)}{m} = L(1,\chi) - \sum_{m>y} \frac{\chi(m)}{m}.$$

To estimate the tail, put $S(x) = \sum_{n \le x} \chi(n)$. Then

$$\sum_{m>y} \frac{\chi(m)}{m} = \int_y^\infty u^{-1}\, dS(u) = -\frac{S(y)}{y} + \int_y^\infty S(u) u^{-2}\, du \ll_\chi y^{-1}.$$

Recall here ~~that~~ $|S(y)| \le q = O_q(1)$. Hence one ~~sees that~~

$$\sum_{n \le x} \frac{\chi(n)\log n}{n} = \sum_{d \le x} \frac{\Lambda(d)\chi(d)}{d}\left(L(1,\chi) + O_\chi\left(1/(x/d)\right)\right)$$

$$= L(1,\chi) \sum_{d \le x} \frac{\Lambda(d)\chi(d)}{d} + O_\chi\Big(\frac{1}{x} \underbrace{\sum_{d \le x} \Lambda(d)}_{=O(x)}\Big).$$

Observe next ~~that~~ $L(1,\chi) \ne 0$, so we have

$$\sum_{d \le x} \frac{\Lambda(d)\chi(d)}{d} = \frac{1}{L(1,\chi)} \sum_{n \le x} \frac{\chi(n)\log n}{n} + O_\chi(1)$$

$$= \frac{1}{L(1,\chi)}\left(-L'(1,\chi) - \sum_{n>x} \frac{\chi(n)\log n}{n}\right) + O(1).$$

Again using R-S integration, we have

$$\sum_{n>x} \frac{\chi(n)\log n}{n} = \int_x^\infty \frac{\log u}{u}\, dS(u)$$

$$= -\frac{S(x)\log x}{x} - \int_x^\infty S(u)(1-\log u)u^{-2}\, du$$

$$\ll \frac{\log x}{x}.$$

Thus $$\sum_{d \le x} \frac{\Lambda(d)\chi(d)}{d} = -\frac{L'(1,\chi)}{L(1,\chi)} + O_\chi(1) \ll_\chi 1. \qquad \square$$

The remaining conclusions follow much as in the argument of Theorem 5.3. //

§9. Dirichlet series and Mellin transforms.

Consider a Dirichlet series $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ with abscissa of convergence σ_c, and consider also the partial sum of coefficients $A(x) = \sum_{1 \le n \le x} a_n$. We saw in Theorem 2.4 that whenever $\sigma > \max\{0, \sigma_c\}$, one has

$$\alpha(s) = s \int_1^{\infty} A(x)\, x^{-s-1}\, dx. \qquad (9.1)$$

This is a special example of a Mellin transform.

Definition 9.1 Suppose that $f : \mathbb{R} \to \mathbb{C}$. Then the Mellin transform $F(s)$ of the function $f(s)$ is defined by

$$F(s) = \int_0^{\infty} f(x)\, x^{s-1}\, dx.$$

By analogy with the theory of Fourier transforms and their inverses, it transpires that under appropriate conditions one may reverse this transformation.

Definition 9.2. When $F : \mathbb{C} \to \mathbb{C}$, we define the inverse Mellin transform $f(x)$ of the function $F(s)$ by putting

$$f(x) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} F(s)\, x^{-s}\, ds$$

(when σ_0 is suitably large).

The special case of the inverse Mellin transform of interest to us is embodied in Perron's formula.

Theorem 9.1 (Perron's formula). Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with abscissa of convergence σ_c. Then whenever $\sigma_0 > \max\{0, \sigma_c\}$ and $x > 0$, one has

$$\sum_{n \le x}{}' a_n = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s) \frac{x^s}{s}\, ds.$$

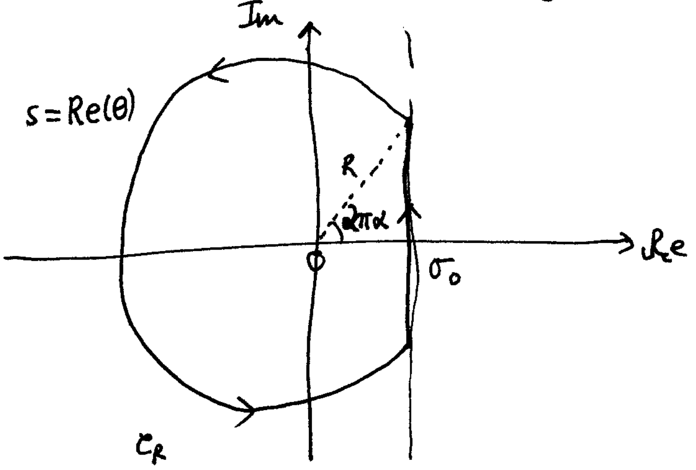
Here, in the summation over a_n, in the situation that $x \in \mathbb{N}$, the final term is counted with weight $1/2$.

Thus $\sum_{n \le x}{}' a_n = a_1 + \cdots + a_{x-1} + \frac{1}{2} a_x$ when $x \in \mathbb{N}$.

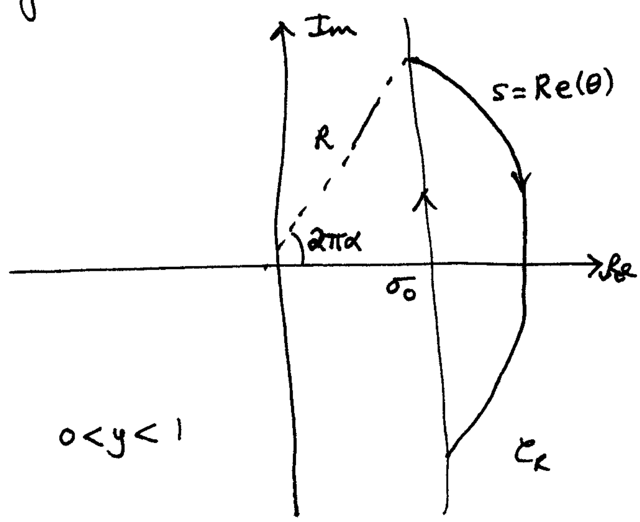
Proof. The key observation is that, by the calculus of residues, one has

$$\frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} y^s \frac{ds}{s} = \begin{cases} 1, & \text{when } y > 1, \\ 0, & \text{when } 0 < y < 1, \end{cases}$$

all provided that $\sigma_0 > 0$. In order to see this, note that we may consider the following contour integrals and take the limit as $R \to \infty$:



$y > 1$



$0 < y < 1$

When $y>1$, we find that with $R\cos(2\pi\alpha)=\sigma_0$, one has that the circular segment of the contour contributes

$$\frac{1}{2\pi i}\int_{\alpha}^{1-\alpha}\frac{y^{Re(\theta)}}{Re(\theta)}\,d\left(Re(\theta)\right)=\int_{\alpha}^{1-\alpha}y^{Re(\theta)}\,d\theta.$$

But $R\sin\left(\frac{\pi}{2}-2\pi\alpha\right)=\sigma_0$, so that $\alpha=\frac{1}{4}+O(1/R)$. Then

$$\left|\int_{\alpha}^{1-\alpha}y^{Re(\theta)}\,d\theta\right|\leq\int_{|\alpha-\frac{1}{4}|\leq C\sqrt{\log R}/R}y^{\sigma_0}\,d\theta+\int_{\substack{\alpha\\|\alpha-\frac{1}{4}|>C\sqrt{\log R}/R}}^{1-\alpha}y^{-\sqrt{\log R}}\,d\theta,$$

for a suitable $C>0$, whence

$$\left|\int_{\alpha}^{1-\alpha}y^{Re(\theta)}\,d\theta\right|\ll_{y,\sigma_0}\sqrt{\log R}/R\longrightarrow 0\quad\text{as } R\longrightarrow\infty.$$

But the function $\frac{y^s}{s}$ is analytic within the contour except for a simple pole at $s=0$, whence

$$\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}y^s\frac{ds}{s}=\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\mathcal{C}_R}y^s\frac{ds}{s}=\lim_{s\to 0}y^s=1.$$

Meanwhile, when $0<y<1$, a similar argument shows that the circular segment of the contour contributes

$$-\frac{1}{2\pi i}\int_{-\alpha}^{\alpha}\frac{y^{Re(\theta)}}{Re(\theta)}\,d\left(Re(\theta)\right)=-\int_{-\alpha}^{\alpha}y^{Re(\theta)}\,d\theta$$

$$\ll\int_{|\alpha-\frac{1}{4}|\leq C\sqrt{\log R}/R}y^{\sigma_0}\,d\theta+\int_{-\alpha}^{\alpha}y^{\sqrt{\log R}}\,d\theta$$

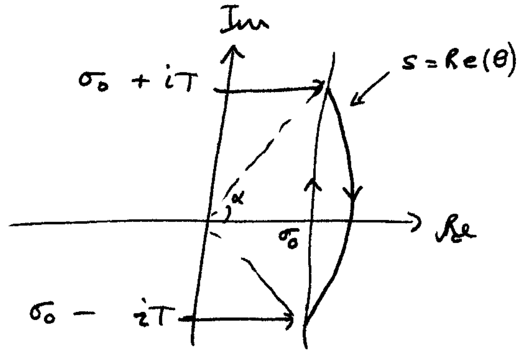
$$\ll_{y,\sigma_0}\sqrt{\log R}/R\longrightarrow 0\quad\text{as } R\to\infty.$$

In this instance, the function y^s/s is analytic within the contour (no poles!), whence

$$\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}y^s\frac{ds}{s}=\lim_{R\to\infty}\frac{1}{2\pi i}\int_{\mathcal{C}_R}y^s\frac{ds}{s}=0.$$

We add ~~to this~~ the Cauchy principal value of the corresponding integral when $y = 1$ and $\sigma_0 > 0$:

$$\lim_{T\to\infty} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{ds}{s} = \frac{1}{2}$$



[Note that as $T \to \infty$, the integral gives $\frac{1}{2\pi i} \int_{-\pi/4}^{\pi/4} \frac{d(Re(\theta))}{Re(\theta)} = \frac{1}{2}$]

We now return to the proof of Perron's formula. The idea is to truncate the series $\alpha(s)$. Thus, if we take N larger than $2x+2$ and $\sigma > \max\{0, \sigma_c\}$, we have

$$\alpha(s) = \sum_{n \leq N} a_n n^{-s} + \sum_{n > N} a_n n^{-s} = \alpha_1(s) + \alpha_2(s), \text{ say.}$$

But then, when $\sigma_0 > \max\{0, \sigma_c\}$, we find ~~that~~

$$\sum_{n \leq x} a_n' = \sum_{n \leq x} a_n \lim_{T\to\infty} \int_{\sigma_0 - iT}^{\sigma_0 + iT} (x/n)^s \frac{ds}{s}$$

$$= \lim_{T\to\infty} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha_1(s) \frac{x^s}{s} ds.$$

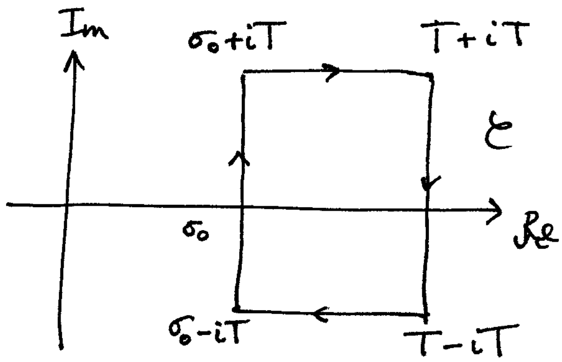
It remains to handle $\alpha_2(s)$, the tail of the Dirichlet series. By Riemann - Stieltjes integration, we see ~~that~~ for $\sigma > \max\{0, \sigma_c\}$,

$$\alpha_2(s) = \int_N^\infty u^{-s} \, d\left(A(u) - A(N)\right) = s \int_N^\infty (A(u) - A(N)) u^{-s-1} du.$$

But whenever $\theta > \max\{0, \sigma_c\}$, we have $A(u) - A(N) \ll u^\theta$, whence for $\sigma > \theta$,

$$\alpha_2(s) \ll |s| \int_N^\infty u^{\theta - \sigma - 1} du \ll \left(\frac{|s|}{\sigma - \theta} \right) N^{\theta - \sigma}$$

We estimate $\displaystyle\int_{\sigma_0-iT}^{\sigma_0+iT} \alpha_2(s)\frac{x^s}{s}\,ds$ for large T by considering the contour:



By the residue theorem, we have $\displaystyle\int_{\mathcal{C}} \alpha_2(s)\frac{x^s}{s}\,ds = 0$, whence

$$\int_{\sigma_0-iT}^{\sigma_0+iT} \alpha_2(s)\frac{x^s}{s}\,ds = -\int_{\sigma_0-iT}^{T-iT} \frac{\alpha_2(s)}{s}x^s\,ds + \int_{\sigma_0+iT}^{T+iT} \alpha_2(s)\frac{x^s}{s}\,ds - \int_{T-iT}^{T+iT} \alpha_2(s)\frac{x^s}{s}\,ds .$$

We have

$$\int_{\sigma_0\pm iT}^{T\pm iT} \alpha_2(s)\frac{x^s}{s}\,ds \ll \frac{N^\theta}{\sigma_0-\theta}\int_{\sigma_0}^{\infty}\left(\frac{x}{N}\right)^{\sigma} d\sigma \ll \frac{N^\theta}{\sigma_0-\theta}\,\frac{(x/N)^{\sigma_0}}{\log(N/x)}$$

and

$$\int_{T-iT}^{T+iT} \alpha_2(s)\frac{x^s}{s}\,ds \ll \frac{N^{\theta-T}}{T-\theta}\int_{-T}^{T} x^T\,d\gamma \ll N^\theta (x/N)^T$$
$$\ll N^\theta (x/N)^{\sigma_0} .$$

Thus

$$\int_{\sigma_0-iT}^{\sigma_0+iT} \alpha_2(s)\frac{x^s}{s}\,ds \ll \frac{N^\theta (x/N)^{\sigma_0}}{\sigma_0-\theta} \ll x^{\sigma_0}N^{\theta-\sigma_0},$$

provided that $\sigma_0 > \theta > \max\{0, \sigma_c\}$.

On combining the contributions from $\alpha_1(s)$ and $\alpha_2(s)$, we conclude that

$$\limsup_{T\to\infty}\left|\sum_{n\le x}{}' a_n - \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT}\alpha(s)\frac{x^s}{s}\,ds\right| \ll x^{\sigma_0}N^{\theta-\sigma_0}.$$

This relation holds for all large N, so on taking $N \to \infty$ we find that

$$\sum_{n \le x}{}' a_n = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s) \frac{x^s}{s} ds.$$

In applications we require a ~~quantitative~~ version of ~~this~~ result useful in ~~obtaining~~ asymptotic relations. This makes use of ~~the~~ sine integral

$$si(x) = -\int_x^\infty \frac{\sin u}{u} du.$$

By integrating by parts, we see that when $x \ge 1$, one has

$$si(x) \ll \min\{1, 1/x\}.$$

Moreover, on evaluating the integral (an exercise in contour integrals), one has

$$si(x) + si(-x) = -\int_{-\infty}^{\infty} \frac{\sin u}{u} du = -\pi.$$

Theorem 9.2. Suppose ~~that~~ $\sigma_0 > \max\{0, \sigma_a\}$ and $x > 0$. Then

$$\sum_{n \le x}{}' a_n = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s) \frac{x^s}{s} ds + R(T),$$

where

$$R(T) = \frac{1}{\pi} \sum_{\frac{x}{2} < n < x} a_n \, si\left(T \log \frac{x}{n}\right) - \frac{1}{\pi} \sum_{x < n \le 2x} a_n \, si\left(T \log \frac{n}{x}\right)$$

$$+ O\left(\frac{4^{\sigma_0} + x^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_0}} \right).$$

We note, as we shall show, ~~that the~~ "error term" $R(T)$ may be estimated more simply as

$$R(T) \ll \sum_{\substack{\frac{x}{2} < n < 2x \\ n \ne x}} |a_n| \min\left\{1, \frac{x}{T|x-n|}\right\} + \frac{4^{\sigma_0} + x^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_0}}.$$

Proof: Since $\alpha(s)$ is absolutely convergent on $[\sigma_0 - iT, \sigma_0 + iT]$, we find that

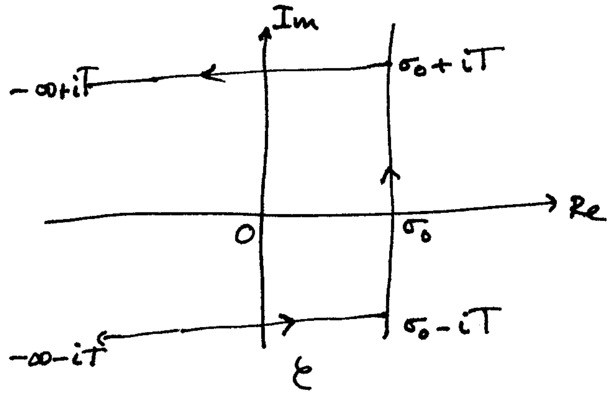
$$\frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s)\,\frac{x^s}{s}\,ds = \sum_n a_n \cdot \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} \left(\frac{x}{n}\right)^s \frac{ds}{s}.$$

It therefore follows ~~that~~ the desired conclusion is a consequence of the formula (when $\sigma_0 > 0$),

$$(*) \quad \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} y^s\,\frac{ds}{s} = \begin{cases} 1 + O(y^{\sigma_0}/T), & \text{when } y \geq 2, \\ 1 + \frac{1}{\pi}\,\mathrm{si}(T\log y) + O(2^{\sigma_0}/T), & \text{when } 1 \leq y \leq 2, \\ -\frac{1}{\pi}\,\mathrm{si}(T\log(\frac{1}{y})) + O(2^{\sigma_0}/T), & \text{when } \frac{1}{2} \leq y \leq 1, \\ O(y^{\sigma_0}/T), & \text{when } y \leq \frac{1}{2}. \end{cases}$$

This we now set about proving, and divide into cases:

(i) Suppose that $y \geq 2$.



By the calculus of residues, we have

$$\frac{1}{2\pi i}\int_{\mathcal{C}} y^s\,\frac{ds}{s} = \lim_{s\to 0} y^s = 1$$

$\uparrow$ simple pole at $s = 0$

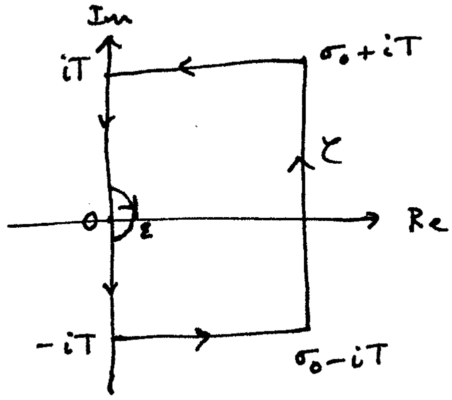
On the other hand, we have

$$\int_{-\infty \pm iT}^{\sigma_0 \pm iT} y^s\,\frac{ds}{s} = \int_{-\infty}^{\sigma_0} \frac{y^{\sigma \pm iT}}{\sigma \pm iT}\,d\sigma \ll \frac{1}{T}\int_{-\infty}^{\sigma_0} y^{\sigma}\,d\sigma = \frac{y^{\sigma_0}}{T\log y}$$

$$\ll y^{\sigma_0}/T. \qquad (\text{since } \log y \geq 2)$$

Thus
$$\frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} y^s\,\frac{ds}{s} = \frac{1}{2\pi i}\left(\int_{\mathcal{C}} - \int_{-\infty - iT}^{\sigma_0 - iT} - \int_{\sigma_0 + iT}^{-\infty + iT} y^s\,\frac{ds}{s}\right)$$

$$= 1 + O(y^{\sigma_0}/T). \quad \square$$



(ii) Suppose that $1 \le y \le 2$.

In this case $\frac{1}{2\pi i}\int_{\mathcal{C}} y^s \frac{ds}{s} = 0$, yet

$$\int_{\pm iT}^{\sigma_0 \pm iT} y^s \frac{ds}{s} \ll \frac{1}{T}\int_0^{\sigma_0} y^\sigma \, d\sigma$$

$$\le \frac{1}{T}\int_0^{\sigma_0} 2^\sigma \, d\sigma \ll \frac{2^{\sigma_0}}{T},$$

while the integral over the semi-circular arc of radius ε (sufficiently small) contributes something asymptotic to

$$\frac{1}{2\pi i}\int_{1/4}^{-1/4} \frac{d(\varepsilon e(\theta))}{\varepsilon e(\theta)} = -\frac{1}{2} \quad \text{as } \varepsilon \to 0.$$

Thus, since

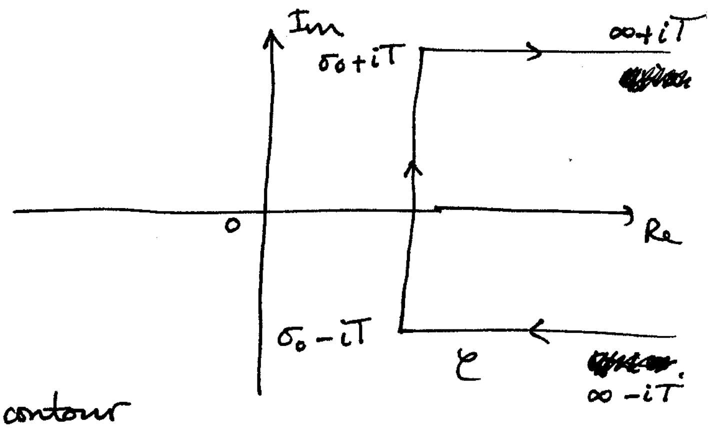
$$\frac{1}{2\pi i}\lim_{\varepsilon\to 0}\left(\int_{i\varepsilon}^{iT} + \int_{-iT}^{-i\varepsilon}\right) y^s \frac{ds}{s} = \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{T} (y^{it} - y^{-it})\frac{dt}{t}$$

$$= \frac{1}{\pi}\int_0^{T\log y} \frac{\sin v}{v}\, dv = \frac{1}{\pi}\left(-\mathrm{si}(0) + \mathrm{si}(T\log y)\right)$$

$$= \frac{1}{2} + \frac{1}{\pi}\mathrm{si}(T\log y),$$

then we deduce that

$$\frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} y^s \frac{ds}{s} = \left(\frac{1}{2} + \frac{1}{\pi}\mathrm{si}(T\log y)\right) - \left(-\frac{1}{2}\right) + O\left(\frac{2^{\sigma_0}}{T}\right). \quad \square.$$



(iii) Suppose that $1/2 \le y \le 1$.

