§1. Introduction: generating functions.

We are familiar with the idea of generating functions of the shape

\[ A(z) = \sum_{\ell=0}^{\infty} a_\ell z^\ell \]  

(formal power series).

The coefficients \( a_\ell \in \mathbb{C} \) can be extracted from the power series \( A(z) \) via Cauchy's formula

\[ a_n = \frac{1}{2\pi i} \oint \frac{A(z)}{z^{n+1}} \, dz, \]

provided \( A(z) \) converges appropriately and a suitable contour of integration is chosen. Moreover, if \( B(z) = \sum_{m=0}^{\infty} b_m z^m \) is a second such power series, then

\[ A(z)B(z) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} a_\ell b_m z^{\ell+m} = \sum_{n=0}^{\infty} c_n z^n, \]

where

\[ c_n = \sum_{\ell=0}^{n} \sum_{m=0}^{n} a_\ell b_m \quad (n \geq 0). \]

Thus (formal) power series are appropriate vehicles for discussing additive structure. The associated Fourier analysis begins with the Fourier series obtained by setting \( z = e^{2\pi i \alpha} \), which we write as \( e(\alpha) \), to obtain the (formal) Fourier series

\[ \hat{A}(\alpha) = \sum_{\ell=0}^{\infty} a_\ell e(\ell \alpha). \]

What about multiplicative structure? The generating functions
of interest here lie at the core of this course. We now consider the (formal) Dirichlet series

$$\alpha(s) = \sum_{l=1}^{\infty} a_l l^{-s}.$$ 

It transpires that the coefficients $a_l \in \mathbb{C}$ may again be extracted via a relative of Cauchy’s formula. Moreover, if $\beta(s) = \sum_{m=1}^{\infty} b_m m^{-s}$ is a second such Dirichlet series, then

$$\alpha(s) \beta(s) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} a_l b_m (lm)^{-s} = \sum_{n=1}^{\infty} C_n n^{-s},$$

where

$$C_n = \sum_{l=1}^{n} \sum_{m=1}^{n} a_l b_m \quad (n \geq 1).$$

This makes it evident that (formal) Dirichlet series are appropriate vehicles for discussing multiplicative structure. The associated Fourier analysis begins with the series obtained by setting $s = it$ ($t \in \mathbb{R}$) to obtain the (formal) series

$$\tilde{\alpha}(it) = \sum_{l=1}^{\infty} a_l e^{-it \log l}.$$ 

Notice that this series can be written instead as

$$\tilde{\alpha}(it) = \sum_{l=1}^{\infty} a_l e\left(-\frac{it}{2\pi} \log l\right)$$

with frequencies $\log l$ ($l \in \mathbb{N}$). As $l \to \infty$, the gaps between these successive frequencies go to 0.
Example 1.1. A central example of a Dirichlet series is the Riemann zeta function

\[ \zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \quad (1.1) \]

Thus far, the right hand side is merely a formal Dirichlet series. However, when \( \Re(s) > 1 \), we have

\[ \sum_{n=1}^{\infty} |n^{-s}| < \sum_{n=1}^{\infty} n^{-\Re(s)} < \infty. \]

By tradition, in this subject we think of \( s \) as being of the shape

\[ s = \sigma + it, \]

where \( \sigma, t \in \mathbb{R} \). Thus, for \( \sigma > 1 \) we have that the series on the right hand side of (1.1) converges absolutely. The Riemann zeta function \( \zeta(s) \) is defined by the series (1.1) for \( \sigma > 1 \), and by analytic continuation (where possible) for \( \sigma \leq 1 \). This is a matter that will occupy our attention in the next 3 classes.

The multiplicative structure of Dirichlet series has the potential to reveal the properties of multiplicative entities. This idea can be seen already with \( \zeta(s) \). Since the series (1.1) converges absolutely for \( \Re(s) > 1 \), we may rearrange terms to see that
\[
\sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} (p_1^{h_1} \ldots p_r^{h_r})^{-s} = \prod_p \left( 1 + p^{-s} + p^{-2s} + \ldots \right),
\]

(All distinct primes

(Prime factorisation)

Whence

\[
\zeta(s) = \prod_p \left( 1 - p^{-s} \right)^{-1}. \quad (\sigma > 1)
\]

(1.2)

This Euler product representation of \( \zeta(s) \) encodes critical multiplicative information.