The treatment is the same, but note that

$$\mathrm{si}(-T\log(\tfrac{1}{y})) = \mathrm{si}(+T\log y)$$
$$\mathrm{si}(-T\log(\tfrac{1}{y})) = -\pi - \mathrm{si}(T\log(\tfrac{1}{y})). \quad \square$$

(iv) Suppose that $y \le \frac{1}{2}$. We take our contour to the right, noting that $\frac{1}{2\pi i}\int_{\mathcal{C}} y^s \frac{ds}{s} = 0$,

$$\int_{+\infty \pm iT}^{\sigma_0 \pm iT} y^s \frac{ds}{s} = \int_{+\infty}^{\sigma_0} \frac{y^{\sigma \pm iT}}{\sigma \pm iT}\, d\sigma \ll \frac{1}{T}\int_{\sigma_0}^{\infty} y^{\sigma}\, d\sigma \ll \frac{y^{\sigma_0}}{T \log(1/y)},$$

so ~~that~~

$$\frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} y^s \frac{ds}{s} \ll y^{\sigma_0}/T. \quad \square$$

In order to obtain the consequent simplification, note ~~that~~

$$\operatorname{si}(T|\log n/x|) \ll \min\left\{1, \frac{1}{T\log|n/x|}\right\}.$$

But $\frac{n}{x} = 1 + \frac{n-x}{x}$, whence $\log|n/x| \asymp \left|\frac{n-x}{x}\right|$ for $-\frac{1}{2} \le \left|\frac{n-x}{x}\right| \le 1$.

Thus
$$\operatorname{si}(T|\log n/x|) \ll \min\left\{1, \frac{x}{T|x-n|}\right\} \quad \text{when } \frac{x}{2} \le n \le 2x.$$

It therefore follows ~~that~~

$$R \ll \sum_{\substack{\frac{x}{2} < n < 2x \\ n \ne x}} |a_n| \min\left\{1, \frac{x}{T|x-n|}\right\} + \frac{4^{\sigma_0} + x^{\sigma_0}}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{n^{\sigma_0}},$$

There are weighted versions of these formulae. Thus, for example, if one has weighted sums of the shape

$$A_w(x) = \sum_{n=1}^{\infty} a_n\, w(n/x),$$

and

$$K(s) = \int_0^{\infty} w(x)\, x^{s-1}\, dx, \qquad \text{(Mellin transform of } w(x)\text{)},$$

then we expect (under appropriate conditions) ~~that~~

$$\alpha(s)K(s) = \int_0^{\infty} A_w(x)\, x^{-s-1}\, dx$$

and

$$A_w(x) = \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \alpha(s)\, K(s)\, x^s\, ds.$$

§10. A zero-free region for $\zeta(s)$.

Our goal in the next section is the proof of the Prime Number Theorem in the shape

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O\left(x e^{-c\sqrt{\log x}}\right) \sim \frac{x}{\log x},$$

for a suitable $c>0$. We have seen a pathway towards this conclusion, since $\pi(x)$ is intimately related to $\psi(x)$, and $\psi(x)$ is a partial sum of $\Lambda(n)$, and hence given via Perron's formula in terms of

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \Lambda(n) n^{-s}.$$

A key question then arises — what are the poles of $\frac{\zeta'}{\zeta}(s)$? We know that there is a pole at $s=1$, but there may also be poles at the zeros of $\zeta(s)$. Thus we spend time seeking a zero-free region for $\zeta(s)$. This entails some review of basic complex analysis.

Our first objective is a relation between the number of zeros of an analytic function in a domain, and its rate of growth. This idea generalises the fact that a polynomial of degree k (growing like $|z|^k$) has at most k zeros.

Lemma 10.1. (Jensen's inequality). Suppose that $f(z)$ is analytic in a domain containing the disc $|z| \leq R$. Suppose also ~~that~~ $|f(z)| \leq M$ in this disc, and that $f(0) \neq 0$. Then, whenever $0 < r < R$, the number of zeros of $f(z)$ in the disc $|z| \leq r$ is at most

$$\frac{\log(M/|f(0)|)}{\log(R/r)}.$$

Proof: Let the zeros of $f(z)$ in the disc $|z| \leq R$ be

$z_1, \ldots, z_n$, included with multiplicity. We define the Blaschke product

$$g(z) = f(z) \prod_{k=1}^{n} \frac{R^2 - z\bar{z}_k}{R(z - z_k)} .$$

Here, the k-th factor in the product has a pole at $z = z_k$, and when $|z| = R$ one has

$$\left| \frac{R^2 - z\bar{z}_k}{R(z - z_k)} \right| = \left| \frac{z(\bar{z} - \bar{z}_k)}{R(z - z_k)} \right| = 1 .$$

Then we see that $g(z)$ is analytic in the disc $|z| \leq R$, and when $|z| = R$ one has

$$|g(z)| = |f(z)| \leq M .$$

Thus, by the maximum modulus principle, we see that

$$|f(0)| \prod_{k=1}^{n} \frac{R}{|z_k|} = |g(0)| \leq M .$$

But $|z_k| \leq R$ for every k, and when $|z_k| \leq r$ one has, further, $R/|z_k| \geq R/r$. Then the number, say N, of zeros of $f(z)$ in the disc $|z| \leq r$, satisfies

$$M \geq |f(0)| \left(\frac{R}{r}\right)^N ,$$

whence

$$N \leq \frac{\log(M/|f(0)|)}{\log(R/r)} .$$

[Ex. Check what happens if one has "infinitely" many zeros of f in $|z| \leq R$.].

The conclusion of Lemma 10.1 is only useful if we are able to provide a bound for $|f(z)|$ throughout the disc $|z| \leq R$. Very often it is simpler to bound the real value of such a function, and so it is useful to infer bounds for $|f(z)|$ from corresponding bounds for $\mathrm{Re}(f(z))$.

<u>Lemma 10.2</u> (The Borel–Carathéodory Lemma). Suppose ~~that~~ $h(z)$ is analytic in a domain containing the disc $|z| \le R$. Suppose also ~~that~~ $h(0)=0$, and ~~that~~ $\mathrm{Re}(h(z)) \le M$ for $|z| \le R$. Then, whenever $|z| \le r < R$, one has

$$|h(z)| \le \frac{2Mr}{R-r} \quad \text{and} \quad |h'(z)| \le \frac{2MR}{(R-r)^2}.$$

<u>Proof.</u> Our strategy is to expand $h(z)$ and $h'(z)$ as Taylor series about 0, and to ~~this~~ end we require estimates for the Taylor series coefficients $h^{(k)}(0)/k!$. Observe ~~that~~

$$0 = h(0) = \frac{1}{2\pi i} \oint_{|z|=R} h(z)\frac{dz}{z} = \int_0^1 h(Re(\theta))\, d\theta.$$

Also, when $k>0$, we have

$$\int_0^1 h(Re(\theta))\, e(k\theta)\, d\theta = \frac{R^{-k}}{2\pi i} \oint_{|z|=R} h(z) z^{k-1}\, dz = 0 \qquad \text{—}(10.1)$$

and

$$\int_0^1 h(Re(\theta))\, e(-k\theta)\, d\theta = \frac{R^k}{2\pi i} \oint_{|z|=R} h(z) z^{-k-1}\, dz = R^k \frac{h^{(k)}(0)}{k!}. \qquad \text{—}(10.2)$$

Combining (10.1) and (10.2) when $k>0$, we see ~~that~~

$$\int_0^1 h(Re(\theta))\,(1+\cos(2\pi(k\theta+\phi)))\, d\theta = \int_0^1 h(Re(\theta))\left(1+\frac{e(k\theta+\phi)+e(-k\theta-\phi)}{2}\right) d\theta$$

$$= \frac{e(-\phi)}{2} \cdot \frac{R^k h^{(k)}(0)}{k!}.$$

Thus, since $\mathrm{Re}(h(z)) \le M$, we see ~~that~~

$$\mathrm{Re}\left(\frac{e(-\phi)}{2} \cdot \frac{R^k h^{(k)}(0)}{k!}\right) \le \int_0^1 M(1+\cos(2\pi(k\theta+\phi)))\, d\theta.$$

We choose ϕ so that $e(-\phi)h^k(0) = |h^{(k)}(0)|$, and hence infer that

$$\frac{|h^k(0)|}{k!} \leq \frac{2M}{R^k}\int_0^1 \Big(1+\cos(2\pi(k\theta+\phi))\Big)\,d\theta = \frac{2M}{R^k}.$$

We may now apply this upper bound in a Taylor expansion to deduce that whenever $|z| \leq r < R$, one has

$$|h(z)| \leq \sum_{k=1}^{\infty} \left|\frac{h^{(k)}(0)}{k!}\right| r^k \leq 2M\sum_{k=1}^{\infty}\left(\frac{r}{R}\right)^k = \frac{2Mr}{R-r}$$

and

$$|h'(z)| \leq \sum_{k=1}^{\infty} \frac{|h^{(k)}(0)|\cdot kr^{k-1}}{k!} \leq \frac{2M}{R}\sum_{k=1}^{\infty} k\left(\frac{r}{R}\right)^{k-1} = \frac{2MR}{(R-r)^2}.$$

//

Finally, in our discussion of general results, we come to a formula for the logarithmic derivative of an analytic function in terms of local zeros.

Lemma 10.3. Suppose that $f(z)$ is analytic in a domain containing the disc $|z| \leq 1$. Suppose also that $|f(z)| \leq M$ in this disc, and $f(0) \neq 0$. Then, whenever r and R are fixed with $0 < r < R < 1$, and $|z| \leq r$, one has

$$\frac{f'}{f}(z) = \sum_{k=1}^{n} \frac{1}{z-z_k} + O_{r,R}\Big(\log\big(M/|f(0)|\big)\Big),$$

where the summation is over all zeros of f for which $|z_k| \leq R$.

Proof. If necessary by slightly increasing the value of R, we may suppose that $f(z)$ has no zeros on the circle $|z| = R$. Note that this assumes that f has a finite number of zeros in $|z| \leq R + \varepsilon$, when ε is small, which follows from Lemma 10.1. We define $g(z)$ by the Blaschke product

$$g(z) = f(z)\prod_{k=1}^{n}\left(\frac{R^2 - z\bar{z}_k}{R(z-z_k)}\right).$$

Note that, as a consequence of Lemma 10.1, one has

$$n \le \frac{\log(M/|f(0)|)}{\log(1/R)} \ll_R \log\left(\frac{M}{|f(0)|}\right). \qquad (10.3)$$

Also, when $|z|=R$, one has

$$\left|\frac{R^2 - z\bar{z}_k}{R(z-z_k)}\right| = 1,$$

whence for $|z| = R$ one has $|g(z)| = |f(z)| \le M$. It follows from the maximum modulus principle ~~that~~ $|g(z)| \le M$ for $|z| \le R$.

Observe next ~~that~~

$$|g(0)| = |f(0)| \prod_{k=1}^{n} \frac{R}{|z_k|} \ge |f(0)| > 0.$$

Since $g(z)$ has no zeros in the disc $|z| \le R$, we may put $h(z) = \log\left(\frac{g(z)}{g(0)}\right)$. Then $h(0) = 0$, and when $|z| \le R$,

$$\mathrm{Re}(h(z)) = \log|g(z)| - \log|g(0)| \le \log M - \log|f(0)|.$$
$$\le \log\left(\frac{M}{|f(0)|}\right).$$

We therefore deduce from ~~the~~ Borel–Carathéodory lemma ~~that~~ when $|z| \le r$, one has

$$h'(z) \ll_{r,R} \log\left(\frac{M}{|f(0)|}\right).$$

But

$$h'(z) = \frac{g'(z)}{g} = \frac{f'}{f}(z) - \sum_{k=1}^{n}\frac{1}{z-z_k} + \sum_{k=1}^{n}\frac{1}{z - R^2/\bar{z}_k}.$$

Thus,

$$\frac{f'}{f}(z) - \sum_{k=1}^{n} \frac{1}{z-z_k} + \sum_{k=1}^{n} \frac{1}{z-R^2/\bar{z}_k} \ll \log\left(\frac{M}{|f(0)|}\right).$$

But $|R^2/\bar{z}_k| \geq R$, so that when $|z| \leq r$, one has $|z - R^2/\bar{z}_k| \geq R-r$, whence

$$\sum_{k=1}^{n} \frac{1}{z-R^2/\bar{z}_k} \leq \frac{n}{R-r} \overset{(10\cdot3)}{\ll_{R,r}} \log\left(\frac{M}{|f(0)|}\right).$$

We may ~~therefore conclude that~~

$$\frac{f'}{f}(z) = \sum_{k=1}^{n} \frac{1}{z-z_k} + O\left(\log\left(\frac{M}{|f(0)|}\right)\right). \quad //$$

At last, we may apply these ideas to obtain a formula for $\frac{\zeta'}{\zeta}(s)$ in terms of zeros of $\zeta(s)$. In preparation for this we derive a crude bound for $\zeta(s)$.

<u>Lemma 10·4.</u> Let $\delta > 0$ be fixed. Then one has

$$\zeta(s) = \frac{1}{s-1} + O(1)$$

uniformly for s in the rectangle $\delta \leq \sigma \leq 2$ and $|t| \leq 1$, and further

$$\zeta(s) \ll (1 + \tau^{1-\sigma}) \min\left(\frac{1}{|\sigma-1|}, \log \tau\right)$$

uniformly for $\delta \leq \sigma \leq 2$ and $|t| \geq 1$. (Here, we note $\tau = |t|+4$).

<u>Proof.</u> For the first assertion we recall ~~that~~

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{s+1}}\,du \qquad (\sigma > 0) \qquad (\text{see } (3\cdot3)).$$

The second term here is $\ll |s| \int_1^\infty u^{-1-\sigma}\,du \ll 1/\delta$, and the desired conclusion follows. □

Let us turn to the second assertion. We recall the conclusion of

(77)

Theorem 3.5, namely ~~that~~ when $s \neq 1$ and $\sigma > 0$, $x > 0$, one has

$$\zeta(s) = \sum_{n \le x} n^{-s} + \frac{x^{1-s}}{s-1} + \frac{\{x\}}{x^s} - s\int_x^\infty \frac{\{u\}}{u^{s+1}}\,du .$$

The integral here ~~satisfies~~

$$\int_x^\infty \frac{\{u\}}{u^{s+1}}\,du \ll \int_x^\infty u^{-1-\sigma}\,du \ll x^{-\sigma}/\sigma,$$

whence

$$\zeta(s) = \sum_{n \le x} n^{-s} + \frac{x^{1-s}}{s-1} + O\left(\frac{|s|}{\sigma}\, x^{-\sigma}\right)$$

$$= \sum_{n \le x} n^{-s} + \frac{x^{1-s}}{s-1} + O(\tau x^{-\sigma}). \qquad (10.4)$$

We may suppose ~~that~~ $x \ge 2$, whence

$$\sum_{n \le x} n^{-s} \ll \sum_{n \le x} n^{-\sigma} \ll 1 + \int_1^x u^{-\sigma}du \qquad (\sigma \ge 0).$$

When $|\sigma - 1| \le 1/\log x$ and $1 \le u \le x$, one has $u^{-\sigma} \asymp u^{-1}$, and then $\int_1^x u^{-\sigma}du \ll \int_1^x u^{-1}du = \log x$. Meanwhile, when $(\sigma - 1) > 1/\log x$, one has instead

$$\int_1^x u^{-\sigma}du < \int_1^\infty u^{-\sigma}du = 1/\sigma-1,$$

and when $0 \le \sigma \le 1 - 1/\log x$,

$$\int_1^x u^{-\sigma}du = \frac{x^{1-\sigma}-1}{1-\sigma} < \frac{x^{1-\sigma}}{1-\sigma}.$$

Hence

$$\sum_{n \le x} n^{-s} \ll (1 + x^{1-\sigma}) \min\left\{\frac{1}{|1-\sigma|}, \log x\right\}$$

uniformly for $0 \le \sigma \le 2$. On ~~substituting this estimate into~~ (10.4), we conclude ~~that~~ with $x = \tau$ one has

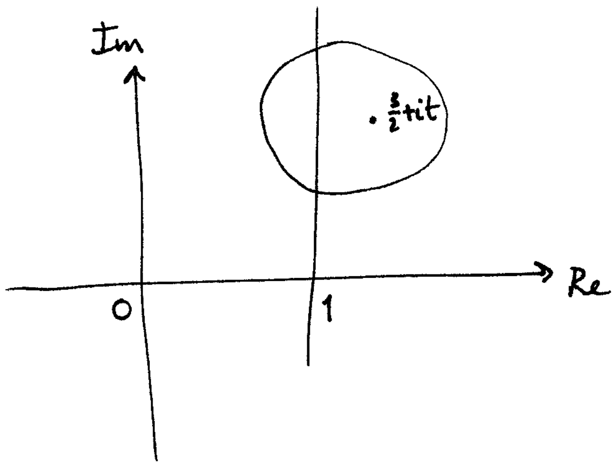
$$\zeta(s) \ll (1 + \tau^{1-\sigma}) \min\left\{\frac{1}{|\sigma - 1|}, \log\tau\right\}. \quad //$$

Lemma 10.5. Suppose that $|t| \geq \frac{7}{8}$ and $\frac{5}{6} \leq \sigma \leq 2$. Then one has

$$\frac{\zeta'}{\zeta}(s) = \sum_{\rho} \frac{1}{s-\rho} + O\Big(\log \underbrace{(|t|+4)}_{\tau} \Big),$$

where the summation is over all zeros ρ of $\zeta(s)$ satisfying $|\rho - (\frac{3}{2}+it)| \leq \frac{5}{6}$.

[Recall that $s = \sigma + it$].



Proof: We apply Lemma 10.3 to the function

$$f(z) = \zeta\big(\underbrace{z + (\tfrac{3}{2}+it)}_{s} \big),$$

with $R = 5/6$ and $r = 2/3$. We have the absolutely convergent product formula

$$f(0) = \zeta(\tfrac{3}{2}+it) = \prod_p \left(1 - p^{-\frac{3}{2}-it}\right)^{-1},$$

whence

$$|f(0)| \geq \prod_p \left(1 - p^{-3/2}\right)^{-1} > 0.$$

Thus $|f(0)| \gg 1$. Moreover, by Lemma 10.4, we have

$$|f(z)| \ll \tau^{1/2} \log \tau \ll \tau \quad \text{for} \quad |z| \leq 1.$$

We therefore conclude that

$$\frac{f'}{f}(z) = \sum_{\rho} \frac{1}{(z+\frac{3}{2}+it)-\rho} + O(\log \tau),$$

so that $$\frac{\zeta'}{\zeta}(s) = \sum_{\rho} \frac{1}{s-\rho} + O(\log \tau). \quad //$$

How now do we establish a zero-free region for $\zeta(s)$? Suppose that $\zeta(s)$ has a zero very close to the line $\sigma = 1$, say with imaginary part γ. As a thought experiment, suppose this zero is at $s = 1+i\gamma$. Then with $\delta \to 0+$, we would have

$$\frac{\zeta'}{\zeta}(1+\delta+i\gamma) \sim \frac{m}{\delta},$$

where m is the multiplicity of this zero. Notice here that

$$\mathrm{Re}\left(\frac{\zeta'}{\zeta}(1+\delta+i\gamma)\right) = -\sum_{n=1}^{\infty} \Lambda(n) n^{-1-\delta} \cos(\gamma \log n)$$

$$\leq \sum_{n=1}^{\infty} \Lambda(n) n^{-1-\delta} = -\frac{\zeta'}{\zeta}(1+\delta) \sim \frac{1}{\delta},$$

so necessarily $m \leq 1$. If this occurs with something close to equality, it follows that $\cos(\gamma \log n) \approx -1$ for most $n = p^h$, whence $p^{i\gamma} \approx -1$ for most p. But then

$$p^{2i\gamma} \approx +1 \quad \text{for most } p,$$

whence

$$\frac{\zeta'}{\zeta}(1+\delta+2i\gamma) \sim -\sum_{n=1} \Lambda(n) n^{-1-\delta} \sim -\frac{1}{\delta}. \quad \left(\text{pole at } 1+2i\gamma\right)$$

This heuristic argument shows ~~that~~ the existence of a zero very close to the 1-line implies the existence of a pole very close to the 1-line distinct from ~~that~~ at $s=1$, yielding a contradiction.

We now set about making this heuristic rigorous, and this involves relating the values of $-\frac{\zeta'}{\zeta}(s)$ with $s = \sigma + it$ to the value at $s = \sigma + 2it$.