A hint of the hidden content of the relation (1.2) is revealed by a proof of the infinitude of primes. One has

\[
\sum_{n \leq x} \frac{1}{n} = \log x + O(1)
\]

\[
\land
\]

\[
\prod_{p \leq x} \left( 1 + \frac{1}{p} + \frac{1}{p^2} + \ldots \right) = \prod_{p \leq x} \left( 1 - \frac{1}{p} \right)^{-1}.
\]

Thus

\[
- \sum_{p \leq x} \log \left( 1 - \frac{1}{p} \right) \geq \log \log x + O \left( \frac{1}{\log x} \right)
\]

\[
\Rightarrow
\]

\[
\sum_{p \leq x} \left( \frac{1}{p} + \frac{1}{2p^2} + \ldots \right) = \sum_{p \leq x} \frac{1}{p} + O(1).
\]

We therefore conclude that

\[
\sum_{p \leq x} \frac{1}{p} \geq \log \log x + O(1) \to \infty
\]
as \( x \to \infty \). In particular, there are infinitely many prime numbers. Using the standard notation

\[
\pi(x) = \sum_{p \leq x} 1,
\]

we see that if \( \pi(x) = n \in \mathbb{N} \), then crudely

\[
\sum_{m \leq n} \frac{1}{m} \geq \sum_{p \leq x} \frac{1}{p} \geq \log \log x + O(1),
\]

whence \( \log n \geq \log \log x + O(1) \). Then \( \pi(x) \gg (\log x)^\theta \) for large \( x \), meaning that there is a positive number \( C \) with the property that

\[
\pi(x) \geq C \log x
\]

for all large \( x \).

We shall work harder with these ideas, and one of our early objectives will be the proof that

\[
\pi(x) \sim \frac{x}{\log x},
\]

and more precisely

\[
\pi(x) = \text{li}(x) + O \left( x \exp(-cN^{1/2}) \right),
\]

where

\[
\text{li}(x) = \int_2^x \frac{dt}{\log t} = x \sum_{k=1}^{K-1} \frac{(k-1)!}{(\log x)^k} + O_K \left( \frac{x}{(\log x)^K} \right),
\]

\((K \in \mathbb{N})\)

with \( c \) a suitable positive constant.

Next: Riemann–Stieltjes integration and analytic properties of Dirichlet series...

Recall that we were discussing Dirichlet series of the shape
\[ \alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}. \]  
(2.1)

Just as it makes sense to consider the radius of convergence of a power series \( \sum_{n=0}^{\infty} b_n z^n \), so it is desirable to understand the region of the complex plane in which the series \( \alpha(s) \) converges.

**Definition 2.1.** (i) The abscissa of convergence of a Dirichlet series \( \alpha(s) \) as in (2.1) is
\[ \sigma_c = \inf \left\{ \sigma \in \mathbb{R} : \sum_{n=1}^{\infty} a_n n^{-s} \text{ converges for all } s \text{ with } \Re(s) > \sigma \right\}; \]

(ii) the line \( \sigma = \sigma_c \) is the line of convergence of \( \alpha(s) \);

(iii) the half-plane of convergence of \( \alpha(s) \) is
\[ H_c = \{ s \in \mathbb{C} : \Re(s) > \sigma_c \}. \]

**Theorem 2.2.** Suppose that \( \alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s} \) converges at the point \( s = s_0 \). Then whenever \( H > 0 \), the series \( \alpha(s) \) is uniformly convergent in the sector
\[ S = \{ s \in \mathbb{C} : \sigma > \sigma_0, \quad |t - t_0| \leq H (\sigma - \sigma_0) \} \]

\[ C \]

\[ \sigma_0 \]

\[ \text{Re}(s) \]

\[ \text{Im}(s) \]
By taking $H$ as large as necessary, it follows that $\alpha(s)$ converges for all $s \in \mathbb{C}$ with $\text{Re}(s) > \sigma_0$.

**Corollary 2.3.** The abscissa of convergence of a Dirichlet series $\alpha(s)$ has the property that $\alpha(s)$ converges for all $s \in \mathbb{C}$ with $\sigma > \sigma_c$, and for no $s \in \mathbb{C}$ with $\sigma < \sigma_c$. Further, when $s_0 \in \mathbb{C}$ satisfies $\sigma_0 > \sigma_c$, then there is a neighbourhood of $s_0$ in which $\alpha(s)$ converges uniformly.

**Proof of Theorem 2.2:** We'll use properties of Riemann–Stieltjes integration, so we take a diversion. We are interested in defining integrals that accommodate differentials of functions with jump discontinuities. This is all fairly intuitive, and agrees with Riemann integration where applicable.

Define $\int_a^b f(x) \, d\, g(x)$ as a limit of Riemann sums $\sum_{n=1}^{N} f(\xi_n) \Delta g(\xi_n)$.

When $a < b$ and $a = x_0 \leq x_1 \leq \ldots \leq x_N = b$ is a partition of $[a, b]$, and for each $n$ we have a real number $\xi_n$ satisfying $x_{n-1} \leq \xi_n \leq x_n$, we put

$$S(x, \xi) = \sum_{n=1}^{N} f(\xi_n) (g(x_n) - g(x_{n-1})).$$

We say that the Riemann–Stieltjes integral $\int_a^b f(x) \, d\, g(x)$ exists with value $I$ if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|S(x, \xi) - I| < \varepsilon,$$

whenever

$$\text{mesh } \{x_n \xi\} = \max_{1 \leq n \leq N} (x_n - x_{n-1}) \leq \delta.$$
One can check as exercises the following properties:

(A) The Riemann-Stieltjes integral \( \int_a^b f(x) \, dg(x) \) exists whenever \( f \) is continuous on \([a, b]\) and \( g \) is of bounded variation on \([a, b]\).

[Thus, \( \text{Var}_{[a,b]}(g) = \sup_{x} \sum_{n=1}^{N} |g(x_n) - g(x_{n-1})| < \infty \).]

(B) If \( \int_a^b f \, dg \) exists, then so too does \( \int_a^b g \, df \), and
\[
\int_a^b g \, df = f(b)g(b) - f(a)g(a) - \int_a^b f \, dg.
\]

(C) If \( g' \) is continuous on \([a, b]\), then
\[
\int_a^b f \, dg = \int_a^b f g' \, dx
\]

(D) Suppose that \( f \) is continuous on \([0, M]\) and \((a_n)_{n=1}^{\infty}\) is a sequence of complex numbers. Then on interval
\[
A(x) = \sum_{1 \leq m \leq x} a_m,
\]

we have
\[
\sum_{m=1}^{M} a_m f(m) = \int_0^M f(x) \, dA(x).
\]

Moreover, if \( f' \) is continuous on \([0, M]\), then
\[
\sum_{m=1}^{M} a_m f(m) = A(M) f(M) - \int_0^M A(x) f'(x) \, dx.
\]

Properties (A), (B), (C) are not completely trivial formal exercises — the readers will profit by convincing themselves of their validity. As for the first of the conclusions in property (D), observe that in the situation described in the definition of
Riemann-Stieltjes integration, with mesh $|x_j - x_{j-1}| < \delta$, we have

$$A(x_n) - A(x_{n-1}) = 0$$

whenever $x_n, x_{n-1}$ both lie within an interval between successive integers, and

$$A(x_n) - A(x_{n-1}) = a_m$$

when $x_{n-1} < m < x_n$ with $m \in \mathbb{Z} \cap (0,M]$. Thus, on noting that the continuity of $f$ implies that

we may suppose that $|f(\frac{x_j}{2}) - f(m)| < \varepsilon^2$ in the latter situation, we find that