<u>Lemma 10.6.</u> When $\sigma > 1$, one has

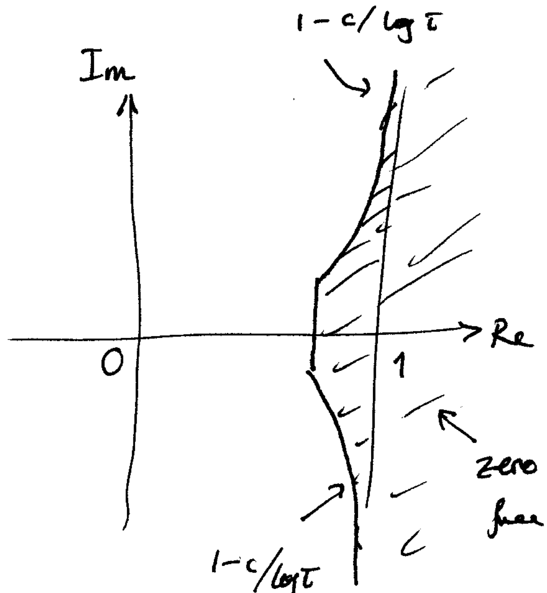
$$\mathrm{Re}\left(-3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{\zeta'}{\zeta}(\sigma+it) - \frac{\zeta'}{\zeta}(\sigma+2it)\right) \geq 0.$$

<u>Proof.</u> The left hand side here is

$$\sum_{n=1}^{\infty} \Lambda(n)\, n^{-\sigma}\left(3 + 4\cos(t\log n) + \cos(2t\log n)\right).$$

But $3 + 4\cos\theta + \cos(2\theta) = 2(1+\cos\theta)^2 \geq 0$ for all θ, and so the desired conclusion follows. //

<u>Theorem ~~Lemma~~ 10.7.</u> There is an absolute constant $c > 0$ such ~~that~~ $\zeta(s) \neq 0$ for $\sigma \geq 1 - c/\log\tau$.



<u>Remark:</u> The sharpest know zero-free region is due to Korobov and Vinogradov (1958, separately):

$$\sigma \geq 1 - c(\log\tau)^{-2/3}(\log\log\tau)^{-1/3}.$$

<u>Riemann Hypothesis:</u> $\sigma > \frac{1}{2}$.

<u>Proof</u> (of Lemma 10.7): When $\sigma > 1$, one has

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} \geq \prod_p (1 - p^{-\sigma})^{-1} > 0,$$

by absolute convergence of the product. Thus we may restrict attention to the situation with $\sigma \leq 1$.

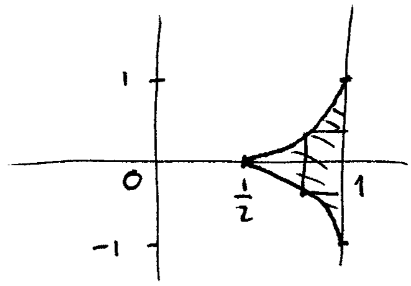
Next, from the formula

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{s+1}}\,du,$$

we find that

$$\left|\zeta(s) - \frac{s}{s-1}\right| \leq |s|\int_1^\infty \frac{du}{u^{\sigma+1}} = \frac{|s|}{\sigma}. \qquad (\sigma > 0).$$

Thus $\zeta(s) \neq 0$ when $\sigma > |s-1|$. This condition is satisfied when



$$\sigma^2 > (\sigma-1)^2 + t^2$$

$$\leftrightarrow \qquad \sigma > (1+t^2)/2.$$

Then $\zeta(s) \neq 0$ for $|t| \leq 7/8$ and $\frac{8}{9} \leq \sigma \leq 1$. $\left(\frac{1}{2}\left(\left(\frac{7}{8}\right)^2+1\right) = 1 - \frac{15}{128} < \frac{8}{9}\right)$.

Suppose next ~~that~~ there is a zero $\rho_0 = \beta_0 + i\gamma_0$ of $\zeta(s)$ with $\frac{5}{6} \leq \beta_0 \leq 1$ and $|\gamma_0| \geq \frac{7}{8}$. Since $Re(\rho) \leq 1$ for all ~~zeros~~ ρ of $\zeta(s)$, we have

$$Re\left(\frac{1}{s-\rho}\right) > 0 \qquad \text{whenever} \quad \sigma > 1.$$

We apply Lemma 10.5 with $s = 1+\delta+i\gamma_0$ to see ~~that~~ for small positive values of δ, and for a suitable $c_1 > 0$, one has

$$-Re\left(\frac{\zeta'}{\zeta}(1+\delta+i\gamma_0)\right) \leq -\frac{1}{1+\delta-\beta_0} + c_1\log(|\gamma_0|+4).$$

(notice – additional zeros, if any, make an even more negative contrib^n).

Similarly, but now with $s = 1+\delta+2i\gamma_0$, we obtain

$$-Re\left(\frac{\zeta'}{\zeta}(1+\delta+2i\gamma_0)\right) \leq c_1\log(|2\gamma_0|+4).$$

Finally, we have

$$-\frac{\zeta'}{\zeta}(1+\delta) = \frac{1}{\delta} + O(1).$$

Hence, on combining these estimates via Lemma 10.6, we discern that for a suitable $c_2 > 0$, one has

$$\frac{3}{\delta} - \frac{4}{1+\delta-\beta_0} + c_2(\log(|\gamma_0|+4)) \geq 0$$

Whence on putting $\delta = \dfrac{1}{2c_2 \log(|\gamma_0|+4)}$ we obtain

$$7c_2 \log(|\gamma_0|+4) \geq 4/(1+\delta-\beta_0).$$

This, in turn, implies that

$$1+\delta-\beta_0 = 1 + \frac{1}{2c_2\log(|\gamma_0|+4)} - \beta_0 \geq \frac{4}{7c_2\log(|\gamma_0|+4)},$$

so that

$$1-\beta_0 \geq \frac{1}{14c_2\log(|\gamma_0|+4)} = \frac{1}{14c_2\log\tau}.$$

This completes the proof of the theorem. //

When it comes to applying Perron's formula to give explicit versions of the prime number theorem, it is useful to have available bounds for $1/\zeta(s)$, $\zeta'/\zeta(s)$ and $\log\zeta(s)$ in the vicinity of the line $\sigma=1$.

<u>Theorem 10.8:</u> Suppose that $c>0$ is an absolute constant with the property that $\zeta(s)\neq 0$ for $\sigma\geq 1-c/\log\tau$. Then whenever $\sigma > 1 - c/(2\log\tau)$ and $|t|\geq 7/8$, one has

$$\zeta(s) \ll \log\tau, \quad \frac{\zeta'}{\zeta}(s) \ll \log\tau, \quad \frac{1}{\zeta(s)} \ll \log\tau, \qquad \text{(10.5)}$$

and

$$|\log\zeta(s)| \leq \log\log\tau + O(1). \qquad \text{(10.6)}$$

<u>Proof.</u> The first bound in (10.5) follows from Lemma 10.4, since for $\sigma \geq 1-\frac{1}{2}c/\log\tau$, one has

$$\zeta(s) \ll (1+\tau^{1-\sigma}) \min\left\{\frac{1}{|\sigma-1|}, \log\tau\right\}, \qquad \text{for } |t|\geq 1$$

$$\ll \log\tau,$$

whilst

$$\zeta(s) = \frac{1}{s-1} + O(1) \ll \log\tau, \qquad \text{for } \tfrac{7}{8}\leq |t| \leq 1. \quad \square$$

We next consider the second estimate claimed in (10.5). When $\sigma > 1$, one has

$$\left|\frac{\zeta'}{\zeta}(s)\right| \leq \sum_{n=1}^{\infty} \Lambda(n) n^{-\sigma} = -\frac{\zeta'}{\zeta}(\sigma) \ll \frac{1}{\sigma-1}$$

(using the Laurent series for $\zeta(\sigma)$). Thus, whenever $\sigma \geq 1 + 1/\log\tau$, it follows immediately that $\frac{\zeta'}{\zeta}(s) \ll \log\tau$. We now extrapolate from this bound for $\frac{\zeta'}{\zeta}(s_1)$ when $s_1 = 1 + 1/\log\tau + it$ to obtain bounds for $\frac{\zeta'}{\zeta}(s)$ inside the line $\sigma = 1$. Observe ~~that~~

$$\frac{\zeta'}{\zeta}(s_1) \ll \log\tau,$$

and further, from Lemma 10.5, whenever $|t| \geq 7/8$ and $1 - \frac{1}{2}c/\log\tau \leq \sigma \leq 1 + 1/\log\tau$,

$$\frac{\zeta'}{\zeta}(s) = \sum_\rho \frac{1}{s-\rho} + O(\log\tau), \quad \text{where we sum over zeros } \rho \text{ with } |\rho - (\tfrac{3}{2}+it)| \leq 5/6.$$

In particular, one has

$$\log\tau \gg \frac{\zeta'}{\zeta}(s_1) = \sum_\rho \frac{1}{s_1-\rho} + O(\log\tau) \Rightarrow \mathrm{Re}\left(\sum_\rho \frac{1}{s_1-\rho}\right) \ll \log\tau.$$

Given any $s = \sigma + it$ with $1 - \frac{1}{2}c/\log\tau \leq \sigma \leq 1 + 1/\log\tau$, one therefore sees that

$$\frac{\zeta'}{\zeta}(s) - \frac{\zeta'}{\zeta}(s_1) = \sum_\rho \left(\frac{1}{s-\rho} - \frac{1}{s_1-\rho}\right) + O(\log\tau).$$

But

$$\frac{1}{s-\rho} - \frac{1}{s_1-\rho} = \frac{s_1-s}{(s-\rho)(s_1-\rho)} \ll \frac{1}{|s_1-\rho|^2 \log\tau}$$

Note here that since $\zeta(s) \neq 0$ for $\sigma > 1 - c/\log\tau$, ~~then~~ $\mathrm{Re}(\rho) \leq 1 - c/\log\tau$ for all zeros ρ counted in the sum, whence $\mathrm{Re}(s-\rho) \gg 1/\log\tau$ at the same ~~time~~ as one has $|s - s_1| \ll 1/\log\tau$. This justifies the assertion ~~that~~ $|s-\rho| \asymp |s_1-\rho|$ employed above. Since also $|s_1 - \rho| \gg 1/\log\tau$, one concludes ~~that~~

$$\frac{1}{s-\rho} - \frac{1}{s_1-\rho} \ll \frac{1}{|s_1-\rho|} \ll \mathrm{Re}\left(\frac{1}{s_1-\rho}\right),$$

whence

$$\sum_\rho \left(\frac{1}{s-\rho} - \frac{1}{s_1-\rho}\right) \ll \mathrm{Re}\left(\sum_\rho \frac{1}{s_1-\rho}\right) \ll \log\tau.$$

$\cancel{+O(\log\tau)}$

We ~~therefore~~ conclude ~~that~~

$$\frac{\zeta'}{\zeta}(s) = \frac{\zeta'}{\zeta}(s_1) + O(\log\tau) \ll \log\tau. \quad \square$$

The third estimate of (10.5) (and indeed the first) can be obtained from (10.6), since $\log(1/|\zeta|) = -\mathrm{Re}(\log\zeta) \leq -\log\log\tau + O(1)$ implies $1/|\zeta(s)| \ll \log\tau$. Thus we may concentrate now on the proof of (10.6). Here we proceed similarly in moving from $\sigma = 1 + 1/\log\tau$ to the left of the line $\sigma = 1$. Observe ~~that~~ when $\sigma > 1$, one has

$$|\log\zeta(s)| \leq \sum_{n=2}^\infty \frac{\Lambda(n)}{\log n} n^{-\sigma} = \log\zeta(\sigma).$$

But

$$\zeta(\sigma) = \frac{\sigma}{\sigma-1} - \sigma\int_1^\infty \frac{\{u\}}{u^{\sigma+1}}\,du < \frac{\sigma}{\sigma-1} \quad (\text{by } (3.3)),$$

so when $\sigma \geq 1 + 1/\log\tau$ one has

$$\log(\zeta(s)) < \log(\log\tau).$$

It follows ~~that~~ $|\log\zeta(s_1)| \leq \log\log\tau$ ~~...~~ when $s_1 = 1 + 1/\log\tau + it$.

In order to move from s_1 to the point $s = \sigma + it$, with $1 - \frac{1}{2}c/\log\tau \le \sigma \le 1 + 1/\log\tau$, we observe that

$$\log\zeta(s) - \log\zeta(s_1) = \int_{s_1}^{s} \frac{\zeta'}{\zeta}(z)\,dz$$

$$\ll |s - s_1| \sup_{z\in[s_1,s]} \left|\frac{\zeta'}{\zeta}(z)\right|$$

$$\ll \frac{1}{\log\tau}\cdot\log\tau \qquad (\text{using } (10.5)).$$

But then $|\log\zeta(s)| = |\log\zeta(s_1)| + O(1) \le \log\log\tau + O(1)$. $\square$ //

§11. The Prime Number Theorem.

We are now equipped to apply Perron's formula to obtain an asymptotic formula for

$$\psi(x) = \sum_{n\le x}\Lambda(n)$$

via bounds for $\frac{\zeta'}{\zeta}(s)$.

Theorem 11.1. There is a positive constant c having the property that

$$\psi(x) = x + O(x\exp(-c\sqrt{\log x})),$$

$$\theta(x) = x + O(x\exp(-c\sqrt{\log x})),$$

and

$$\pi(x) = li(x) + O(x\exp(-c\sqrt{\log x})),$$

uniformly for $x \ge 2$. [Here $li(x) = \int_2^x \frac{dt}{\log t}$ is the logarithmic integral, and satisfies $li(x) = x\sum_{k=1}^{K-1}\frac{(k-1)!}{(\log x)^k} + O_K\left(\frac{x}{(\log x)^K}\right)$.]

Proof.

Recall that

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty}\Lambda(n)n^{-s}.$$

Thus, it follows from the quantitative version of Perron's formula (Theorem 9.2) that for $\sigma_0 > 1$,

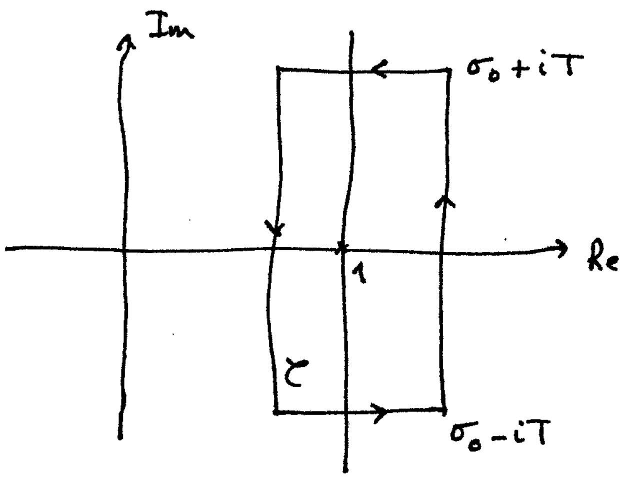
$$\psi(x) = -\frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0+iT}\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds + R(T),$$

where

$$R(T) \ll \left(\sum_{\frac{x}{2} < n < 2x} \Lambda(n) \min\left\{1, \frac{x}{T|x-n|}\right\} \right) + \left(\frac{(4x)^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma_0}} \right).$$

In Qn 4 of Problem Sheet 4, one shows that with $2 \le T \le x$ and $\sigma_0 = 1 + 1/\log x$, one has $R(T) \ll \frac{x}{T}(\log x)^2$, whence

$$\psi(x) = -\frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\zeta'}{\zeta}(s) \frac{x^s}{s} \, ds + O\left(\frac{x}{T}(\log x)^2\right).$$



$$\sigma_0 = 1 + 1/\log x, \qquad \sigma_1 = 1 - b/\log T.$$

We now turn our attention towards the calculation of the integral in this formula for $\psi(x)$. We close the contour $[\sigma_0 - iT, \sigma_0 + iT]$ by using a vertical contour that avoids including any of the zeros of $\zeta(s)$. We know from Theorem 10.7 that there is a positive constant $b > 0$ such that

$\zeta(s)$ has no zeros s with $\sigma > 1 - 2b/\log t$. We put $\sigma_1 = 1 - b/\log T$, and close the contour as indicated in the diagram. It follows that $-\frac{\zeta'}{\zeta}(s)$ has only a simple pole at $s = 1$ with residue 1, and otherwise is analytic ~~within~~ $\mathcal{C}$. Hence

$$\frac{1}{2\pi i} \int_{\mathcal{C}} \left(-\frac{\zeta'}{\zeta}(s)\right) \frac{x^s}{s}\, ds = \lim_{s \to 1} \frac{x^s}{s} = x.$$

Also, one has

$$\int_{\sigma_1 \pm iT}^{\sigma_0 \pm iT} \left(-\frac{\zeta'}{\zeta}(s)\right) \frac{x^s}{s}\, ds \overset{\text{Thm 10.8}}{\ll} \log T \cdot \frac{x^{\sigma_0}}{T} (\sigma_0 - \sigma_1) \ll \frac{x}{T} \quad (\text{since } x \geq T)$$

and

$$\int_{\sigma_1 - iT}^{\sigma_1 + iT} \left(-\frac{\zeta'}{\zeta}(s)\right) \frac{x^s}{s}\, ds \overset{\text{Thm 10.8}}{\ll} \log T \cdot x^{\sigma_1} \int_{-T}^{T} \frac{dt}{1+|t|} + x^{\sigma_1} \int_{-1}^{1} \frac{dt}{|\sigma_1 + it - 1|}$$

$$\ll x^{\sigma_1} (\log T)^2 + \frac{x^{\sigma_1}}{1 - \sigma_1}$$

$$\ll x^{\sigma_1} (\log T)^2.$$

$\frac{\zeta'(s)}{\zeta(s)} \ll \frac{1}{s-1}$ as $|s-1|<1$.

Thus, we conclude that

$$\frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \left(-\frac{\zeta'}{\zeta}(s)\right) \frac{x^s}{s}\, ds = x + O\left(\frac{x}{T} + x^{\sigma_1} (\log T)^2\right),$$

Whence

$$\psi(x) = x + O\left(x (\log x)^2 \left(\frac{1}{T} + x^{-b/\log T}\right)\right).$$

We have now only to optimise the size of the error term here, and this is achieved by choosing T so that

$$T \asymp x^{b/\log T} \longleftrightarrow \log T \asymp b \frac{\log x}{\log T}.$$

Then we may take $\log T = \sqrt{\log x}$, or equivalently,

$$T = \exp(\sqrt{\log x}).$$

Then we have

$$x(\log x)^2\left(\frac{1}{T} + x^{-b/\log T}\right) \ll x(\log x)^2\left(\exp(-\sqrt{\log x}) + \exp(-b\sqrt{\log x})\right)$$

$$\ll x\exp(-c\sqrt{\log x}),$$

for a suitable $c > 0$. This proves ~~that~~

$$\psi(x) = x + O(x\exp(-c\sqrt{\log x})).$$

We have shown (cf. Problems 3, ex 1) ~~that~~

$$\theta(x) = \psi(x) + O(\sqrt{x}),$$

and thus

$$\theta(x) = x + O(x\exp(-c\sqrt{\log x})).$$

Finally, by Riemann-Stieltjes ~~integration~~, we have

$$\pi(x) = \int_{2^-}^{x} \frac{1}{\log u}\, d\,\theta(u) = \int_{2^-}^{x}\frac{du}{\log u} + \int_{2^-}^{x}\frac{d(\theta(u)-u)}{\log u}$$

$$= li(x) + \left[\frac{\theta(u)-u}{\log u}\right]_{2^-}^{x} + \int_{2^-}^{x}\frac{\theta(u)-u}{u(\log u)^2}\,du,$$

whence

$$\pi(x) - li(x) \ll x\exp(-c\sqrt{\log x}) + \int_2^x \exp(-c\sqrt{\log u})\,du$$

$$\ll x\exp(-c\sqrt{\log x}).$$

(consider $\int_t^{2t} \exp(-c\sqrt{\log u})\,du$ for $t = 2^j$).

Hence

$$\pi(x) = li(x) + O(x\exp(-c\sqrt{\log x})). \quad \square$$

//

Exercise: Prove that for a suitable $c>0$, one has

$$\sum_{n\le x} \mu(n) \ll x\exp(-c\sqrt{\log x}).$$

World Record: Korobov - Vinogradov, 1958: $\pi(x) = li(x) + O(x\exp(-c(\log x)^{3/5}(\log\log x)^{-1/5}))$

Riemann Hypothesis: $\pi(x) = li(x) + O(x^{1/2}\log x)$.

§12. The distribution of smooth numbers. (a.k.a. "psixyology").

Integers $n \in \mathbb{N}_{>1}$, all of whose prime factors are small are frequently called "smooth numbers" (occasionally "friable numbers"). They play a prominent role in factorisation algorithms, hence cryptography, as well as being of interest in their own right. They are also used in the circle method (Waring's problem, etc).

Definition 12.1. Let

$$S(x,y) := \{n \in \mathbb{Z} \cap [1,x] : p|n \ \& \ p \text{ prime} \Rightarrow p \leq y\},$$

$$\psi(x,y) := \operatorname{card}(S(x,y)).$$

We first investigate the ~~situation~~ in which y is a power of x in size. Here, we have in mind (in particular) the scenario where $y = x^{\eta}$ with $\eta > 0$ very small.