$$\left| S(x, \frac{1}{2}) - \sum_{m=1}^{M} a_m f(m) \right| < \varepsilon^2 \sum_{m=1}^{M} |a_m| < \varepsilon,$$

say. The desired conclusion follows. \qed

Back to the proof of Theorem 2.2. We consider the tail of the infinite series defining $\alpha(s)$. Put

$$R(u) = \sum_{n>u} a_n n^{-s_0}.$$ We can apply Riemann-Stieltjes integration to infer what happens at a general argument $s$ in place of $s_0$. Thus,

$$\sum_{n=M+1}^{N} a_n n^{-s} = - \int_{M}^{N} u^{s_0-s} d R(u)$$

[The -ve sign here: notice that as $u$ increases in $R(u)$, we are removing terms]

\begin{align*}
&= - u^{s_0-s} R(u) \bigg|_{M}^{N} + \int_{M}^{N} R(u) d u^{s_0-s} \\
&= M^{s_0-s} R(M) - N^{s_0-s} R(N) + (s_0-s) \int_{M}^{N} R(u) u^{s_0-s-1} d u. \tag{2.2}
\end{align*}
Since \( \alpha(s) \) converges at \( s = s_0 \), we may suppose that \( M \) is chosen with \( |R(u)| \leq \varepsilon \) for all \( u \geq M \). Then, whenever \( \sigma > \sigma_0 \), we deduce that

\[
\left| \sum_{n=M+1}^{N} a_n n^{-s} \right| \leq 2M^{\sigma_0 - \sigma} + |s_0 - s| \int_{M}^{\infty} |R(u)| u^{\sigma_0 - \sigma - 1} \, du
\]

\[
\leq 2 \varepsilon + \varepsilon |s_0 - s| \int_{M}^{\infty} u^{\sigma_0 - \sigma - 1} \, du
\]

\[
\leq \left( 2 + \frac{|s_0 - s|}{\sigma - \sigma_0} \right) \varepsilon.
\]

When \( |t - t_0| \leq H(\sigma - \sigma_0) \), we have

\[
|s - s_0| \leq \sigma - \sigma_0 + |t - t_0| \leq (H+1)(\sigma - \sigma_0),
\]

whence

\[
\left| \sum_{n=M+1}^{N} a_n n^{-s} \right| \leq (H+3) \varepsilon. \quad (\text{so "tail" of } \alpha(s) \text{ is small})
\]

We conclude that \( \alpha(s) \) converges uniformly in \( S \).

It follows from Theorem 2.2 that \( \alpha(s) \) converges in the half-plane \( \Re(s) > \sigma_0 \). The first conclusion of Corollary 2.3 follows at once.

Two easy consequences of Corollary 2.3:

- \( \sum_{n=1}^{\infty} n^{-s} (n \in \mathbb{N}) \) is an analytic function of \( s \), and
  \( e^{-\gamma s} \),

\[
\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}
\]

is locally uniformly convergent, it follows that \( \alpha(s) \) is analytic for \( \sigma > \sigma_c \).
Likewise, the differentiated series
\[ \alpha'(s) = -\sum_{n=1}^{\infty} a_n \frac{(\log n)n^{-s}}{n^s} \]
is locally uniformly convergent for \( \sigma > \sigma_c \).

How do we compute \( \sigma_c \) from the sequence \( (a_n)_{n=1}^{\infty} \)?

**Theorem 2.14.** Let \( \alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s} \) and write \( A(x) = \sum_{n \leq x} a_n \).

(i) When \( \sigma_c > 0 \), one has
\[ \sigma_c = \lim \sup_{x \to \infty} \frac{\log |A(x)|}{\log x} ; \]

(ii) When \( \sigma_c < 0 \), the function \( A(x) \) is bounded;

(iii) When \( \sigma > \max \{ \sigma_c, 0 \} \), one has
\[ \sum_{n=1}^{\infty} a_n n^{-s} = s \int_{1}^{\infty} \frac{A(x)}{x^{s+1}} \, dx . \quad (2.3) \]

**Proof.** Put
\[ \sigma^* = \lim \sup_{x \to \infty} \frac{\log |A(x)|}{\log x} . \]

If \( \sigma_c > \sigma^* \), then \( A(x) \leq x^\sigma \) (with an implicit constant depending on \( \sigma \) and \( \sigma_c \)). Then whenever \( \sigma > \sigma_c \), we see that
\[ \int_{1}^{\infty} \left| \frac{A(x)}{x^{s+1}} \right| \, dx \leq \int_{1}^{\infty} \frac{x^\sigma}{x^{s+1}} \, dx < \infty . \]

Meanwhile, by Riemann–Stieltjes integration, one has
\[ \sum_{n=1}^{N} a_n n^{-s} = \int_{1}^{N} x^{-s} \, dA(x) = A(x) x^{-s} \bigg|_{1}^{N} - \int_{1}^{N} A(x) \, d(x^{-s}) \]
\[ = A(N) N^{-s} + s \int_{1}^{N} A(x) x^{-s-1} \, dx . \]
The second term on the right-hand side here is absolutely convergent.
as $N \to \infty$ whenever $\sigma > \sigma_0$, and likewise the limit satisfies
\[ |A(N)N^{-\sigma}| \leq C N^{\sigma_0 - \sigma} \to 0 \quad \text{as} \quad N \to \infty. \]

Thus, whenever $\sigma > \sigma^*$, one finds that
\[ \sum_{n=1}^{\infty} a_n n^{-s} = \int_1^{\infty} A(x)x^{-s-1}dx. \quad \text{(2.4)} \]

We now divide into cases.

(a) Suppose that $\sigma_c < 0$. Then Corollary 2.3 (with $s = 0$) shows that $A(x)$ converges as $x \to \infty$, whence $\sigma^* \leq 0$. We may then conclude that $A(x)$ is bounded and (2.3) holds for $\sigma > 0$. This proves (ii).

(b) Suppose that $\sigma_c > 0$. Then Corollary 2.3 shows that $A(x)$ diverges when $\sigma < \sigma_c$, whence $\sigma^* \geq \sigma_c$. To show that $\sigma^* \leq \sigma_c$, consider $\sigma_0 > \sigma_c$ and observe that (2.2) delivers the estimate
\[ A(N) = -R(N)N^{\sigma_0} + \sigma_0 \int_0^{N} R(u)u^{-\sigma_0-1}du \quad (N \to \infty), \]
in which $R(u) = \sum_{n>u} a_n n^{-\sigma_0}$. This series is bounded (because $\tilde{A}(\sigma)$ converges), and hence $A(N) \leq N^{\sigma_0}$. But then $\sigma^* \leq \sigma_c$.

Since this relation holds whenever $\sigma_0 > \sigma_c$, we must have $\sigma^* \leq \sigma_c$. Thus we have $\sigma_c = \sigma^*$, and this proves (i), and also (in view of (2.4)) it proves (iii).

\[ \text{Definition 2.5.} \quad \text{The} \quad \text{abscissa of absolute convergence} \quad \text{of} \quad \text{a} \quad \text{Dirichlet series} \quad \alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s} \quad \text{is defined to be} \]
\[ \sigma_a = \inf \{ \sigma \in \mathbb{R} : \sum_{n=1}^{\infty} |a_n| n^{-\sigma} < \infty \}. \]

[This is the abscissa of convergence of $\sum_{n=1}^{\infty} |a_n| n^{-s}$]

\[ \text{Theorem 2.6.} \quad \text{One has} \quad \sigma_c \leq \sigma_a \leq \sigma_c + 1. \]

Proof. Problem set 1.
Theorem 2.7. Suppose that \( \alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s} \) has abscissa of convergence \( \sigma_c \). Suppose also that \( \delta > 0 \) and \( \sigma > \sigma_c + \delta \).