It transpires that a special function, called the Dickman function $\rho(u)$, plays a distinguished role in the subject. This function $\rho : [0,\infty] \to \mathbb{R}$ is defined to be the unique continuous function satisfying the differential-delay equation

$$u\rho'(u) = -\rho(u-1) \quad (\text{for } u > 1),$$

subject to

$$\rho(u) = 1 \qquad (0 \leq u \leq 1).$$

It is useful to note before proceeding further ~~that~~ since

$$\rho'(u) = -\frac{\rho(u-1)}{u} \qquad (u>1),$$

we see that whenever $1 \leq u \leq v$, one has

$$\rho(v) - \rho(u) = -\int_u^v \frac{\rho(t-1)}{t}\,dt. \qquad (12.1)$$

In order to get a sense for how $\rho(u)$ behaves, observe that when $u>1$, one has

$$(u\rho(u))' = u\rho'(u) + \rho(u) = \rho(u) - \rho(u-1),$$

whence

$$u\rho(u) = \int_{u-1}^{u} \rho(v)\,dv + C,$$

for $u \geq 1$, where $C = 1\cdot\rho(1) - \int_0^1 \rho(v)dv = 1 - 1 = 0.$

One can now proceed by induction to show that $\rho(u) > 0$ for all $u \geq 0$ from the relation

$$\rho(u) = \frac{1}{u}\int_{u-1}^{u} \rho(v)\,dv.$$

It therefore follows from the relation $\rho'(u) = -\frac{1}{u}\rho(u-1)$ $(u>1)$ that $\rho(u)$ is decreasing for $u>1$, whence by induction

$$\rho(n) \leq \frac{1}{n!} = \frac{1}{\Gamma(n+1)}, \qquad (n \in \mathbb{N})$$

A more careful analysis shows that for $u \geqslant 0$, one has

$$\frac{1}{2\Gamma(2u+1)} \leq \rho(u) \leq \frac{1}{\Gamma(u+1)}.$$

Thus, when u is large, we have $\rho(u) \lessapprox u^{-u}e^{u}$ roughly speaking.

<u>Theorem 12.2.</u> (Dickman, 1930). For each $U \geqslant 0$, one has

$$\psi(x, x^{1/u}) = \rho(u)x + O\left(\frac{x}{\log x}\right) \qquad (0 \leq u \leq U,\ x \geqslant 2).$$

<u>Proof.</u> We apply a variation of the inclusion-exclusion principle sometimes referred to as Buchstab's identity. This is easy to follow in the initial stages of the argument. First, trivially, when $0 \leq u \leq 1$, one

(91)

has $$\psi(x, x^{1/u}) = \operatorname{card}\{1 \le n \le x\} = \lfloor x \rfloor = x + O(1).$$
$$= \rho(u)x + O(1). \quad \square$$

Suppose next ~~that~~ $1 < u \le 2$ and put $y = x^{1/u}$. We observe ~~that~~ if $n \le x$, then n can have at most one prime factor p with $p > x^{1/u} \geqslant \sqrt{x}$, whence

$$\psi(x,y) = \sum_{1 \le n \le x} 1 - \sum_{y < p \le x} \sum_{\substack{n \le x \\ p \mid n}} 1$$

$$= \lfloor x \rfloor - \sum_{y<p\le x} \left\lfloor \frac{x}{p} \right\rfloor$$

$$= x - x \sum_{y<p\le x} \frac{1}{p} + O(\pi(x))$$

$$= x - x\left(\log\log x - \log\log y + O\left(\frac{1}{\log y}\right)\right) + O\left(\frac{x}{\log x}\right)$$

$$= x\left(1 - \log\left(\frac{\log x}{\log y}\right)\right) + O\left(\frac{x}{\log x}\right).$$

$$= x(1 - \log u) + O\left(\frac{x}{\log x}\right).$$

But when $1 < u \le 2$, one has

$$\underset{\rho(u)-1}{\underset{\parallel}{\rho(u) - \rho(1)}} = -\int_1^u \frac{\rho(t-1)}{t}\,dt = -\int_1^u \frac{dt}{t} = -\log u,$$

whence $\rho(u) = 1 - \log u$. Thus

$$\psi(x,y) = x\rho(u) + O\left(\frac{x}{\log x}\right),$$

when $y = x^{1/u}$ with $1 < u \le 2$. $\square$

We now proceed by induction on U, supposing that the conclusion of the theorem holds for $0 \le u \le U$. This has been established with $U = 2$ as the basis of the induction, and we now seek to establish the conclusion for $0 \le u \le U+1$. Suppose then that when $0 \le u \le U$, one has

$$\psi(x, x^{1/u}) = \rho(u)\, x + O\left(\frac{x}{\log x}\right).$$

Given any integer n with $1 \le n \le x$, let $P(n)$ denote the largest prime factor of n. Then one has

$$\psi(x,y) = 1 + \sum_{p \le y} \operatorname{card}\{2 \le n \le x : P(n) = p\}$$

$$= 1 + \sum_{p \le y} \psi(x/p, p).$$

Thus, when $y = x^{1/u}$ with $U < u \le U+1$, one sees that

$$\psi(x,y) - \psi(x, x^{1/U}) = \sum_{p \le y} \psi(x/p, p) - \sum_{p \le x^{1/U}} \psi(x/p, p)$$

$$= - \sum_{y < p \le x^{1/U}} \psi(x/p, p).$$

We may now apply the inductive hypothesis, since $x/p = p^{u_p}$, with

$$u_p = \frac{\log(x/p)}{\log p} = \frac{\log x - \log p}{\log p} \le u - 1 \le U.$$

Thus,

$$\sum_{y<p\le x^{1/U}} \psi(x/p, p) = \sum_{y<p\le x^{1/U}} \left(\rho\left(\frac{\log x}{\log p} - 1\right)\frac{x}{p} + O\left(\frac{x/p}{\log(x/p)}\right)\right)$$

$$= x \sum_{y<p\le x^{\frac{1}{U}}} \frac{\rho\left(\frac{\log x}{\log p} - 1\right)}{p} + O\Bigg(\frac{x}{\log x} \underbrace{\sum_{y<p\le x^{1/U}} \frac{1}{p}}_{O(1)}\Bigg)$$

since $\sum_{y<p\le x^{1/U}} \frac{1}{p} = \log\left(\frac{\log x^{1/U}}{\log y}\right) = \log\left(\frac{u}{U}\right) = O(1)$.

We estimate the main term by applying Riemann-Stieltjes integration. Put $A(z) = \sum_{p\le z} \frac{1}{p}$, so by Mertens' theorem we have $A(z) = \log\log z + c + r(z)$, where $r(z) = O\left(\frac{1}{\log z}\right)$. Thus,

$$\sum_{y<p\le x^{\frac{1}{U}}} \frac{\rho\left(\frac{\log x}{\log p}-1\right)}{p} = \int_y^{x^{1/U}} \rho\left(\frac{\log x}{\log z}-1\right) d\,A(z)$$

$$= \int_{y^{\frac{1}{u}}}^{x^{1/U}} \rho\left(\frac{\log x}{\log z}-1\right) d\,\log\log z + \int_{y^{1/u}}^{x^{1/U}} \rho\left(\frac{\log x}{\log z}-1\right) d\,r(z)$$

$$= \int_U^u \rho(t-1)\,\frac{dt}{t} + \left[\rho\left(\frac{\log x}{\log z}-1\right) r(z)\right]_{y^{1/u}}^{x^{1/U}} - \int_{x^{1/u}}^{x^{\frac{1}{U}}} r(z)\,d\rho\left(\frac{\log x}{\log z}\right)$$

$$\left(t = \frac{\log x}{\log z}\right)$$

$$\Downarrow$$

$$\frac{dt}{t} = -\frac{\log x\;dz}{z(\log z)^2\cdot(\log x)/(\log z)} = d(\log\log z)$$

$$\overset{(12\cdot 1)}{=} \rho(U)-\rho(u) + O\left(\frac{1}{\log x}\left(1+\int_{x^{1/u}}^{x^{1/U}} d\left|\rho\left(\frac{\log x}{\log z}-1\right)\right|\right)\right)$$

$$= \rho(U)-\rho(u) + O\left(\frac{1}{\log x}\right).$$

We conclude that

$$\psi(x,y) - \psi(x, x^{\frac{1}{U}}) = -x(\rho(U)-\rho(u)) + O(x/\log x)$$

$$= x\rho(u) + O(x/\log x),$$

since $\psi(x, x^{\frac{1}{U}}) = x\rho(U) + O(x/\log x)$.

This confirms the inductive hypothesis when $U \le u \le U+1$, and so the conclusion of the theorem follows by induction. //

The behaviour of $\psi(x,y)$ is very different when y is extremely small, and this is of ~~interest~~ for cryptographic purposes. One can obtain asymptotic information here, but to illustrate ideas we obtain a lower bound useful in many applications.

Theorem 12.3. Suppose ~~that~~ $\log x \le y \le x$. Then one has

$$\psi(x,y) \gg \frac{x}{y} \exp\left(-u \log\log x + u/2\right),$$

where $u = \dfrac{\log x}{\log y}$.

Corollary 12.4. One has $\psi(x, (\log x)^a) \gg x^{1-\frac{1}{a}+o(1)}$ for $a \ge 1$.

[In fact, also, $\psi(x, (\log x)^a) \ll x^{1-1/a+o(1)}$].

Proof. From Theorem 12.3, one has

$$\psi(x, (\log x)^a) \gg x(\log x)^{-a} \exp\left(-\frac{\log x}{a\log\log x}\cdot \log\log x + \frac{1}{2}\frac{\log x}{a\log\log x}\right)$$

$$= x^{1-\frac{1}{a}} \exp\left(\frac{1}{2}\frac{\log x}{a\log\log x} - a\log\log x\right) = x^{1-\frac{1}{a}+o(1)} \quad //$$

Corollary 12.5. When $c>0$ is fixed, one has

$$\psi(x, \exp(c\sqrt{\log x \log\log x})) \gg x\exp\left(-(c+\tfrac{1}{c}+o(1))\sqrt{\log x\log\log x}\right).$$

Proof. From Theorem 12.3, one has

$$\psi(x, \exp(c\sqrt{\log x\log\log x})) \gg x\exp\left(-c\sqrt{\log x\log\log x} - \frac{\log x}{c\sqrt{\log x\log\log x}}\cdot\log\log x + \frac{1}{2}\frac{\log x}{c\sqrt{\log x\log\log x}}\right)$$

$$= x\exp\left(-(c+\tfrac{1}{c})\sqrt{\log x\log\log x} + \frac{1}{2c}\sqrt{\frac{\log x}{\log\log x}}\right)$$

//

Proof. (of Theorem 12·3). Write $r = \pi(y)$, and let the first r prime numbers be $p_1, \dots, p_r$. Then if $n \in S(x,y)$, one has

$$\log n = a_1 \log p_1 + \dots + a_r \log p_r ,$$

for some non-negative integers $a_1, \dots, a_r$ with $a_i \leq \frac{\log n}{\log p_i}$. Each choice of $\underline{a}$ with $a_1 + \dots + a_r \leq \frac{\log n}{\log p_r}$ gives a unique element of $S(x,y)$. Thus

$$\psi(x,y) \geq \# \{ a_i \in \mathbb{Z}_{\geq 0} : a_1 + \dots + a_r \leq k := \lfloor \tfrac{\log n}{\log y} \rfloor \}.$$

$$= \# \{ a_i \in \mathbb{Z}_{\geq 0} : a_0 + \dots + a_r = k \}$$

$$= \text{coeff. of } t^k \text{ in } \Big(\sum_{a=0}^{\infty} t^a \Big)^{r+1} = (1-t)^{-r-1} = \sum_{k=0}^{\infty} \binom{r+k}{k} t^k$$

$$= \binom{r+k}{k} .$$

By Stirling's formula, we obtain

$$\psi(x,y) \gg \left(\frac{r+k}{k} \right)^k \left(\frac{r+k}{r} \right)^r \frac{1}{\sqrt{k}} \qquad \text{since } r \gg k.$$

Put $z = \frac{y}{k \log y} = \frac{r}{k}(1+o(1))$. Then we see ~~that~~

$$\psi(x,y) \gg \left(1 + \frac{y}{k \log y} \right)^k \left(1 + \frac{k \log y}{y} \right)^{y/\log y} \frac{1}{\sqrt{k}} \geq \left(z(1+1/z)^z \right)^k .$$

We have ~~that~~ $z(1+1/z)^z = \exp\left(z \log(1+1/z) + \log z \right)$ is an increasing function of z for $z \geq 1$, whence

$$\psi(x,y) \gg \left(w(1+1/w)^w \right)^k \geq \left(w(1+1/w)^w \right)^{u-1} ,$$

where $w = \frac{y}{u \log y}$, and where $u = \frac{\log x}{\log y}$. But $w \leq y/\sqrt{k}$,

whence

$$\psi(x,y) \gg \frac{1}{y} \left(\frac{y}{u \log y} \right)^u \left(1 + \frac{u \log y}{y} \right)^{y/\log y}$$

$$= \frac{1}{y}\, e^{u \log y} \exp\left(- u \log\log x + \frac{y}{\log y} \log\left(1 + \frac{\log x}{y} \right)\right)$$

$$= \frac{x}{y} \exp\left(- u \log\log x + \frac{1}{2} \underbrace{\frac{\log x}{\log y}}_{= u} \right).$$

//

We now turn to some applications and open problems concerning smooth numbers. First an application. Suppose ~~that~~ q is large, and ~~that~~ χ is a non-principal character modulo q. Then we have

$$\sum_{1 \le n \le q} \chi(n) = 0,$$

so ~~that~~ $\chi(n) \neq 1$ for some integer n with $1 \le n \le q$. But we might expect ~~that~~ $\chi(n) \neq 1$ for a much smaller value of q — ~~this~~ value of n is a "witness" to χ being non-principal. To illustrate ideas, consider a large prime number q.

Suppose ~~that~~ $\chi(n) = 1$ for all integers n with $1 \le n \le y$. Then in particular, one has $\chi(\pi) = 1$ for all prime numbers $1 \le \pi \le y$, whence $\chi(m) = 1$ for $m \in S(q, \pi)$. We deduce ~~that~~

$$0 = \sum_{1 \le n \le q} \chi(n) = \sum_{m \in S(q,y)} \chi(m) + \sum_{\substack{1 \le n \le q \\ n \notin S(q,y)}} \chi(n)$$

$$\geq \psi(q,y) - (q - 1 - \psi(q,y)) = 2\psi(q,y) - (q-1).$$

Hence

$$\psi(q,y) \le \tfrac{1}{2}(q-1) \Rightarrow \frac{\psi(q,y)}{q} \le \frac{1}{2} + O\left(\frac{1}{q}\right).$$

We now may apply Theorem 12.2, noting ~~that~~ when $y = q^{1/\sqrt{e}}$, one has

$$\rho\left(\frac{\log q}{\log y} \right) = \rho(\sqrt{e}) = 1 - \log \sqrt{e} = 1/2.$$

Thus, the relation $\frac{\psi(q,y)}{q} \leq \frac{1}{2} + O(1/q)$ ensures that $y \leq q^{1/\sqrt{e} + o(1)}$. Then necessarily, one finds that

$$\chi(\pi) \neq 1 \quad \text{for some prime } \pi \text{ with } \pi \leq q^{1/\sqrt{e} + o(1)}.$$

<u>Theorem 12.6.</u> When p is a large prime number and χ is a non-principal character modulo p, there is a prime number π with $\pi \leq p^{1/\sqrt{e} + o(1)}$ for which $\chi(\pi) \neq 1$.

We'll see later that this exponent can be improved — Vinogradov conjectured that in fact the bound of Theorem 12.6 can be replaced by $\pi \leq p^{\varepsilon}$ for any $\varepsilon > 0$.

The Twin Prime Conjecture — that there are infinitely many primes p for which $p+2$ is also prime — remains wide open. On the other hand, the Twin Smooth-number Conjecture is a theorem:

<u>Theorem 12.7.</u> There are infinitely many integers n for which n and $n+1$ are both n^{ε}-smooth, for any fixed $\varepsilon > 0$.

<u>Proof.</u> We take $n+1 = m^k$ and $n = m^k - 1$ with a suitable choice for k, and with $m = 2$, say. It follows that $n+1$ is n^{ε}-smooth, and we may factor n as a product of cyclotonic polynomials

$$n = \prod_{d \mid k} \Phi_d(m),$$

where $\deg \Phi_d(t) = \phi(d)$.

The largest prime dividing n has size n^{θ}, where

$$\theta = \frac{\log \Phi_d(m)}{\log m^k} \leq \frac{\phi(k)}{k} = \prod_{\pi | k} (1 - 1/p).$$

Then if we take $k = \prod_{p \leq y} p$, we see ~~that~~

$$\frac{\phi(k)}{k} = \prod_{p \leq y} \left(1 - \frac{1}{p}\right) \asymp \frac{1}{\log y},$$

whence

$$\theta \ll \frac{1}{\log y} \asymp \frac{1}{\log\log k} \asymp \frac{1}{\log\log\log n}.$$

Since $\theta < \varepsilon$ whenever n is sufficiently large, it follows ~~that~~ n and $n+1$ are both n^ε-smooth. //

Similar arguments show ~~that~~ one can find integers n infinitely often with

$$n, n+1, \ldots, n + t(n) \quad \text{all } n^\varepsilon\text{-smooth},$$

where $t(n) \gg_\varepsilon \dfrac{\log\log\log\log n}{\log\log\log\log\log n}$. (Balog & W.).

Also, given $\varphi(t) = at^2 + bt + c \in \mathbb{Z}[t]$, it follows ~~that~~ ~~there~~ are infinitely many integers n so ~~that~~ $\varphi(n)$ is n^ε-smooth (Baker, Fretwell, Martin & W.). But no such result is known for cubic or higher degree polynomials.

§13. The functional equation for $\zeta(s)$.

We saw in Corollary 3·6 that the Riemann zeta function is analytic in $\mathrm{Re}(s) > 0$ except for a simple pole at $s = 1$ with residue 1. We now seek to extend this continuation to cover the whole complex plane. This requires new ideas, and leads to an elegant formulation via what is known as the functional equation.

Theorem 13·1 The function

$$\xi(s) = \tfrac{1}{2} s(s-1)\zeta(s)\Gamma(\tfrac{1}{2}s)\pi^{-s/2}$$

is entire, and one has

$$\xi(s) = \xi(1-s) \quad \text{for all } s \in \mathbb{C} \qquad (\text{the functional equation})$$

The proof of this assertion requires some preparation — perhaps the most elegant route goes via the application of theta functions. There is a kind of reciprocity for theta functions that follows via Poisson summation.

Theorem 13·2. Let $\alpha \in \mathbb{R}$ and $z \in \mathbb{C}$ ~~satisfy~~ $\mathrm{Re}(z) > 0$. Then one has

$$\sum_{n=-\infty}^{\infty} e^{-\pi(n+\alpha)^2 z} = z^{-\frac{1}{2}} \sum_{k=-\infty}^{\infty} e(k\alpha)\, e^{-\pi k^2/z}. \qquad (13·1)$$

Moreover, under the same conditions,

$$\sum_{n=-\infty}^{\infty} (n+\alpha)\, e^{-\pi(n+\alpha)^2 z} = -i z^{-3/2} \sum_{k=-\infty}^{\infty} k\, e(k\alpha)\, e^{-\pi k^2/z}, \qquad (13·2)$$

in which the branch of $z^{1/2}$ is determined by taking $1^{1/2} = 1$.