Then for each \( \varepsilon > 0 \) with \( \varepsilon < \delta \), one has

\[
\alpha(s) \ll (1 + \varepsilon)^{1 - \delta - \varepsilon}.
\]

Proof. Put \( s_0 = \sigma_c + \varepsilon \) and note that by (2.2) one has

\[
\alpha(s) = \sum_{n=1}^{M} a_n n^{-s} + R(M) M^{\sigma_c + \varepsilon - s} + (\sigma_c + \varepsilon - s) \int_{M}^{\infty} \frac{R(u)}{u^{\sigma_c + \varepsilon - s}} du.
\]

Since \( \alpha(\sigma_c + \varepsilon) \) converges, we have \( a_n \ll n^{\sigma_c + \varepsilon} \), whence \( R(u) \ll 1 \).

Thus

\[
\alpha(s) \ll \sum_{n=1}^{M} n^{-\delta + \varepsilon} + M^{-\delta + \varepsilon} + \frac{|\sigma_c + \varepsilon - s|}{\sigma - \sigma_c - \varepsilon} M^{\sigma_c + \varepsilon - \sigma}.
\]

Thus, on putting \( M = [t \ln t] \) we obtain the desired conclusion. //

Finally, we have uniqueness of Dirichlet series expansions.

Theorem 2.8. Suppose that for all \( s \) with \( \sigma > \sigma_0 \), one has

\[
\sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} b_n n^{-s}.
\]

Then \( a_n = b_n \) for all \( n \in \mathbb{N} \).

Proof. Put \( c_n = a_n - b_n \) and consider \( \sum_{n=1}^{\infty} c_n n^{-s} \). We prove by induction that \( c_n = 0 \) for all \( n \). We may suppose that \( \sigma > \sigma_0 \) and that \( \sum_{n=1}^{\infty} c_n n^{-\sigma} = 0 \). Thus

\[
c_0 = -\sum_{n=2}^{\infty} c_n n^{-\sigma}.
\]

The right-hand side here is absolutely convergent for \( \sigma > \sigma_0 + 1 \), and by dominated convergence one has
Then $c_1 = 0$, establishing the base of the induction. If we have $c_n = 0$ for $n < N$ and $N \geq 2$, meanwhile, then

$$|c_N| \leq \sum_{n > N} |c_n| (N/n)^{\sigma} \to 0 \quad \text{as} \quad \sigma \to \infty,$$

in like manner. Thus $c_N = 0$ and the induction is complete.

§3. The Riemann zeta function: basic properties.

Observe first that the multiplicative property of (formal) Dirichlet series asserted in our first class is justified in regions of absolute convergence.

**Theorem 3.1.** Let

$$\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s} \quad \text{and} \quad \beta(s) = \sum_{n=1}^{\infty} b_n n^{-s}$$

be Dirichlet series, and put

$$\gamma(s) = \sum_{n=1}^{\infty} c_n n^{-s}, \quad \text{where} \quad c_n = \sum_{l m = n} a_l b_m.$$

Then proved that $\alpha(s)$ and $\beta(s)$ are both absolutely convergent at $s \in \mathbb{C}$, so too is $\gamma(s)$, and $\gamma(s) = \alpha(s) \beta(s)$.

**Proof.** The absolute convergence of $\alpha(s)$ and $\beta(s)$ implies that we may rearrange terms to see that

$$\alpha(s) \beta(s) = \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} a_l b_m (lm)^{-s} = \sum_{n=1}^{\infty} c_n n^{-s} = \gamma(s),$$

with

$$\sum_{n=1}^{\infty} |c_n n^{-s}| \leq \sum_{n=1}^{\infty} \sum_{l m = n} |a_l b_m| (lm)^{-\sigma}.$$
Motivated by this multiplicative relation, we return to an earlier observation concerning $\zeta(s)$.