Proof. Write $f(u) = e^{-\pi u^2 z}$, so that the left hand side of (13·1) can be written as

$$\sum_{n\in\mathbb{Z}} f(n+\alpha).$$

Notice here that ~~since~~ $\mathrm{Re}(z)>0$, ~~this~~ sum is absolutely convergent for each fixed value of α, and also uniformly convergent for α in any compact set. The Fourier transform of f is given by

$$\hat{f}(t) := \int_{\mathbb{R}} f(u)\, e(-tu)\, du,$$

so ~~that~~ if $g(u) = f(u+\alpha)$, we have

$$\hat{g}(t) = \int_{\mathbb{R}} f(u+\alpha)\, e(-tu)\, du = \int_{\mathbb{R}} f(v)\, e(-t(v-\alpha))\, dv$$

$$= e(t\alpha)\hat{f}(t).$$

Moreover,

$$\hat{f}(t) = \int_{-\infty}^{\infty} \exp(-\pi u^2 z - 2\pi i t u)\, du$$

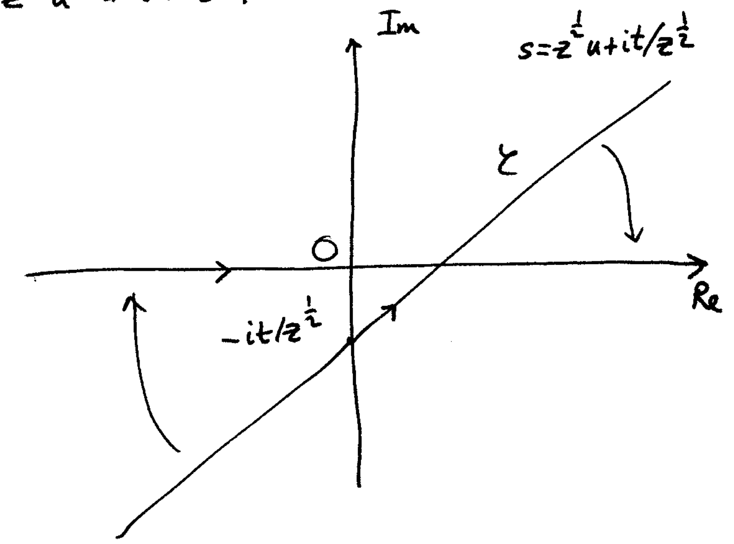
$$\left(u^2 z + 2itu = (u + it/z)^2 z + t^2/z\right)$$

$$= e^{-\pi t^2/z} \int_{-\infty}^{\infty} e^{-\pi (u+it/z)^2 z}\, du$$

$$= e^{-\pi t^2/z} \int_{-\infty}^{\infty} e^{-\pi (z^{1/2} u + it/z^{1/2})^2}\, du.$$

$$s := z^{\frac{1}{2}} u + it/z^{\frac{1}{2}}.$$

$$= z^{-\frac{1}{2}} e^{-\pi t^2/z} \int_{\mathcal{C}} e^{-\pi s^2}\, ds$$



We can move the contour $\mathcal{C}$ defined by $s = z^{\frac{1}{2}} u + it/z^{\frac{1}{2}}$ to the real line, since when $|u|$ is large, say $|u|>R$, we have $|e^{-\pi s^2}|$ very small,

$$|e^{-\pi s^2}| = e^{-\pi u^2 \mathrm{Re}(z)} = O(e^{-R^2 \cdot \mathrm{Re}(z)}).$$

Thus the contribution from this manoeuvre at ∞ is 0, and one obtains

$$\hat{f}(t) = z^{-\frac{1}{2}} e^{-\pi t^2/z} \int_{-\infty}^{\infty} e^{-\pi s^2}\, ds = z^{-\frac{1}{2}} e^{-\pi t^2/z}.$$

We may conclude thus ~~far that~~

$$\sum_{k\in\mathbb{Z}} \hat{g}(k) = \sum_{k\in\mathbb{Z}} e(k\alpha)\hat{f}(k) = \sum_{k\in\mathbb{Z}} e(k\alpha)\cdot z^{-\frac{1}{2}} e^{-\pi k^2/z}.$$

Since $\mathrm{Re}(z) > 0$, ~~the~~ Fourier transforms $\hat{g}(k) = e(k\alpha)\hat{f}(k)$ are rapidly decaying as $|k| \to \infty$, and thus by Poisson summation we have

$$\sum_{k \in \mathbb{Z}} \hat{g}(k) = \sum_{n \in \mathbb{Z}} g(n) = \sum_{n \in \mathbb{Z}} f(n+\alpha) = \sum_{n \in \mathbb{Z}} e^{-\pi(n+\alpha)^2 z}.$$

Thus,

$$\sum_{n \in \mathbb{Z}} e^{-\pi(n+\alpha)^2 z} = z^{-\frac{1}{2}} \sum_{k \in \mathbb{Z}} e(k\alpha)\, e^{-\pi k^2/z}. \quad \square$$

In order to obtain the relation (13.2), we may differentiate (13.1) with respect to α noting ~~that~~ ~~the~~ series are uniformly convergent for α in compact sets. Thus

$$\frac{d}{d\alpha}\left(\sum_{n=-\infty}^{\infty} \exp(-\pi(n+\alpha)^2 z)\right) = \frac{d}{d\alpha}\left(z^{-\frac{1}{2}} \sum_{k=-\infty}^{\infty} e(k\alpha)\exp(-\pi k^2/z)\right)$$

$$\sum_{n=-\infty}^{\infty} -2\pi(n+\alpha) z \exp(-\pi(n+\alpha)^2 z) \qquad = \qquad z^{-\frac{1}{2}} \sum_{k=-\infty}^{\infty} 2\pi i k\, e(k\alpha)\exp(-\pi k^2/z).$$

$$\Downarrow$$

$$\sum_{n \in \mathbb{Z}} (n+\alpha) e^{-\pi(n+\alpha)^2 z} = -i z^{-3/2} \sum_{k \in \mathbb{Z}} k\, e(k\alpha) e^{-\pi k^2/z}.$$

$\square$

Define $\Gamma(s,a) = \int_a^\infty e^{-w} w^{s-1}\, dw$ (incomplete gamma function). $(a \in \mathbb{C})$

Theorem 13.3. Suppose that $s \in \mathbb{C} \setminus \{0,1\}$ and $\mathrm{Re}(z) \geq 0$. Then one has

$$\zeta(s)\Gamma(s/2)\pi^{-s/2} = \pi^{-s/2} \sum_{n=1}^{\infty} \Gamma(s/2, \pi n^2 z) n^{-s} + \pi^{(s-1)/2} \sum_{n=1}^{\infty} \Gamma((1-s)/2, \pi n^2/z) n^{s-1} + \frac{z^{(s-1)/2}}{s-1} - \frac{z^{s/2}}{s}.$$

Proof. Recall that when $\sigma > 0$, one has

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx.$$

Then by the change of variable $x = \pi n^2 u$, we see that

$$\Gamma(s/2) = \int_0^\infty e^{-x} x^{s/2-1}\,dx = n^s \pi^{s/2} \int_0^\infty e^{-\pi n^2 u} u^{s/2-1}\,du.$$

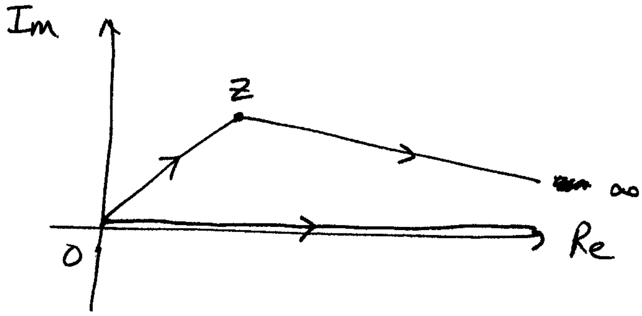
We therefore deduce that when $\sigma > 1$ (required to obtain absolute conv. of $\sum n^{-s}$)

$$\sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 u} u^{s/2-1}\,du \overset{\text{abs. conv.}}{=} \int_0^\infty u^{s/2-1}\left(\sum_{n=1}^\infty e^{-\pi n^2 u}\right) du$$

$$\|$$

$$\sum_{n=1}^\infty n^{-s}\pi^{-s/2}\Gamma(s/2) = \zeta(s)\pi^{-s/2}\Gamma(s/2).$$

We next adjust the integral on the right hand side so as to rewrite it in terms of incomplete gamma functions (with improved convergence properties). When $\mathrm{Re}(z) \geq 0$, we may move the path of integration $[0,\infty)$ to $[0,z] \cup [z,\infty)$. Then on writing



$$\theta_1(u) = \sum_{n=1}^\infty e^{-\pi n^2 u},$$

we see that

$$\int_0^\infty \theta_1(u) u^{s/2-1}\,du = \underbrace{\int_0^z \theta_1(u) u^{s/2-1}\,du}_{I_1} + \underbrace{\int_z^\infty \theta_1(u) u^{s/2-1}\,du}_{I_2}.$$

Now, reversing course, we see that

$$I_2 = \sum_{n=1}^\infty \int_z^\infty e^{-\pi n^2 u} u^{s/2-1}\,du \qquad u = x/(n^2\pi)$$

$$= \sum_{n=1}^\infty n^{-s}\pi^{-s/2} \int_{\pi n^2 z}^\infty e^{-x} x^{s/2-1}\,dx$$

$$= \pi^{-s/2} \sum_{n=1}^\infty n^{-s}\,\Gamma(s/2, \pi n^2 z).$$

Thus far we have shown that

$$\zeta(s)\pi^{-s/2}\Gamma(s/2) = \pi^{-s/2}\sum_{n=1}^\infty n^{-s}\,\Gamma(s/2, \pi n^2 z) + I_1.$$

In order to analyse I_1, we put $\theta(u) = 1 + 2\theta_1(u)$, so ~~that~~

$$\theta(u) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 u}.$$

Then

$$I_1 = \int_0^z \tfrac{1}{2}(\theta(u)-1)\, u^{s/2-1}\,du = \tfrac{1}{2}\int_0^z \theta(u)\, u^{s/2-1}\,du - \underbrace{\tfrac{1}{2}\int_0^z u^{s/2-1}\,du}_{\frac{1}{s} z^{s/2}}.$$

But by Theorem 10.2, we have $\theta(u) = u^{-1/2}\theta(1/u)$. Thus

$$\tfrac{1}{2}\int_0^z \theta(u)\,u^{s/2-1}\,du = \tfrac{1}{2}\int_0^z \theta(1/u)\,u^{(s-3)/2}\,du$$

$$= \underbrace{\int_0^z \theta_1(1/u)\,u^{(s-3)/2}\,du}_{\substack{v=1/u \\ \Vert \\ \int_{1/z}^\infty \theta_1(v)\, v^{-(s+1)/2}\,dv}} + \underbrace{\tfrac{1}{2}\int_0^z u^{(s-3)/2}\,du}_{\frac{1}{s-1} z^{(s-1)/2}}.$$

Then

$$I_1 + \tfrac{1}{s} z^{s/2} - \tfrac{1}{s-1} z^{(s-1)/2} = \sum_{n=1}^\infty \int_{1/z}^\infty e^{-\pi n^2 v}\, v^{-(s+1)/2}\,dv$$

$$\overset{v = x/(n^2\pi)}{=} \sum_{n=1}^\infty n^{s-1}\pi^{(s-1)/2}\int_{\pi n^2/z}^\infty e^{-x} x^{-(s+1)/2}\,dx$$

$$= \pi^{(s-1)/2}\sum_{n=1}^\infty n^{s-1}\,\Gamma((1-s)/2, \pi n^2/z).$$

Then we conclude that

$$\zeta(s)\pi^{-s/2}\Gamma(s/2) = \pi^{-s/2}\sum_{n=1}^\infty n^{-s}\,\Gamma(s/2, \pi n^2 z)$$
$$+ \pi^{(s-1)/2}\sum_{n=1}^\infty n^{s-1}\,\Gamma((1-s)/2, \pi n^2/z)$$
$$- \frac{1}{s} z^{s/2} + \frac{1}{s-1} z^{(s-1)/2}. \qquad (13.3)$$

Thus far, we have established this conclusion for $\mathrm{Re}(s) > 1$.

However, when $s \in \mathbb{C}\setminus\{0,1\}$, the last two terms are entire. Also, for each z with $\operatorname{Re}(z) \geq 0$ and $|z| \geq \varepsilon > 0$, one has

$$\Gamma(s,a) = \int_a^\infty e^{-u} u^{s-1}\, du$$

$$= \int_0^\infty e^{-v-a} (a+v)^{s-1}\, dv$$

$$\uparrow$$

$$|a+v|^{s-1} \ll |a|^{\sigma-1}$$ uniformly for $|\sigma| \leq C$, fixed $C>0$

$$\ll |a|^{\sigma-1}$$

$$\Rightarrow \quad n^{-s}\,\Gamma(s/2, \pi n^2 z) \ll_\varepsilon n^{-\sigma}\cdot (n^2)^{\sigma/2 - 1} = n^{-2}.$$

Then $\displaystyle\sum_{n=1}^\infty n^{-s}\,\Gamma(s/2, \pi n^2 z)$ is uniformly convergent for s in any compact set, whence (Weierstrass) entire. Likewise for

$$\sum_{n=1}^\infty n^{s-1}\,\Gamma((1-s)/2, \pi n^2/z)$$

since $\quad n^{s-1}\,\Gamma((1-s)/2, \pi n^2/z) \ll_\varepsilon n^{\sigma-1}\cdot (n^2)^{\frac{1-\sigma}{2}-1} \ll n^{-2}.$

Then the four terms on the right hand side of (13.1) are all entire, except possibly at $s = 0$ and 1.

By the uniqueness of analytic continuation, therefore, the identity (13.1) holds for all $s \in \mathbb{C}$ except at $s=0$ and 1. //

<u>Proof of Theorem 13.1:</u> The expression on the right hand side in Theorem 13.3 is left unchanged if we interchange

$$s \quad \text{with} \quad 1-s$$

and

$$z \quad \text{with} \quad 1/z.$$

Thus the left hand side is also left unchanged by the same operations, and we have

$$\zeta(s)\,\Gamma(s/2)\,\pi^{-s/2} = \zeta(1-s)\,\Gamma\!\left(\tfrac{1-s}{2}\right)\pi^{-(1-s)/2},$$

except possibly when $s \in \{0,1\}$. If we multiply by $s(1-s)$, then the pole of $\zeta(s)$ is cancelled and the identity is seen to hold even when $s \in \{0,1\}$. //

Corollary 13.4. When $s \neq 1$, one has
$$\zeta(s) = \zeta(1-s) \cdot 2^s \pi^{s-1} \Gamma(1-s) \sin(\tfrac{1}{2}\pi s).$$

Proof: We have $\xi(s) = \xi(1-s)$, and thus for $s \neq 1$ one has
$$\tfrac{1}{2}s(s-1)\zeta(s)\Gamma(\tfrac{1}{2}s)\pi^{-s/2} = \tfrac{1}{2}(1-s)(-s)\zeta(1-s)\Gamma(\tfrac{1}{2}(1-s))\pi^{(s-1)/2}$$
$$\Downarrow$$
$$\zeta(s) = \zeta(1-s) \cdot \frac{\Gamma(\frac{1}{2}(1-s))}{\Gamma(\frac{1}{2}s)} \pi^{s-1/2}.$$
This relation requires some careful consideration when $s=0$ and when $\Gamma(\frac{1}{2}(1-s))$ is unbounded, but this can be justified via analyticity. Moreover, one has the classical relations
$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \quad \text{and} \quad \Gamma(s)\Gamma(s+\tfrac{1}{2}) = \sqrt{\pi}\, 2^{1-2s}\Gamma(2s).$$
Thus
$$\frac{1}{\Gamma(\frac{1}{2}s)} = \frac{\sin(\pi s/2)}{\pi}\Gamma(1-\tfrac{1}{2}s)$$
$$\Rightarrow \frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)} = \frac{\sin(\frac{1}{2}\pi s)}{\pi} \cdot \Gamma(\tfrac{1}{2}-\tfrac{1}{2}s)\Gamma(1-\tfrac{1}{2}s) = \frac{\sin(\frac{1}{2}\pi s)}{\pi}\sqrt{\pi}\, 2^s\,\Gamma(1-s).$$
Then we conclude that $\zeta(s) = \zeta(1-s)\, 2^s \pi^{s-1}\Gamma(1-s)\sin(\tfrac{1}{2}\pi s)$. //

Observe ~~that~~ putting $s = -\sigma$, with $\sigma > 0$, we see ~~that~~
$$\zeta(-\sigma) = \zeta(\sigma+1)2^{-\sigma}\pi^{-\sigma-1}\Gamma(\sigma+1)\sin(-\tfrac{1}{2}\pi\sigma).$$
We know already that $\zeta(s) \neq 0$ for $Re(s) > 0$, and thus it follows that $\zeta(-\sigma) = 0$ if and only if $\sin(\frac{1}{2}\pi\sigma) = 0$, which is to say when σ is an even integers.

Corollary 13.5. Other than the trivial zeros $-2, -4, \ldots$, all zeros of $\zeta(s)$ satisfy $0 < Re(s) < 1$.

We also see that

$$\zeta(0) = \lim_{\sigma \to 0} \zeta(1-\sigma)\, 2^{\sigma} \pi^{\sigma - 1} \Gamma(1-\sigma) \sin(\tfrac{1}{2}\pi\sigma)$$

$$= \pi^{-1} \cdot \tfrac{1}{2}\pi \Gamma(1) \lim_{\sigma \to 0} -\frac{1}{\sigma} \cdot \sigma = -\frac{1}{2}.$$

It is of course conjectured that all non-trivial zeros ρ of $\zeta(s)$ satisfy $\mathrm{Re}(\rho) = 1/2$. We have shown that $\zeta(\rho) = 0$ implies that $\rho \leqslant 1 - c/\log(|t|+4)$ for a suitable $c > 0$. It follows from Corollary 13.4 that if ρ is non-trivial, then in fact

$$\rho \geqslant c/\log(|t|+4).$$

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§14. A zero-free region for Dirichlet L-functions.

We have shown (Corollary 8.4 – Dirichlet's theorem) that when $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a,q) = 1$, then there are infinitely many prime numbers p with $p \equiv a \pmod q$. Further,

$$\sum_{\substack{p \leq x \\ p \equiv a \ (\mathrm{mod}\ q)}} \frac{\log p}{p} = \frac{1}{\phi(q)} \log x + O_q(1).$$

This suggests an equidistribution of primes amongst the residue classes $a \pmod q$ with $(a,q) = 1$, so that we expect that

$$\pi(x;q,a) := \sum_{\substack{p \leq x \\ p \equiv a \ (\mathrm{mod}\ q)}} 1$$

should satisfy

$$\pi(x;q,a) \sim \frac{1}{\phi(q)} \cdot \frac{x}{\log x} \quad \text{as } x \to \infty.$$

It is our objective now to prove this asymptotic formula in as much uniformity as we are able, adopting a strategy that parallels that employed in our proof of the Prime Number Theorem

$$\pi(x) = li(x) + O(x \exp(-c\sqrt{\log x})) \qquad (\text{some } c>0).$$

We begin by deriving a zero-free region. This entails some preliminary examination of the behaviour of the L-functions $L(s,\chi)$, defined for $\sigma > 1$ via the relation

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

in which χ is a Dirichlet character modulo q.

We first seek an analogue of Lemma 10.5 with $L(s,\chi)$ in place of $\zeta(s)$. This entails bounding $L(s,\chi)$ (cf. Lemma 10.4).

<u>Lemma 14.1.</u> Let χ be a non-principal character modulo q, and suppose that $\delta > 0$ is fixed. Then one has

$$L(s,\chi) \ll \left(1 + (q\tau)^{1-\sigma}\right) \min\left\{ \frac{1}{|\sigma-1|}, \log(q\tau) \right\},$$

uniformly for $\delta \leq \sigma \leq 2$.

<u>Proof:</u> Write

$$S(u,\chi) = \sum_{1 \leq n \leq u} \chi(n).$$

Then by Riemann-Stieltjes integration, one finds that

$$L(s,\chi) = \sum_{n\leq x} \chi(n) n^{-s} + \int_{x^-}^{\infty} u^{-s}\, d\, S(u,\chi)$$

$$= \sum_{n\leq x} \chi(n) n^{-s} + \Big[S(u,\chi) u^{-s} \Big]_{x^-}^{\infty} - \int_x^{\infty} S(u,\chi)\, d\, u^{-s}$$

$$= \sum_{n\leq x} \chi(n) n^{-s} - S(x^-,\chi)\, x^{-s} + s\int_x^{\infty} u^{-s-1} S(u,\chi)\, du.$$

Observe that by the orthogonality of non-principal characters, one has

$$\sum_{n=m+1}^{m+q} \chi(n) = 0 \quad \text{for all } m, \text{ whence}$$

$$|S(u,\chi)| \leq q.$$

Then if we take $x = q\tau$, we find ~~that~~ the final ~~two terms contribute~~

$$\ll q \cdot (q\tau)^{-\sigma} + q\tau \left| \int_x^\infty u^{-\sigma-1}\, du \right| \ll (q\tau)^{1-\sigma}.$$

In the proof of Lemma 10·4 we showed ~~that~~

$$\sum_{n \leq x} |n^{-s}| \ll (1 + x^{1-\sigma}) \min\left\{\frac{1}{|1-\sigma|}, \log x\right\},$$

whence

$$\sum_{n \leq x} |\chi(n) n^{-s}| \ll (1 + x^{1-\sigma}) \min\left\{\frac{1}{|1-\sigma|}, \log x\right\}$$

$$\ll (1 + (q\tau)^{1-\sigma}) \min\left\{\frac{1}{|\sigma-1|}, \log(q\tau)\right\}.$$

Hence

$$L(s,\chi) \ll (1 + (q\tau)^{1-\sigma}) \min\left\{\frac{1}{|\sigma-1|}, \log(q\tau)\right\}.$$ //

Now we come to the analogue of Lemma 10·5. Write $E_0(\chi) = \begin{cases} 1, & \text{when } \chi = \chi_0, \\ 0, & \text{when } \chi \neq \chi_0. \end{cases}$

<u>Lemma 14·2.</u> Suppose ~~that~~ χ is a character modulo q. Then whenever $5/6 \leq \sigma \leq 2$, one has

$$-\frac{L'}{L}(s,\chi) = \frac{E_0(\chi)}{s-1} - \sum_\rho \frac{1}{s-\rho} + O(\log(q\tau)),$$

where the sum is over all zeros ρ of $L(s,\chi)$ with $|\rho - (\tfrac{3}{2}+it)| \leq 5/6$.

<u>Proof.</u> Suppose first that $\chi \neq \chi_0$, so ~~that~~ $E_0(\chi) = 0$. In ~~this situation~~ we apply Lemma 10·3 with $f(z) = L(z + (\tfrac{3}{2}+it), \chi)$, $R = 5/6$ and $r = 2/3$. We have from Lemma 14·1 ~~that~~ when $|z| \leq 1$, ~~then~~

$$f(z) \ll (1 + (q\tau)^{1/2}) \log(q\tau) \ll q\tau \qquad (\operatorname{Re}(z) \leq 1/2),$$

and when $\operatorname{Re}(z) > 1/2$ we have

$$L(s,\chi) \ll \sum_{n=1}^\infty n^{-2} \ll 1,$$

so the bound is trivial. Thus we can take $M = O(q\tau)$ in Lemma 10·3.