Recall that an arithmetic function $f : \mathbb{N} \to \mathbb{C}$ is called multiplicative if $f(1) = 1$ and, whenever $(m,n) = 1$ then

$$f(mn) = f(m)f(n).$$

Notice the distinction here between multiplicative functions $f$ and totally multiplicative functions $g : \mathbb{N} \to \mathbb{C}$ for which

$$f(mn) = f(m)f(n) \text{ for all } m, n \in \mathbb{N}.$$  

Examples of multiplicative functions:

(i) $f(n) = \mathbb{1}(n) = \begin{cases} 1, & n \in \mathbb{N} \setminus \{1\} \\ 0, & n = 1 \end{cases}$

(ii) (Möbius function) $\mu(n) = \begin{cases} (-1)^{\omega(n)}, & \text{n squarefree} \\ 0, & \text{n not squarefree} \end{cases}$, where $\omega(n) = \sum_{p \mid n} 1$

(iii) (Liouville lambda function) $\lambda(n) = \begin{cases} (-1)^{\Omega(n)}, & \text{where } \Omega(n) = \sum_{p \mid n} h \end{cases}$

(iv) $\tau(n) = \sum_{d \mid n} 1$. (divisor function)

**Theorem 3.2.** Let $f$ be multiplicative and write

$$\varphi(s) = \sum_{n=1}^{\infty} f(n)n^{-s}.$$

Then whenever $\varphi(s)$ is absolutely convergent, one has the Euler product decomposition

$$\varphi(s) = \prod_{p} \left(1 + \sum_{h=0}^{\infty} \frac{f(p^h)}{p^{hs}}\right) = \prod_{p} \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \ldots\right).$$
Suppose that $\zeta(s)$ is absolutely convergent at $s = \sigma$. Each factor in the Euler product is absolutely convergent, since
\[
\sum_{h=0}^{\infty} \left| \frac{f(p^h)}{p^h} \right| \leq \sum_{n=1}^{\infty} \frac{|f(n)|}{n^\sigma} < \infty.
\]
We may therefore rearrange terms in any finite product of such Euler factors. Put $M(x) = \{ n \in \mathbb{N} : p^{\nu} = p \leq x \}$. Then
\[
\prod_{p \leq x} \sum_{h=0}^{\infty} \frac{f(p^h)}{p^{hs}} = \sum_{n \in M(x)} \frac{f(n)}{n^\sigma},
\]
whence
\[
\left| \sum_{n=1}^{\infty} f(n) n^{-s} - \prod_{p \leq x} \sum_{h=0}^{\infty} \frac{f(p^h)}{p^{hs}} \right| \leq \sum_{n \in M(x)} \frac{|f(n)|}{n^\sigma} \leq \sum_{n > x} \frac{|f(n)|}{n^\sigma}.
\]
But given any $\varepsilon > 0$, we can take $x$ large enough that the right hand side is smaller than $\varepsilon$ (since $\zeta(s)$ is absolutely convergent). Thus we see that
\[
\lim_{x \to \infty} \prod_{p \leq x} \sum_{h=0}^{\infty} \frac{f(p^h)}{p^{hs}} = \sum_{n=1}^{\infty} \frac{f(n)}{n^\sigma} = \zeta(s).
\]

**Corollary 3.3.** When $\sigma > 1$, one has

(i) \[
\sum_{n=1}^{\infty} n^{-s} = S(s) = \prod_{p} \left( 1 - p^{-s} \right)^{-1};
\]

(ii) \[
\sum_{n=1}^{\infty} \mu(n)n^{-s} = 1/S(s) = \prod_{p} \left( 1 - p^{-s} \right); \]

(iii) \[
\sum_{n=1}^{\infty} \lambda(n)n^{-s} = \frac{S(2s)}{S(s)} = \prod_{p} \left( 1 + p^{-s} \right)^{-1};
\]
(iv) \( \sum_{n=1}^{\infty} \tau(n) n^{-s} = \zeta(s)^2 = \prod_p \left(1 - p^{-s}\right)^{-2} \).

Proof. When \( s > 1 \), the series in (i), (ii), (iii) are absolutely convergent, since \( \sum_{n=1}^{\infty} n^{-s} < \infty \). The series in (iv) is therefore absolutely convergent by virtue of Theorem 3.1.