Moreover, one has

$$|f(0)| = |L(\tfrac{3}{2}+it,\chi)| = \prod_p \left(1-\chi(p)p^{-3/2-it}\right)^{-1}$$

$$\geqslant \prod_p \left(1+p^{-3/2}\right)^{-1} \gg 1.$$

Then

$$-\frac{L'}{L}(s,\chi) = -\sum_\rho \frac{1}{s-\rho} + \underbrace{O\left(\log\left(\frac{M}{|f(0)|}\right)\right)}_{\ll \log(q\tau)},$$

where the summation is over ~~zeros~~ ρ of $L(s,\chi)$ with $|\rho-(\tfrac{3}{2}+it)| \leq 5/6$. □

Suppose next ~~that~~ $\chi=\chi_0$. Then we have

$$L(s,\chi_0) = \zeta(s)\prod_{p|q}(1-p^{-s}) \qquad \text{for } \sigma>1.$$

But the right hand side is analytic for $s\in\mathbb{C}$ except for a simple pole at $s=1$. The function $L(s,\chi_0)$ is ~~therefore~~ given by ~~the~~ right hand side for all $s\in\mathbb{C}\setminus\{1\}$ by uniqueness of ~~analytic~~ ~~continuation~~. The zeros of $L(s,\chi_0)$ are ~~therefore~~ given by the zeros of $\zeta(s)$ ~~together~~ with the ~~zeros~~ of $1-p^{-s}$ (for $p|q$). The latter are given by

$$s = \frac{2\pi i}{\log p}k \qquad (k\in\mathbb{Z}).$$

All such zeros ~~satisfy~~

$$\left|\frac{2\pi i}{\log p}k - \left(\tfrac{3}{2}+it\right)\right| \geqslant \tfrac{3}{2},$$

so that they are excluded from the sum over ρ in the ~~statement~~ of the lemma. Thus

$$\frac{L'}{L}(s,\chi_0) = \frac{\frac{d}{ds}\left(\zeta(s)\prod_{p|q}(1-p^{-s})\right)}{\zeta(s)\prod_{p|q}(1-p^{-s})}$$

$$= \frac{\zeta'}{\zeta}(s) + \sum_{p|q}\frac{\log p}{p^s-1}.$$

Here, when $\sigma \geqslant 5/6$, we have

$$\sum_{p|q} \frac{\log p}{p^s - 1} \ll \sum_{p|q} \frac{\log p}{p^\sigma} \ll \omega(q) \ll \log q .$$

Thus, making use of Lemma 10·5 (Problem Sheet 4, Qn 3(ii)) we deduce that

$$-\frac{L'}{L}(s,\chi_0) = \frac{1}{s-1} - \sum_\rho \frac{1}{s-\rho} + O(\log \tau) + O(\log q)$$

$$= \frac{E_0(\chi_0)}{s-1} - \sum_\rho \frac{1}{s-\rho} + O(\log(q\tau)). \quad \square //$$

We also require an analogue of Lemma 10·6.

<u>Lemma 14·3.</u> When $\sigma > 1$, one has

$$\mathrm{Re}\left(-3\frac{L'}{L}(\sigma,\chi_0) - 4\frac{L'}{L}(\sigma+it,\chi) - \frac{L'}{L}(\sigma+2it,\chi^2)\right) \geqslant 0.$$

<u>Proof.</u> The left hand side here is

$$\mathrm{Re}\left(\sum_{\substack{n=1\\(n,q)=1}}^\infty \frac{\Lambda(n)}{n^\sigma}\left(3 + 4\chi(n)n^{-it} + \chi(n)^2 n^{-2it}\right)\right) . \qquad (14·1)$$

When $(n,q)=1$, one has

$$\chi(n)n^{-it} = \exp\left(-it\log n + 2\pi i \xi_n\right),$$

for a suitable real number ξ_n. Thus, for some $\theta_n \in \mathbb{R}$, we have $\chi(n)n^{-it} = e^{i\theta_n}$, whence (14·1) may be rewritten as

$$\sum_{\substack{n=1\\(n,q)=1}}^\infty \frac{\Lambda(n)}{n^\sigma}\big(\underbrace{3 + 4\cos\theta_n + \cos 2\theta_n}_{=\,2(1+\cos\theta_n)^2 \,\geqslant\, 0}\big).$$

The non-negativity of the left hand side therefore follows at once //

<u>Theorem 14·4.</u> There is an absolute constant $c>0$ with the following property:

(i) If χ is <u>not</u> a quadratic character modulo q, then $L(s,\chi)$ has no zeros in the region

$$\sigma > 1 - \frac{c}{\log(q\tau)}.$$

(ii) If χ is a quadratic character modulo q, then $L(s,\chi)$ has at most one zero in the region

$$\sigma > 1 - \frac{c}{\log(q\tau)}.$$

If such a zero β exists, then it is necessarily real and satisfies $\beta < 1$.

<u>Proof.</u> When $\chi = \chi_0$, we have

$$L(s,\chi_0) = \zeta(s) \prod_{p\mid q} (1-p^{-s}),$$

so that the zeros of $L(s,\chi_0)$ are those of $\zeta(s)$ together with zeros of the factors $1-p^{-s}$ $(p \mid q)$. There are no zeros of the former with

$$\sigma > 1 - \frac{c_1}{\log \tau},$$

for a suitable $c_1 > 0$. As for the latter factors, these have zeros precisely where

$$\exp(s \log p) = 1 \iff s = \frac{2k\pi i}{\log p} \quad (k \in \mathbb{Z}),$$

and again, none of these zeros satisfy $\sigma > 1 - c/\log(q\tau)$. (they have real part 0).

Henceforth, we may suppose that χ is non-principal. Here we observe first that in view of the Euler product

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \qquad (\sigma > 1),$$

we have $\operatorname{Re}\left(L(s,\chi)\right) \geqslant \prod_p (1+p^{-\sigma})^{-1} > 0 \qquad (\sigma > 1).$

Thus $L(s,\chi) \neq 0$ for $\sigma > 1$. We therefore seek to show that no zeros exist with real part small and to the left of the 1-line. By way of deriving a contradiction, suppose that $L(s,\chi)$ has a zero $\rho_0 = \beta_0 + i\gamma_0$ with $\gamma_0 \in \mathbb{R}$, and $\frac{12}{13} \leqslant \beta_0 \leqslant 1$. We divide into cases:

(i) <u>Suppose that χ is complex.</u> Thus $\chi(n)$ is <u>not</u> real-valued.

We seek to apply Lemma 14.3, this asserting that

$$\operatorname{Re}\left(-3\frac{L'}{L}(\sigma,\chi_0) - 4\frac{L'}{L}(\sigma+it,\chi) - \frac{L'}{L}(\sigma+2it,\chi^2)\right) \geqslant 0,$$

and apply the argument underlying the proof of Theorem 10.7. Thus, we consider a small positive number δ and note that when $s = \sigma + it$ and $\sigma = 1+\delta$, then for all zeros ρ of $L(s,\chi)$ we have $\operatorname{Re}(s-\rho) > 0$ and $\operatorname{Re}\left(\frac{1}{s-\rho}\right) > 0$. Hence, by Lemma 14.2, for a suitable positive number c_1, we have

$$-\operatorname{Re}\left(\frac{L'}{L}(1+\delta,\chi_0)\right) = \operatorname{Re}\Bigg(\frac{1}{\delta} - \sum_{\substack{\rho \\ |\rho-(\frac{3}{2}+it)| \leqslant 5/6}} \frac{1}{1+\delta-\rho}\Bigg) + O(\log q)$$

$$\leqslant \frac{1}{\delta} + c_1 \log q,$$

$$-\operatorname{Re}\left(\frac{L'}{L}(1+\delta+i\gamma_0,\chi)\right) = \operatorname{Re}\Bigg(-\sum_{\substack{\rho \\ |\rho-(\frac{3}{2}+it)| \leqslant 5/6}} \frac{1}{1+\delta+i\gamma_0-\rho}\Bigg) + O\left(\log(q(|\gamma_0|+4))\right)$$

$$\leqslant \underset{\substack{\uparrow \\ \text{from } \rho = \beta_0 + i\gamma_0}}{\frac{-1}{1+\delta-\beta_0}} + c_1 \log(q(|\gamma_0|+4)),$$

and

$$-\operatorname{Re}\left(\frac{L'}{L}(1+\delta+2i\gamma_0,\chi^2)\right) \leqslant \operatorname{Re}\Bigg(-\sum_{\substack{\rho' \\ |\rho'-(\frac{3}{2}+it)| \leqslant 5/6}} \frac{1}{1+\delta+2i\gamma_0-\rho'}\Bigg) + O\left(\log(q(2|\gamma_0|+4))\right)$$

2 (14.2)

$$\leq c_1 \log(q(|\gamma_0|+4)).$$

In the last expression, we sum over zeros ρ' of $L(s,\chi^2)$ with $|\rho' - (\frac{3}{2}+it)| \leq 5/6$. Notice that $\chi^2 \neq \chi_0$, so our application of Lemma 14·2 avoids a pole from the term $\frac{E_0(\chi^2)}{s-1}$ $(=0!)$.

By taking the evident linear combination of these terms, we see that from Lemma 14·3,

$$0 \leq \operatorname{Re}\left(-3\frac{L'}{L}(\sigma,\chi_0) - 4\frac{L'}{L}(\sigma+it,\chi) - \frac{L'}{L}(\sigma+2it,\chi^2)\right)$$

$$\leq \frac{3}{\delta} - \frac{4}{1+\delta-\beta_0} + 8c_1\log(q(|\gamma_0|+4)).$$

It is apparent that $\beta_0 < 1$, for otherwise, if one were to have $\beta_0 = 1$, we obtain

$$-\frac{1}{\delta} + 8c_1 \log(q(|\gamma_0|+4)) \geq 0,$$

which yields a contradiction on taking δ sufficiently small. We may therefore take $\delta = 6(1-\beta_0)$, whence

$$\frac{3}{6(1-\beta_0)} - \frac{4}{7(1-\beta_0)} + 8c_1\log(q(|\gamma_0|+4)) \geq 0.$$

Equivalently,

$$1-\beta_0 \geq \frac{1}{14\cdot 8c_1 \log(q|\gamma_0|+4)}.$$

Then $L(s,\chi)$ has no zeros in the region $\sigma > 1 - c/\log(q\tau)$, provided that $c < 1/112c_1$. □

The situation in which χ is real and non-principal, which is to say, quadratic, is more complicated, and requires a further subdivision into cases.

(ii) Suppose that χ is quadratic and $|\gamma_0| \geq 6(1-\beta_0)$.

 In this situation with $|\gamma_0|$ "large", a very similar argument applies. Note that we have proved earlier that $L(1,\chi) \neq 0$ (in our proof of Dirichlet's theorem). If one were to have $\gamma_0 = 0$, then our initial hypothesis implies that $1-\beta_0 \leq \frac{1}{6}|\gamma_0| = 0$, whence $\beta_0 \geq 1$, and this yields $\beta_0 = 1$. #. Then we have $\gamma_0 \neq 0$.

We now have $\chi^2 = \chi_0$, so (14·2) must be replaced by the bound

$$-\operatorname{Re}\left(\frac{L'}{L}(1+\delta+2i\gamma_0, \underset{\chi_0}{\underset{\parallel}{\chi^2}})\right) \leq \operatorname{Re}\left(\frac{1}{\delta+2i\gamma_0}\right) + c_1 \log(q(|\gamma_0|+4))$$

$$= \frac{\delta}{\delta^2+4\gamma_0^2} + c_1 \log(q(|\gamma_0|+4)).$$

By Lemma 14·3, we now obtain

$$0 \leq \frac{3}{\delta} - \frac{4}{1+\delta-\beta_0} + \frac{\delta}{\delta^2+4\gamma_0^2} + 8c_1 \log(q(|\gamma_0|+4)).$$

Once again, if one were to have $\beta_0 = 1$, then we obtain a contradiction on taking $\delta \to 0+$. Thus we may suppose that $\beta_0 < 1$. Since $|\gamma_0| \geq 6(1-\beta_0)$, it follows that on taking $\delta = 6(1-\beta_0)$, we obtain the bound

$$0 \leq \frac{1}{1-\beta_0}\left(\frac{3}{6} - \frac{4}{7} + \frac{6}{6^2+4\cdot 6^2}\right) + 8c_1 \log(q(|\gamma_0|+4)),$$

whence $$\frac{1}{1-\beta_0}\left(-\frac{1}{2} + \frac{4}{7} - \frac{1}{30}\right) \leq 8c_1 \log(q(|\gamma_0|+4)),$$

which is to say

$$1-\beta_0 \geq \frac{1}{\frac{105}{4}\cdot 8c_1 \log(q(|\gamma_0|+4))} = \frac{1}{210 c_1 \log(q(|\gamma_0|+4))}.$$

Then $L(s,\chi)$ has no zeros in the region $\sigma > 1 - c/\log(q\tau)$, provided that $c < 1/210c_1$, subject to the condition $|\gamma_0| \geq 6(1-\beta_0)$. □

(iii) Suppose that χ is quadratic and $0 < |\gamma_0| \leq 6(1-\beta_0)$.

We now consider the situation in which there is a non-real zero $\rho_0 = \beta_0 + i\gamma_0$ in the putative zero-free region, with γ_0 "small".

In this situation we are able to derive a second such zero by conjugation, for by the Schwarz reflection principle (noting that $L(s,\chi)$ is real when s is real), we have $L(\beta_0 - i\gamma_0, \chi) = \overline{L(\beta_0 + i\gamma_0, \chi)} = 0$. Thus, both $\beta_0 + i\gamma_0$ and $\beta_0 - i\gamma_0$ are zeros of $L(s,\chi)$. We therefore deduce from Lemma 14.2 that when $1 < \sigma \leq 2$, one has

$$\mathrm{Re}\left(-\frac{L'}{L}(\sigma,\chi)\right) \leq -\mathrm{Re}\left(\frac{1}{\sigma-\rho_0} + \frac{1}{\sigma-\overline{\rho}_0}\right) + c_1 \log(4q),$$

for a suitable positive constant $c_1 > 0$. Hence

$$\mathrm{Re}\left(-\frac{L'}{L}(\sigma,\chi)\right) \leq -\frac{2(\sigma-\beta_0)}{(\sigma-\beta_0)^2+\gamma_0^2} + c_1\log(4q). \qquad (14\cdot3)$$

In this case we can take a simpler path to showing that this putative zero lies outside $\mathrm{Re}(s) > 1 - c/\log(q\tau)$ by allowing the close pair of zeros here to conspire with each other. Note that when $\sigma > 1$, one has

$$-\frac{L'}{L}(\sigma,\chi_0) - \frac{L'}{L}(\sigma,\chi) = \sum_{\substack{n=1\\(n,q)=1}}^{\infty} \frac{\Lambda(n)}{n^\sigma}(1+\chi(n)) \geq 0.$$

Put $\sigma = 1 + \overbrace{13(1-\beta_0)}^{\delta}$ and recall that

$$-\mathrm{Re}\left(\frac{L'}{L}(1+\delta,\chi_0)\right) \leq \frac{1}{\delta} + c_1\log q.$$

Then we see that

$$0 \leq \frac{1}{\delta} - \frac{2(\sigma-\beta_0)}{(\sigma-\beta_0)^2+\gamma_0^2} + 2c_1\log(4q),$$

Whence

$$\frac{1}{1-\beta_0}\Bigg(\frac{1}{13}-\underbrace{\frac{2(13+1)}{(13+1)^2+6^2}}_{\frac{7}{7^2+3^2}}\Bigg)+2c_1\log(4q)\geq 0.$$

Thus

$$1-\beta_0 \geq \frac{1}{\frac{754}{33}\cdot 2c_1\log(4q)}.$$

Then $L(s,\chi)$ has no ~~zeros~~ in ~~the~~ region $\sigma > 1 - c/\log(4q)$, provided that $c < 1/46c_1$, subject to the condition $0<|\gamma_0|\leq 6(1-\beta_0)$. $\square$

(iv) Suppose ~~that~~ χ is quadratic and $\gamma_0=0$: so ρ_0 is real.

If β_0 is a real zero of $L(s,\chi)$, ~~then~~ $\beta_0<1$ as a consequence of our proof ~~that~~ $L(1,\chi)\neq 0$. Suppose ~~that~~ β_0 and β_1 are two such ~~zeros~~, say with $\beta_0\leq\beta_1<1$. Then it follows from Lemma 14.2 ~~that~~
when $1<\sigma\leq 2$, one has

$$\operatorname{Re}\left(-\frac{L'}{L}(\sigma,\chi)\right)\leq -\frac{1}{\sigma-\beta_0}-\frac{1}{\sigma-\beta_1}+c_1\log(4q)$$

$$\leq -\frac{2}{\sigma-\beta_0}+c_1\log(4q).$$

Thus, proceeding as in case (iii), we find ~~that~~

$$-\frac{L'}{L}(\sigma,\chi_0)-\frac{L'}{L}(\sigma,\chi)\geq 0,$$

whence with $\sigma=1+\delta$,

$$0\leq\frac{1}{\delta}-\frac{2}{\sigma-\beta_0}+2c_1\log(4q).$$

Put $\delta=2(1-\beta_0)$. Then we obtain

$$0\leq\frac{1}{1-\beta_0}\left(\frac{1}{2}-\frac{2}{3}\right)+2c_1\log(4q),$$

and ~~thus~~

$$1-\beta_0\geq\frac{1}{12c_1\log(4q)},$$

Then $L(s,\chi)$ has at most one zero β_1 in the region $\sigma > 1 - c/\log(4q)$ provided ~~that~~ $c < 1/12c_1$. $\square$

This completes the proof of the theorem. //

We now turn to an analogue of Theorem 10.8 for $L(s,\chi)$.

<u>Theorem 14.5.</u> Let χ be a non-principal character modulo q. Let $c>0$ be an absolute constant with the property that $L(s,\chi) \neq 0$ for $\sigma \geqslant 1 - c/\log(q\tau)$ where, in the case that χ is quadratic, we permit a possible exceptional zero $\beta \in \mathbb{R}$ with $\beta < 1$. Then whenever $\sigma \geqslant 1 - c/(2\log(q\tau))$, one has the following:

(i) when $L(s,\chi)$ has no exceptional zero, and when β_1 is an exceptional zero of $L(s,\chi)$ ~~satisfying~~ $|s-\beta_1| \geqslant 1/\log q$, one has

$$\frac{L'}{L}(s,\chi) \ll \log(q\tau), \quad \frac{1}{L(s,\chi)} \ll \log(q\tau) \quad \text{and} \quad |\log L(s,\chi)| \leqslant \log\log(q\tau) + O(1);$$

(ii) when β_1 is an exceptional zero of $L(s,\chi)$ and $|s-\beta_1| \leqslant 1/\log q$, then

$$\frac{L'}{L}(s,\chi) = \frac{1}{s-\beta_1} + O(\log q) \qquad (s \neq \beta_1),$$

$$|\arg L(s,\chi)| \leqslant \log\log q + O(1) \qquad (s \neq \beta_1),$$

and

$$|s-\beta_1| \ll |L(s,\chi)| \ll (s-\beta_1)(\log q)^2.$$

<u>Proof.</u> We begin by establishing the bound $\frac{L'}{L}(s,\chi) \ll \log(q\tau)$ in (i). This is simple when $\sigma \geqslant 1 + 1/\log(q\tau)$, since then we have

$$\left|\frac{L'}{L}(s,\chi)\right| \leqslant \sum_{n=1}^{\infty} \Lambda(n) n^{-\sigma} = -\frac{\zeta'}{\zeta}(\sigma)_{1+1/\log(q\tau)} \ll \frac{1}{\sigma-1} \ll \log(q\tau).$$

In order to handle general $s = \sigma + it$ with $\sigma \geqslant 1 - c/2\log(q\tau)$, we seek to shift from $s_1 = 1 + 1/\log(q\tau) + it$ to s. We have

$$\frac{L'}{L}(s_1, \chi) \ll \log(q\tau),$$

whence Lemma 14.2 yields

$$+\sum_\rho \frac{1}{s_1-\rho} + O(\log(q\tau)) = \frac{L'}{L}(s_1,\chi) \ll \log(q\tau),$$

where the summation is over ~~zeros~~ ρ of $L(s,\chi)$ with $|\rho - (\frac{3}{2}+it)| \le 5/6$. Hence

$$\sum_\rho \frac{1}{s_1-\rho} \ll \log(q\tau).$$

This implies, in turn, that

$$\sum_\rho \frac{1}{s-\rho} = \sum_\rho \left(\frac{1}{s-\rho} - \frac{1}{s_1-\rho}\right) + O(\log(q\tau)),$$

where for each ρ one has

$$\frac{1}{s-\rho} - \frac{1}{s_1-\rho} = \frac{1+1/\log(q\tau) - \sigma}{(s-\rho)(s_1-\rho)} \ll \frac{1}{|s_1-\rho|^2\log(q\tau)} \ll \mathrm{Re}\left(\frac{1}{s_1-\rho}\right).$$

[Note: $\mathrm{Re}(s_1-\rho) = \mathrm{Re}\left((1+1/\log(q\tau)+it) - (1-c/(2\log(q\tau)))\right) \gg 1/\log(q\tau)$ for $\rho \ne \beta_1$, and $\mathrm{Re}(s_1-\beta_1) \gg 1/\log q$]. Thus we deduce that

$$\sum_\rho \frac{1}{s-\rho} \ll \mathrm{Re}\left(\sum_\rho \frac{1}{s_1-\rho}\right) + O(\log(q\tau)),$$

so from Lemma 14.2 we have

$$\begin{aligned}\frac{L'}{L}(s,\chi) &= \sum_\rho \frac{1}{s-\rho} + O(\log(q\tau)) \ll \mathrm{Re}\left(\sum_\rho \frac{1}{s_1-\rho}\right) + O(\log(q\tau)) \\ &= \mathrm{Re}\left(\frac{L'}{L}(s_1,\chi)\right) + O(\log(q\tau)) \ll \log(q\tau). \quad \square\end{aligned}$$

Next we consider the bound $|\log L(s,\chi)| \le \log\log(q\tau) + O(1)$ of (i). Again, when $\sigma \ge 1 + 1/\log(q\tau)$, we have

$$|\log L(s,\chi)| \le \sum_{n=2}^\infty \frac{\Lambda(n)}{\log n} n^{-\sigma} = \log\zeta(\sigma) \le \log\left(\frac{\sigma}{\sigma-1}\right)$$

$$\le \log\log(q\tau).$$

We again put $s_1 = 1 + 1/\log(q\tau) + it$, and note that when $1+1/\log(q\tau) \ge \sigma \ge 1 - c/2\log(q\tau)$, we have

$$\log L(s,\chi) - \log L(s_1,\chi) = \int_{s_1}^s \frac{L'}{L}(z,\chi)\,dz \ll |s_1 - s|\log(q\tau) \ll 1.$$

Hence $\quad |\log L(s,\chi)| \leq \log\log(q\tau) + O(1)$,

and this in turn implies ~~that~~

$$\log\left(\frac{1}{|L(s,\chi)|}\right) = -\mathrm{Re}\,(\log L(s,\chi)) \leq \log\log(q\tau) + O(1)$$

$$\Downarrow$$

$$\frac{1}{L(s,\chi)} \ll \log(q\tau) . \quad \square$$

It remains to handle part (ii) in which we may assume ~~that~~ $L(s,\chi)$ has ~~an~~ exceptional ~~zero~~ β_1 with $|s-\beta_1| \leq 1/\log q$. We may then suppose ~~that~~

$$1 - c/2\log(4q) \leq \sigma \leq 1 + 1/\log q \quad ,$$

so by Lemma 14.2,

$$\frac{L'}{L}(s,\chi) = \frac{1}{s-\beta_1} + \sum_\rho{}' \frac{1}{s-\rho} + O(\log q),$$

in which the summation over ~~zeros~~ ρ of $L(s,\chi)$ is over $|\rho - (\tfrac{3}{2}+it)| \leq \tfrac{5}{6}$ excluding β_1. We have, just as ~~before~~,

$$\sum_\rho{}' \frac{1}{s-\rho} \ll \mathrm{Re}\left(\sum_\rho{}' \frac{1}{s_1-\rho}\right) + O(\log(q\tau)),$$

whence

$$\frac{L'}{L}(s,\chi) = \sum_\rho \frac{1}{s-\rho} + O(\log(q\tau)) = \frac{1}{s-\beta_1} + \mathrm{Re}\left(\frac{L'}{L}(s_1,\chi)\right) + O(\log(q\tau))$$

$$= \frac{1}{s-\beta_1} + O(\log q). \quad \square$$

In a similar fashion to our treatment in (i), we find ~~that~~

$$\log L(s,\chi) - \log L(s_1,\chi) = \int_{s_1}^{s} \frac{1}{z-\beta_1}\,dz + O(|s_1-s|(\log q))$$

$$= \log\left(\frac{s-\beta_1}{s_1-\beta_1}\right) + O(1),$$

whence

$$\left|\log L(s,\chi) - \log\left(\frac{s-\beta_1}{s_1-\beta_1}\right)\right| \leq |\log L(s_1,\chi)| + O(1) \leq \log\log q + O(1).$$

Thus, since $\arg(s-\beta_1) \ll 1$, $\arg(s_1-\beta_1) \ll 1$ and $\log|s_1-\beta_1| = -\log\log q + O(1)$, we deduce that $|\arg L(s,\chi)| \leq \log\log q + O(1)$ and

$$\log\left|\frac{L(s,\chi)}{s-\beta_1}\right| \leq \left|\log|s_1-\beta_1|\right| + \log\log q + O(1) = 2\log\log q + O(1)$$

and

$$\log\left|\frac{L(s,\chi)}{s-\beta_1}\right| \geq -\log|s_1-\beta_1| - \log\log q + O(1) = O(1),$$

Whence

$$1 \ll \left|\frac{L(s,\chi)}{s-\beta_1}\right| \ll (\log q)^2. \qquad \square \quad //$$

§15. The Prime Number Theorem in arithmetic progressions, I.

In this section we explore the consequences of the weak information so far available to us concerning exceptional zeros. We note

$$\pi(x;q,a) = \sum_{\substack{p\leq x \\ p\equiv a \pmod q}} 1, \qquad \theta(x;q,a) = \sum_{\substack{p\leq x \\ p\equiv a \pmod q}} \log p, \qquad \psi(x;q,a) = \sum_{\substack{n\leq x \\ n\equiv a \pmod q}} \Lambda(n),$$

$$\pi(x,\chi) = \sum_{p\leq x} \chi(p), \qquad \theta(x,\chi) = \sum_{p\leq x} \chi(p)\log p, \qquad \psi(x,\chi) = \sum_{n\leq x} \chi(n)\Lambda(n).$$

We are able to relate the former quantities to the latter via orthogonality. Thus, for example,

$$\psi(x;q,a) = \frac{1}{\phi(q)} \sum_{\chi\in X(q)} \bar{\chi}(a)\,\psi(x,\chi).$$

We are able to provide an asymptotic formula for $\psi(x,\chi)$ (and hence also for $\psi(x;q,a)$) by imitating the argument of the proof of Theorem 11.1 using the zero-free region made available in §14.

<u>Theorem 15.1.</u> There is a constant $c>0$ having the following property. Suppose that $q \leq \exp(2c\sqrt{\log x})$. Then, when $L(s,\chi)$ has no

exceptional zero, one has

$$\psi(x,\chi) = E_0(\chi)\,x + O\left(x \exp\left(-c\sqrt{\log x}\right)\right).$$

Meanwhile, when $L(s,\chi)$ has an exceptional zero β_1, one instead has

$$\psi(x,\chi) = -\frac{x^{\beta_1}}{\beta_1} + O\left(x \exp\left(-c\sqrt{\log x}\right)\right).$$

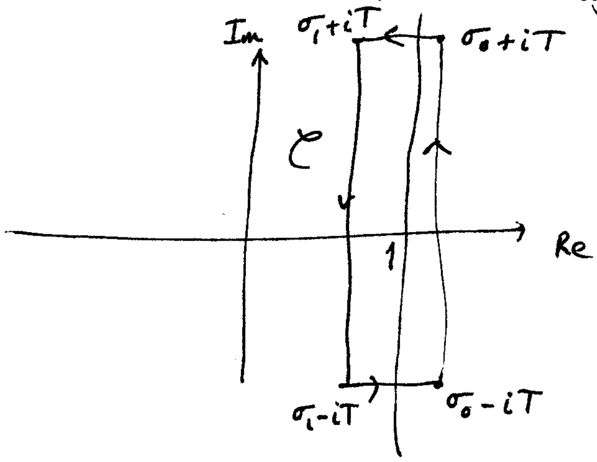
Proof. By our explicit version of Perron's formula,

$$\psi(x,\chi) = -\frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{L'}{L}(s,\chi)\frac{x^s}{s}\,ds + R(T),$$

where $\sigma_0 > 1$ and

$$R(T) \ll \sum_{x/2 < n < 2x} \Lambda(n) \min\left\{1, \frac{x}{T|x-n|}\right\} + \frac{(4x)^{\sigma_0}}{T}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{\sigma_0}}.$$

As in the proof of Theorem 11.1, we suppose that $2 \le T \le x$ and put $\sigma_0 = 1 + 1/\log x$, giving $R(T) \ll \frac{x}{T}(\log x)^2$. The definition of the appropriate contour $\mathcal{C}$ used to complete the path of integration must now take account of the possible existence of the exceptional zero β_1.



(i) Suppose that there is no exceptional zero.

We take $\sigma_1 = 1 - \dfrac{c_1}{5\log(qT)}$, where $c_1 > 0$ is any constant having the property that $L(s,\chi)$ has no zeros in the region $\sigma > 1 - \dfrac{c_1}{\log(qT)}$ (as we are permitted to assume is the case via Theorem 14.4).

When $\chi \neq \chi_0$, it follows that $-\frac{L'}{L}(s,\chi)\frac{x^s}{s}$ is analytic inside and on $\mathcal{C}$, and when $\chi = \chi_0$ then the same is true except for a simple pole at $s=1$ with residue x. We recall in this context that

$$-\frac{L'}{L}(s,\chi_0) = \underbrace{-\frac{\zeta'}{\zeta}(s)}_{\frac{1}{s-1}+O(1)} - \underbrace{\sum_{p\mid q} \frac{\log p}{p^s-1}}_{\text{poles at } s=\frac{2\pi i k}{\log p}\ (k\in\mathbb{Z})}.$$

Hence
$$-\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{L'}{L}(s,\chi)\,\frac{x^s}{s}\,ds = E_0(\chi)\cdot x.$$

We follow the argument of the proof of Theorem 11.1 to handle the left vertical and horizontal segments of $\mathcal{C}$ using Theorem 14.5(i). Thus

$$\psi(x,\chi) - E_0(\chi)\,x \ll x(\log x)^2\left(\frac{1}{T} + \exp\left(-c_1\frac{\log x}{5\log(qT)}\right)\right).$$

By taking $T = \exp(2c\sqrt{\log x})$ and assuming that $q \le \exp(2c\sqrt{\log x})$, with $c = \sqrt{c_1/40}$, we find that

$$\psi(x,\chi) - E_0(\chi)\,x \ll x(\log x)^2\left(\exp(-2c\sqrt{\log x}) + \exp\left(-c_1\frac{\log x}{5\cdot 4c\sqrt{\log x}}\right)\right)$$
$$\ll x\exp(-c\sqrt{\log x}). \qquad \square$$

(ii) <u>Suppose that there is an exceptional zero β_1 and $\beta_1 \ge 1 - c_1/4\log(qT)$.</u> Here, we emphasise that c_1 has the same meaning as in case (i). In this case we may take $\sigma_1 = 1 - c_1/3\log(qT)$. In this case there is a pole of $-\frac{L'}{L}(s,\chi)\frac{x^s}{s}$ at $s=\beta_1$ inside $\mathcal{C}$ having residue $-\frac{x^{\beta_1}}{\beta_1}$. [Recall that $-\frac{L'}{L}(s,\chi) = \frac{-1}{s-\beta_1} + O(1)$ when $s \to \beta_1$].

The remaining parts of the estimation from case (i) remain applicable, and so

$$\psi(x,\chi) = -\frac{x^{\beta_1}}{\beta_1} + O\left(x(\log x)^2 \left(\frac{1}{T} + \exp\left(-c_1 \frac{\log x}{5\log(qT)}\right)\right)\right)$$

$$= -\frac{x^{\beta_1}}{\beta_1} + O\left(x \exp\left(-c\sqrt{\log x}\right)\right). \quad \square.$$

(iii) Suppose that there is an exceptional zero β_1, and $\beta_1 < 1 - \frac{c_1}{4\log(qT)}$.

In this case we again take $\sigma_1 = 1 - c_1/5\log(qT)$ and proceed as in case (i). In this case the exceptional zero does not lie within or on $\mathcal{C}$, and we again obtain

$$\psi(x,\chi) - E_0(\chi)\, x \ll x(\log x)^2 \left(\frac{1}{T} + \exp\left(-c_1 \frac{\log x}{5\log(qT)}\right)\right).$$

But in this case we have

$$-\frac{x^{\beta_1}}{\beta_1} \ll \exp\left(\beta_1 \log x\right) < x\exp\left(-\frac{c_1 \cdot \log x}{4\log(qT)}\right) \ll x\exp\left(-c\sqrt{\log x}\right),$$

whence

$$\psi(x,\chi) = -\frac{x^{\beta_1}}{\beta_1} + O\left(x\exp\left(-c\sqrt{\log x}\right)\right). \quad \square$$

We can obtain an asymptotic formula for $\psi(x;q,a)$ from here by gluing together the corresponding estimates for $\psi(x,\chi)$ for $\chi \in X(q)$. But how many exceptional zeros might there be as we consider the $\varphi(q)$ L-functions $L(s,\chi)$ $(\chi \in X(q))$? This is a matter ~~that~~ we now address.

We again make use of a positivity lemma.

Lemma 15.2. (Landau) Suppose that χ_1 and χ_2 are quadratic characters. Then whenever $\sigma > 1$, one has

$$-\frac{\zeta'}{\zeta}(\sigma) - \frac{L'}{L}(\sigma, \chi_1) - \frac{L'}{L}(\sigma, \chi_2) - \frac{L'}{L}(\sigma, \chi_1\chi_2) \geq 0.$$

Proof. The left hand side here may be written as the Dirichlet series

$$\sum_{n=1}^{\infty} a_n \Lambda(n) n^{-\sigma},$$

where

$$\begin{aligned} a_n &= 1 + \chi_1(n) + \chi_2(n) + \chi_1\chi_2(n) \\ &= (1+\chi_1(n))(1+\chi_2(n)) \geq 0. \end{aligned}$$

The desired conclusion therefore follows via absolute convergence. //

Now we use putative ~~zeros~~ of $L(s,\chi_1)$ and $L(s,\chi_2)$ against eachother.

Theorem 15.3. (Landau) There is a constant $c > 0$ with the following property. Suppose that χ_i is a quadratic character modulo q_i for $i = 1$ and 2, and further suppose that $\chi_1\chi_2$ is non-principal. Then $L(s,\chi_1)L(s,\chi_2)$ has at most one real zero β with $1 - \frac{c}{\log(q_1q_2)} < \beta < 1$.

Proof. It follows from Theorem 14.4 that, should any zero β of the type described exist, then there can be at most one associated with $L(s,\chi_1)$, and one associated with $L(s,\chi_2)$.

[Note that $1 - \frac{c}{\log(q_1q_2)} > \max\left\{1 - \frac{c}{\log q_1}, 1 - \frac{c}{\log q_2}\right\}$.]

Suppose ~~that~~ both $L(s,\chi_1)$ and $L(s,\chi_2)$ have such a zero, say β_1 and β_2 respectively. There is no loss of generality in supposing ~~that~~ $5/6 \le \beta_i \le 1$ $(i=1,2)$. We may also suppose ~~that~~ $\chi_1\chi_2$ is non-principal modulo q_1q_2. Then we deduce from Lemma 14·2 ~~that~~ when $0<\delta\le 1$, and for a suitable constant $c_1>0$,

$$-\frac{\zeta'}{\zeta}(1+\delta) \le \frac{1}{\delta} + c_1 \log 4,$$

$$-\frac{L'}{L}(1+\delta,\chi_i) \le -\frac{1}{1+\delta-\beta_i} + c_1 \log q_i \qquad (i=1,2),$$

$$-\frac{L'}{L}(1+\delta,\chi_1\chi_2) \le c_1 \log(q_1q_2).$$

Then by substitution into the conclusion of Lemma 15·2, we infer ~~that~~

$$0 \le \frac{1}{\delta} - \frac{1}{1+\delta-\beta_1} - \frac{1}{1+\delta-\beta_2} + 3c_1 \log(q_1q_2).$$

If we suppose ~~that~~ $\beta_1 \le \beta_2$ and put $\delta = 2(1-\beta_1)$, it follows that

$$0 \le \frac{1}{\delta} - \frac{2}{1+\delta-\beta_1} + 3c_1 \log(q_1q_2)$$

$$= \frac{1}{1-\beta_1}\left(\frac{1}{2} - \frac{2}{3}\right) + 3c_1 \log(q_1q_2),$$

whence

$$1-\beta_1 \ge \frac{1}{18c_1 \log(q_1q_2)}.$$

We ~~therefore~~ conclude ~~that~~ whenever $c < 1/18c_1$, ~~then~~ $L(s,\chi_1)L(s,\chi_2)$ has at most one real zero β with $1-\beta > \dfrac{c}{\log(q_1q_2)}$. //

<u>Corollary 15·4</u> (Landau) There is a constant $c>0$ with the property that $\prod_{\chi \in X(q)} L(s,\chi)$ has at most one zero in the region $\sigma > 1 - \frac{c}{\log(q\tau)}$. If such a zero exists, then it is necessarily real and the associated character is quadratic.

<u>Proof.</u> Suppose, by way of contradiction, that there are two zeros or more in the region $\sigma > 1 - \frac{c}{\log(q\tau)}$. As a consequence of Theorem 14·4, when c is sufficiently small, this is impossible unless both zeros are real and the associated characters χ_1 and χ_2 are quadratic. But then Theorem 15·3 applies to give a contradiction. //

Notice that there may be several quadratic characters modulo q, so the fact that there is at most one exceptional zero is significant. By making use of this conclusion in combination with Theorem 15·1, we derive our first version of the Prime Number Theorem in arithmetic progressions.

<u>Theorem 15·5</u> (Page). There is a constant $c>0$ having the following property. Suppose that $q \leq \exp(2c\sqrt{\log x})$ and $a \in \mathbb{Z}$ satisfy $(a,q)=1$.

(i) When there is no exceptional character modulo q, one has
$$\psi(x;q,a) = \frac{x}{\varphi(q)} + O\left(x \exp\left(-c\sqrt{\log x}\right)\right);$$

(ii) When there is an exceptional character χ_1 modulo q, and β_1 is the associated exceptional zero of $L(s,\chi_1)$, one has instead

$$\psi(x;q,a) = \frac{x}{\varphi(q)} - \frac{\chi_1(a)\, x^{\beta_1}}{\varphi(q)\,\beta_1} + O\left(x\exp\left(-c\sqrt{\log x}\right)\right).$$

Proof. We have

$$\psi(x;q,a) = \frac{1}{\varphi(q)} \sum_{\chi\in X(q)} \bar{\chi}(a)\,\psi(x,\chi),$$

and so the desired estimates are immediate from the relations derived in Theorem 15.1, namely

$$\psi(x,\chi) = E_0(\chi)\,x + O\left(x\exp\left(-c\sqrt{\log x}\right)\right) \qquad (\chi\neq\chi_1)$$

and

$$\psi(x,\chi) = -\frac{x^{\beta_1}}{\beta_1} + O\left(x\exp\left(-c\sqrt{\log x}\right)\right) \qquad (\chi=\chi_1). \quad //$$

Notice that the claimed asymptotic formulae would be worse than trivial when $q > \exp\left(2c\sqrt{\log x}\right)$, for then

$$\psi(x;q,a) \leq \left(\frac{x}{q}+1\right)\log x \ll x\exp\left(-2c\sqrt{\log x}\right)\cdot \log x,$$

which is an estimate smaller than the respective error terms in the statements of the theorem.

In order to say more concerning the exceptional zeros β_1, we must say more concerning characters χ modulo q, and in particular we focus on so-called <u>primitive</u> characters.

In the final part of the course, we aim to prove an important theorem of Siegel that provides an upper bound on β_1 showing that it cannot lie too close to 1.

<u>Theorem 15.6.</u> (Siegel). For each $\varepsilon > 0$, there is a positive constant $C_1(\varepsilon)$ such that, when χ is a quadratic character modulo q, then

$$L(1, \chi) > C_1(\varepsilon) q^{-\varepsilon}.$$

<u>Corollary 15.7.</u> For any $\varepsilon > 0$, there is a positive constant $C_2(\varepsilon)$ such that, when χ is a quadratic character modulo q and β is a real zero of $L(s, \chi)$, then $\beta < 1 - C_2(\varepsilon) q^{-\varepsilon}$.

<u>Proof.</u> We may suppose that there is a constant $c > 0$ having the property that there is at most one character χ modulo q (quadratic) having a real zero β ~~satisfying~~ $1 - c/\log q < \beta < 1$. If such a zero exists, then it follows from Theorem 14.5 (ii) that

$$L(1, \chi) \ll (1 - \beta)(\log q)^2.$$

Then since $L(1, \chi) > C_1(\varepsilon/2) q^{-\varepsilon/2}$, it follows that there is a constant C_3 with the property that

$$\begin{aligned} (1 - \beta)(\log q)^2 &> C_3^{-1} \cdot C_1(\varepsilon/2) q^{-\varepsilon/2} \\ \wedge \\ C_3 (1 - \beta) q^{\varepsilon/2} \end{aligned} \quad \Rightarrow \quad 1 - \beta > C_2(\varepsilon) q^{-\varepsilon},$$

with $C_2(\varepsilon) = C_1(\varepsilon/2) / C_3^2$. //

This yields an alternative to Theorem 15.5 in which the role of the exceptional zero is suppressed.

<u>Theorem 15.8.</u> There is a constant $c > 0$ having the following property. Let $A > 0$ be fixed. Then whenever $x \geq x_0(A)$ and $q \leq (\log x)^A$, one has

$$\psi(x, \chi) = E_0(\chi)\, x + O(x \exp(-c\sqrt{\log x})).$$

<u>Proof.</u> This is a consequence of Theorem 15.1. All ~~that~~ is required is to handle the possible exceptional ~~zero~~ β_1 of $L(s,\chi)$ for a quadratic character χ. In such a situation, we have

$$\psi(x, \chi) = -\frac{x^{\beta_1}}{\beta_1} + O(x \exp(-c_1\sqrt{\log x})), \qquad (c_1 > 0)$$

where (from Corollary 15.7), we may suppose ~~that~~ for a suitable $c_2(\varepsilon) > 0$, we have $\beta_1 < 1 - c_2(\varepsilon) q^{-\varepsilon}$.

But

$$x^{\beta_1} = x \exp(-(1-\beta_1)\log x) \leq x \exp(-c_2(\varepsilon) q^{-\varepsilon} \log x)$$

$$\ll x \exp(-c_2(\varepsilon)(\log x)^{1-A\varepsilon}).$$

If we take $\varepsilon = 1/(3A)$, then we deduce ~~that~~

$$\frac{x^{\beta_1}}{\beta_1} \ll 2x \exp(-c_2(\varepsilon)(\log x)^{2/3}) < x \exp(-c_1\sqrt{\log x}).$$

Thus we conclude ~~that~~

$$\psi(x, \chi) = O(x \exp(-c_1\sqrt{\log x}))$$

even in the case that an exceptional zero exists. //

<u>Corollary 15.9</u> (Siegel – Walfisz theorem) There is a constant $c > 0$ with the following property. Let $A > 0$ be fixed. Then whenever $x \geq x_0(A)$ and $q \leq (\log x)^A$, and $(a, q) = 1$, one has

$$\psi(x; q, a) = \frac{x}{\varphi(q)} + O_A(x \exp(-c\sqrt{\log x})).$$

§ 16. Primitive Dirichlet characters and Gauss sums.

<u>Definition 16.1.</u> Suppose ~~that~~ $d \mid q$, ~~that~~ χ^* is a Dirichlet character modulo d, and ~~that~~ χ is a Dirichlet character modulo q. Then we say ~~that~~ <u>χ^* induces χ</u> when

$$\chi(n) = \begin{cases} \chi^*(n), & \text{when } (n,q)=1, \\ 0, & \text{when } (n,q)>1. \end{cases}$$

Thus, if χ_0 is the principal character modulo q, we have $\chi = \chi^* \chi_0$. Notice ~~that~~ $\chi(n) \neq \chi^*(n)$ whenever $(n,q)>1$ and $(n,d)=1$. Moreover, it is evident ~~that~~

$$L(s,\chi) = L(s,\chi^*) \prod_{p \mid q} \left(1 - \frac{\chi^*(p)}{p^s}\right).$$

How do we identify this phenomenon — a character $\chi \pmod q$ induced by (and morally equal to) a character $\chi^* \pmod d$ with $d \mid q$ & $d<q$?

<u>Definition 16.2.</u> Let χ be a Dirichlet character modulo q. We say ~~that~~ d is a <u>quasiperiod</u> of χ if $\chi(m) = \chi(n)$ whenever $m \equiv n \pmod d$ and $(mn, q) = 1$.

Note that the least quasiperiod of χ is a divisor of q. For if d is a quasiperiod of χ, one can show that $e = (d,q) \mid q$ is also a quasiperiod. Indeed, whenever $m \equiv n \pmod e$ and $(mn,q)=1$, one has for some $l \in \mathbb{Z}$ ~~that~~

$$m - n = le = l(d,q) = l(dx+qy), \quad \text{some } x,y \in \mathbb{Z}.$$

Hence $\chi(m) = \chi(m - ly\cdot q) = \chi(n + lx\cdot d) = \chi(n)$, so ~~that~~ e is indeed a quasi-period of χ.

A similar argument shows ~~that~~ when d_1, d_2 are both quasiperiods of χ, then so ~~too~~ is (d_1,d_2), and hence the least quasiperiod of χ divides all other quasiperiods, and hence also q.

<u>Definition 16.3.</u> (i) The least quasi-period d of a Dirichlet character χ is called the <u>conductor</u> of χ.

(ii) A character χ modulo q is called <u>primitive</u> when q is the least quasi-period of χ.

<u>Theorem 9.4.</u> Let χ be a Dirichlet character modulo q, and let d be the conductor of χ. Then $d \mid q$, and there is a unique character χ^* modulo d that induces χ.

<u>Proof.</u> We have seen already that the least quasiperiod of χ, namely d, is a divisor of q. We begin by showing that χ is induced by a character χ^* modulo d. Suppose that $n \in \mathbb{Z}$ satisfies $(n, d) = 1$. Let $d_0 = (d^\infty, q) := \lim_{m\to\infty} (d^m, q)$, and put $r = q/d_0$, so that $(r, d) = 1$. Then there exists $k \in \mathbb{Z}$ with $(n + kd, r) = 1$, whence $1 = (n + kd, rd_0) = (n + kd, q)$. For any such integer k, we define

$$\chi^*(n) := \chi(n + kd).$$

Notice that since d is a quasiperiod of χ, then the choice of k does not impact the definition of $\chi^*(n)$. This defines $\chi^*(n)$ when $(n, d) = 1$, and when $(n, d) > 1$ we put $\chi^*(n) = 0$.

It is evident that χ^* has period d. Also, when $(n, m) = 1$ one has for suitable integers u and v that

$$\chi^*(n)\chi^*(m) = \chi(n + ud)\chi(m + vd) = \chi(nm + (um + vn + uvd)d)$$
$$= \chi^*(nm).$$ (Notice issues on coprimality with q here).

 Thus χ^* is multiplicative with period d, and satisfies $\chi^*(n) = 0$ for $(n,d)>1$, and thus is a Dirichlet character modulo d. Indeed, if χ_0 is the principal character modulo q, one has
$$\chi(n) = \chi_0(n)\chi^*(n),$$
and χ^* induces χ.

If χ^* were to have a quasiperiod smaller than d, then so too would χ, contradicting minimality. Then χ^* is primitive. Finally, to see that χ^* is unique we observe that if some other character χ^+ modulo d also induces χ, then for any $n \in \mathbb{Z}$ with $(n,d)=1$, we could find $k \in \mathbb{Z}$ with
$$\chi^*(n) = \chi^*(n+kd) = \chi(n+kd) = \chi^+(n+kd) = \chi^+(n),$$
so that $\chi^* = \chi^+$. //

We can decompose primitive characters according to the prime power decomposition of the conductor q.

Lemma 16.5. Suppose that $q_1, q_2 \in \mathbb{N}$ and $(q_1,q_2)=1$. Suppose also that χ_1 and χ_2 are characters modulo q_1 and q_2, respectively, and put $\chi(n) = \chi_1(n)\chi_2(n)$. Then the character χ is primitive modulo q_1q_2 if and only if both χ_1 and χ_2 are primitive.

Proof. ($\Rightarrow$) Put $q = q_1q_2$, and suppose that χ is primitive modulo q. Also, write $d_i = \mathrm{cond}(\chi_i)$ $(i=1,2)$. We seek to show that $q = d_1d_2$, whence $d_1 = q_1$ and $d_2 = q_2$, so that both χ_1 and χ_2 are primitive. To see this, observe that whenever $m \equiv n \pmod{d_1d_2}$ and $(mn,q)=1$, one has $\chi_i(m) = \chi_i(n)$ $(i=1,2)$, whence $\chi(m) = \chi(n)$. Thus d_1d_2 is a quasi-period of χ, so by primitivity of χ, it follows that $q = d_1d_2$. □

(⇐) Suppose that χ_i is primitive modulo q_i $(i=1,2)$, and let $d = \operatorname{cond}(\chi)$. We aim to show that $d=q$, whence χ is primitive. Put $d_i = (d, q_i)$ $(i=1,2)$. Then whenever $m \equiv n \pmod{d_1}$ and $(mn, q_1)=1$, we may choose integers m', n' with $m' \equiv m \pmod{q_1}$ and $n' \equiv n \pmod{q_1}$ with $m' \equiv n' \equiv 1 \pmod{q_2}$. We then have $m' \equiv n' \pmod{d_1 d_2}$, that is $m \equiv n \pmod{d}$, and also $(m'n', q)=1$. Hence

$$\begin{array}{ccc} \chi(m') & = & \chi(n') \\ \| & & \| \\ \chi_1(m')\chi_2(m') & & \chi_1(n')\chi_2(n') \\ \| & & \| \\ \chi_1(m)\,\chi_2(1) & & \chi_1(n)\chi_2(1) \end{array},$$

whence $\chi_1(m) = \chi_1(n)$, so that d_1 is a quasiperiod of χ_1. But χ_1 is primitive, so $d_1 = q_1$. A similar argument shows that $d_2 = q_2$, and consequently $\underbrace{d_1 d_2}_{=q_1q_2} = d$, which gives $d = q$. So χ is primitive. □

It follows from Lemma 16.5 that primitive characters modulo q are determined by primitive characters modulo p^h for $p^h \| q$.

<u>p odd</u>: $$\chi(n) = e\left(\frac{k \operatorname{ind}_g n}{\varphi(p^h)}\right)$$ for g primitive modulo p^h.

<u>$h=1$</u>: χ primitive if and only if $\chi \neq \chi_0$, so $(p-1) \nmid k$.

<u>$h>1$</u>: χ primitive if and only if $p \nmid k$.

<u>p even</u>: $(p=2)$.

<u>$h=1$</u>: $\chi = \chi_0$ not primitive

<u>$h=2$</u>: χ either principal (not primitive) or not, in which case $\chi(4k+1)=1$, $\chi(4k-1) = -1$, and this is primitive.

<u>$h \geq 3$</u>: $$\chi(n) = e\left(\frac{ju}{2} + \frac{kv}{2^{h-2}}\right)$$ with $n \equiv (-1)^u 5^v \pmod{2^h}$.

χ primitive if and only if k odd.

Observation: (i) If χ (mod q) is primitive, then whenever $d \mid q$ and $d < q$, there exists $c \equiv 1 \pmod d$ with $(c,q)=1$ such that $\chi(c) \neq 1$.

Proof. There exist $m \equiv n \pmod d$ with $\chi(m) \neq \chi(n)$ and $\chi(mn) \neq 0$. Then with any $c \in \mathbb{Z}$ s.t. $(c,q)=1$ and $cm \equiv n \pmod q$, we have

$$\underset{\displaystyle \chi(c)\chi(m)}{\overset{\shortparallel}{\chi(cm)}} = \chi(n) \quad \Rightarrow \quad \chi(c) = \chi(n)\overline{\chi(m)} \neq 1. \quad \square$$

(ii) If further $d \mid q$ and $d < q$, then for all $a \in \mathbb{Z}$ we have

$$\sum_{\substack{n=1 \\ n \equiv a \pmod d}}^{q} \chi(n) = 0.$$

Proof. We have an integer c satisfying (i), and then

$$\sum_{\substack{n=1 \\ n \equiv a \pmod d}}^{q} \chi(n) = \sum_{k=1}^{q/d} \chi(a+kd) = \sum_{k=1}^{q/d} \chi(\underbrace{c(a+kd)}_{\substack{\text{permutes this arithmetic} \\ \text{progression } a \pmod d \\ \text{working modulo } q}})$$

$$= \chi(c) \sum_{\substack{n=1 \\ n \equiv a \pmod d}}^{q} \chi(n)$$

$$\Rightarrow \sum_{\substack{n=1 \\ n \equiv a \pmod d}}^{q} \chi(n) = 0 \quad \text{since } \chi(c) \neq 1. \qquad \square.$$

§17. Proof of Siegel's Lemma.

We have seen a proof of the fact that $L(1,\chi) \neq 0$ using Landau's lemma in our proof of Dirichlet's theorem. We now work harder so as to obtain more quantitative information. For this purpose, we require a lemma of Estermann.

<u>Lemma #.1</u> (Estermann). Suppose ~~that~~ $f(s)$ is analytic for $|s-2| \leq 3/2$, and ~~that~~ $|f(s)| \leq M$ for s ~~within this~~ disc. Put $F(s) = \zeta(s) f(s)$, suppose ~~that~~ $F(s)$ can be written as a Dirichlet series for $\sigma > 1$, and define the coefficients $r(n)$ by means of the notation
$$F(s) = \sum_{n=1}^{\infty} r(n) n^{-s} \qquad (\sigma > 1).$$
Suppose in addition ~~that~~ $r(1) = 1$ and $r(n) \geq 0$ $(n \in \mathbb{N})$. Then whenever there exists $\sigma \in [19/20, 1)$ such ~~that~~ $f(\sigma) \geq 0$, one has
$$f(1) \geq \tfrac{1}{4}(1-\sigma) M^{-3(1-\sigma)}.$$

<u>Proof.</u> We expand $F(s)$ as a power series in $(2-s)$. Thus, we have
$$F(s) = \sum_{k=0}^{\infty} b_k (2-s)^k,$$
for $|s-2| < 1$, say. By Cauchy's formula, we see ~~that~~
$$b_k = \frac{(-1)^k}{k!} F^{(k)}(2) = \frac{1}{k!} \sum_{n=1}^{\infty} r(n) n^{-2} (\log n)^k.$$
Then our hypotheses on $r(n)$ ensure that $b_k \geq 0$ for all k, and moreover
$$b_0 = \sum_{n=1}^{\infty} r(n) n^{-2} \geq 1.$$
On noting ~~that~~
$$\frac{1}{s-1} = \frac{1}{1-(2-s)} = \sum_{k=0}^{\infty} (2-s)^k \qquad (|s-2|<1),$$
we deduce ~~that~~
$$F(s) - \frac{f(1)}{s-1} = \sum_{k=0}^{\infty} (b_k - f(1))(2-s)^k \qquad (|s-2|<1). \qquad (\#.1)$$

The lhs of this ~~last~~ relation is analytic for $|s-2| \leq 3/2$, so the series on the rhs converges in this disc (which goes beyond the disc $|s-2| < 1$!). Our goal now is to estimate the terms appearing here so as to obtain a lower bound on $f(1)$. We begin by bound the coefficients $b_k - f(1)$ in modulus by bounding the lhs of (16·1). Observe ~~that~~ when $|s-2| = 3/2$, we have $|s-1| \geq \frac{1}{2}$, $|s| \leq 7/2$ and $\sigma \geq \frac{1}{2}$. Also,

$$|\zeta(s)| = \left| 1 + \frac{1}{s-1} + s\int_1^\infty \frac{[u]-u}{u^{s+1}}\,du \right|$$

$$\leq 1 + \frac{1}{|s-1|} + \frac{|s|}{\sigma} \leq 1 + 2 + \frac{7/2}{1/2} = 10,$$

and $|f(1)| \leq M$, so ~~that~~

$$\left| \frac{f(1)}{s-1} \right| \leq \frac{M}{1/2} = 2M.$$

We ~~therefore~~ deduce ~~that~~

$$\left| F(s) - \frac{f(1)}{s-1} \right| \leq |\zeta(s) f(s)| + \left| \frac{f(1)}{s-1} \right| \leq 10M + 2M = 12M,$$

whence from the Cauchy coefficient inequalities, we see ~~that~~

$$|b_k - f(1)| \leq \left(\sup_{|s-2| = 3/2} \left| F(s) - \frac{f(1)}{s-1} \right| \right) \Big/ \left(\frac{3}{2} \right)^k$$

$$\leq 12M \left(\frac{2}{3} \right)^k.$$

We now have a means of bounding the rhs of (16·1). It transpires that it is most convenient to truncate the infinite series at a cut-off K to be chosen later. Thus, whenever $\frac{1}{2} < \sigma \leq 2$, we deduce ~~that~~

$$\zeta(\sigma) f(\sigma) - \frac{f(1)}{\sigma - 1} \geq \sum_{k=0}^{K} (b_k - f(1))(2-\sigma)^k - 12M \sum_{k>K} \left(\tfrac{2}{3}(2-\sigma) \right)^k. \tag{16·2}$$

Since $b_0 \geq 1$, $b_k \geq 0$ $(k \geq 1)$ and $\frac{2}{3}(2-\sigma) \leq \frac{7}{10}$ for $\frac{19}{20} \leq \sigma < 1$, we see that the rhs here is

$$\geqslant 1 - f(1)\ \frac{1-(2-\sigma)^{K+1}}{1-(2-\sigma)} - \left(\tfrac{7}{10}\right)^{K+1} \frac{12M}{1-\frac{7}{10}},$$

Whence from (16.2),

$$1 \leq -f(1)\cdot \frac{1-(2-\sigma)^{K+1}}{1-\sigma} + \zeta(\sigma) f(\sigma) - \frac{f(1)}{\sigma-1} + 40M\left(\tfrac{7}{10}\right)^{K+1}.$$

$$= f(1)\ \frac{(2-\sigma)^{K+1}}{1-\sigma} + \zeta(\sigma) f(\sigma) + 40M\left(\tfrac{7}{10}\right)^{K+1}.$$

At ~~this~~ point we choose K in such a way ~~that the~~ final term is not troublesome. If we take

$$K = \left[\frac{\log(80M)}{\log(\frac{10}{7})}\right],$$

then the final term is at most $1/2$. Also, since $\frac{19}{20} \leq \sigma < 1$, we have $\zeta(\sigma) > 0$, and further $f(\sigma) \geqslant 0$ by hypothesis.

Thus
$$f(1)\ \frac{(2-\sigma)^{K+1}}{1-\sigma} \geqslant 1 - \frac{1}{2},$$

whence
$$\begin{aligned} f(1) &\geqslant \tfrac{1}{2}(1-\sigma)(2-\sigma)^{-K-1} \\ &\geqslant \tfrac{1}{2}(1-\sigma)(2-\sigma)^{-([\log(80M)/\log(10/7)]+1)} \\ &\geqslant \tfrac{10}{21}(1-\sigma)(80M)^{-\log(2-\sigma)/\log(10/7)} \\ &\geqslant \tfrac{1}{2}(80)^{-\log(21/20)/\log(10/7)}\ .\ M^{-\log(2-\sigma)/\log(10/7)} \\ &\geqslant \tfrac{1}{4} M^{-\log(2-\sigma)/\log(10/7)}. \end{aligned}$$

But $\log(10/7) \geqslant 1/3$ and $\log(2-\sigma) \leqslant 1-\sigma$, so
$f(1) \geqslant \frac{1}{4} M^{-3(1-\sigma)}$, as required. //

This lemma of Estermann paves the way for Siegel's theorem.

<u>Theorem 15.6</u> (Siegel). For each $\varepsilon > 0$, there is a positive constant $c_1(\varepsilon)$ such that, when χ is a quadratic character modulo q, then

$$L(1,\chi) > c_1(\varepsilon)\, q^{-\varepsilon}.$$

<u>Proof.</u> Consider any positive number ε with $\varepsilon \le 1/5$. We observe first that there is no loss of generality in restricting to primitive characters χ. For if χ is not primitive modulo q, then χ is induced by some primitive character χ^* modulo d with $d \mid q$, say $q = dr$. But then, assuming the conclusion of the theorem for χ^*, we see that

$$L(1,\chi) = L(1,\chi^*) \prod_{p \mid r}\left(1 - \frac{\chi^*(p)}{p}\right) \geq L(1,\chi^*)\,\frac{\varphi(r)}{r}$$

$$\geq c_1(\varepsilon)\, d^{-\varepsilon} \cdot r^{-\varepsilon} = c_1(\varepsilon)\, q^{-\varepsilon}. \;\square$$

Suppose then that χ is primitive. We divide into two cases.

(i) Suppose that there exists <u>no</u> quadratic character χ_1 such that $L(s,\chi_1)$ has a real zero $\beta_1 \in [1-\varepsilon/4, 1)$.

In this case we take $f(s) = L(s,\chi)$ and $\sigma = 1 - \varepsilon/4$. By Lemma 14.1, we have

$$\sup_{|s-2| \le 3/2} |L(s,\chi)| \ll \sup_{\substack{1/2 \le \sigma \le 7/2 \\ |t| \le 3/2}} \left(1 + (q\tau)^{1-\sigma}\right) \min\left\{\frac{1}{|\sigma - 1|}, \log(q\tau)\right\}$$

$$\ll q^{1/2},$$

$\boxed{\zeta(s)L(s,\chi) = \sum_{n=1}^{\infty} r(n) n^{-s} \text{ with } r(n) = \sum_{d \mid n} \chi(d) \geq 0 \text{ just as in proof that } L(1,\chi) > 0.}$

so we may apply Lemma 17.1 with $M \ll q^{1/2}$. Thus we deduce that

$$|L(1,\chi)| \geq \tfrac{1}{4}(\varepsilon/4)\, M^{-3\varepsilon/4} \gg \varepsilon\, q^{-3\varepsilon/8}.$$

The conclusion $L(1,\chi) > c_1(\varepsilon) q^{-\varepsilon}$ then follows for a suitable $c_1(\varepsilon)>0$. $\square$

(ii) Suppose that there exists a quadratic character χ_1 such that $L(s,\chi_1)$ has a real zero $\beta_1 \geq 1 - \varepsilon/4$.

In this case, since $L(1,\chi_1)>0$, there is a constant $c_2(\varepsilon)>0$ such that $L(1,\chi_1) \geqslant c_2(\varepsilon) q_1^{-\varepsilon}$ (just take $c_2(\varepsilon) = q_1^{\varepsilon} L(1,\chi_1)$).

We now investigate the implications of this assumption for other primitive Dirichlet characters χ modulo q with $\chi \neq \chi_1$. We apply Lemma 17.1 with

$$f(s) = L(s,\chi) L(s,\chi_1) L(s,\chi_1\chi).$$

We must check in this instance that $f(s)\zeta(s)$ has a Dirichlet series with non-negative coefficients. To verify this property, note that

$$\log(\zeta(s) f(s)) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\log n} (1+\chi(n)+\chi_1(n)+\chi\chi_1(n)) n^{-s}$$

$$= \sum_{n=1}^{\infty} \overbrace{\frac{\Lambda(n)}{\log n} (1+\chi(n))(1+\chi_1(n))}^{c(n)} n^{-s}$$

has non-negative coefficients. Thus, on exponentiation we find that

$$\zeta(s) f(s) = \exp\left(\sum_{n=1}^{\infty} c(n) n^{-s}\right)$$

$$= \sum_{h=0}^{\infty} \left(\sum_{n=1}^{\infty} c(n) n^{-s}\right)^h / h! = \sum_{m=1}^{\infty} g(m) m^{-s}$$

non-negative ~~positive~~ coefficients, since $c(n) \geq 0$.

has non-negative coefficients.

Observe next that

$$\sup_{|s-2| \leq 3/2} |f(s)| \ll q^{1/2} \cdot q_1^{1/2} \cdot (qq_1)^{1/2} \ll qq_1.$$

Then we may apply Lemma 17.1 with $M = c_3 qq_1$, say, and $\sigma = \beta_1$.

Thus $$f(1) \geqslant \tfrac{1}{4}(C_3 qq_1)^{-3(1-\beta_1)} \geqslant \tfrac{1}{4}(C_3 qq_1)^{-3\varepsilon/4}.$$

Moreover, also from Lemma 14.1,
$$f(1) = L(1,\chi)L(1,\chi_1)L(1,\chi\chi_1) \ll L(1,\chi)(\log q_1)(\log(q_1 q)),$$

whence
$$L(1,\chi)(\log(qq_1))^2 \gg (qq_1)^{-3\varepsilon/4},$$

and thus $$L(1,\chi) \gg_{q_1,\varepsilon} q^{-3\varepsilon/4}/\log q \gg q^{-\varepsilon}.$$

We consequently conclude also in this case that $L(1,\chi) \geqslant C_1(\varepsilon)q^{-\varepsilon}$, for a suitable $C_1(\varepsilon)>0$. $\square$ //