The coefficients of the Dirichlet series (i) - (iv) are all multiplicative, so in cases (i) and (ii) we obtain the Euler products:

\[
\zeta(s) = \prod_p \sum_{h=0}^{\infty} \frac{\mu(p^h)}{p^{hs}} = \prod_p \left(1 + \frac{1}{p^{-s}} + \frac{1}{p^{-2s}} + \ldots \right) = \prod_p \left(1 - p^{-s}\right)^{-1},
\]

\[
\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p \sum_{h=0}^{\infty} \frac{\mu(p^h)}{p^{hs}} = \prod_p \left(1 - p^{-s}\right).
\]

Thus \( \zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = 1 \implies \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}. \)

In case (iii) we obtain similarly:

\[
\sum_{h=0}^{\infty} \frac{\lambda(h)}{h^s} = \prod_p \sum_{h=0}^{\infty} \frac{\lambda(p^h)}{p^{hs}} = \prod_p \left(1 - p^{-s} + p^{-2s} - \ldots \right) = \prod_p \left(1 + p^{-s}\right)^{-1}.
\]

Thus \( \zeta(s) \sum_{h=0}^{\infty} \frac{\lambda(h)}{h^s} = \prod_p \left(1 - p^{-s}\right)^{-1} = \zeta(2s) \).

Finally, one has:

\[
\sum_{n=1}^{\infty} \tau(n) n^{-s} = \prod_p \sum_{h=0}^{\infty} \frac{\tau(p^h)}{p^{hs}} = \prod_p \left(1 + \frac{2}{p^{-s}} + \frac{3}{p^{-2s}} + \ldots \right)
\]

\[= \prod_p \left(1 - p^{-s}\right)^{-2} = \zeta(s)^2. \]
Connection with Möbius inversion formula: Suppose f, g arithmetic functions.

Recall that \( \sum_{d \mid n} g(n/d) = \sum \mu(d) f(n). \)

Corresponding (formal) analogue in Dirichlet series:

\[
\sum_{n=1}^{\infty} f(n) n^{-s} = \sum_{n=1}^{\infty} \left( \sum_{d \mid n} g(d) \right) n^{-s} = \sum_{n=1}^{\infty} g(n) n^{-s}.
\]

\[
\sum_{n=1}^{\infty} f(n) n^{-s} = \frac{1}{\zeta(s)} \sum_{n=1}^{\infty} \phi(n) n^{-s}.
\]

Define the von Mangoldt function \( \Lambda(n) \):

\[
\Lambda(n) = \begin{cases} 
\log p & \text{when } n = p^k, \ k \geq 1, \ p \text{ prime, } \\
0 & \text{otherwise}.
\end{cases}
\]

Corollary 3.4. When \( \sigma > 1 \), one has

\[
\log \zeta(s) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s}.
\]

And

\[
\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s} \quad \text{(logarithmic derivative of } \zeta(s)\text{)}.
\]

Proof. Using the Euler product for \( \zeta(s) \), one has

\[
\log \zeta(s) = -\sum_{p} \log (1 - p^{-s}) = \sum_{p} \sum_{k=1}^{\infty} k^{-1} p^{-ks} = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s};
\]

Thus, by differentiating,

\[
\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} \cdot \log n \cdot n^{-s}. \]

Notice that

\[
\sum_{n=1}^{\infty} \log(n) n^{-s} = \int_{1}^{\infty} \frac{1}{x^s} \log x \, dx = \frac{\zeta'(s)}{\zeta(s)}
\]

Whence

\[
\sum_{d \mid n} \Lambda(d) = \log n.
\]
Analytic continuation of the Riemann zeta function.

Thus far we have defined $\zeta(s)$ as an analytic function for $\sigma > 1$ by means of the series $\sum_{n=1}^{\infty} n^{-s}$. It is natural to attempt to analytically continue $\zeta(s)$ into the plane $\sigma < 1$.

**Theorem 3.5.** Suppose that $s \in \mathbb{C}$ satisfies $\sigma \neq 1$, $\sigma > 0$, and that $x > 0$. Then one has

$$
\zeta(s) = \sum_{n \leq x} n^{-s} + \frac{x^{1-s}}{s-1} + \frac{x^s}{s} - s \int_{x}^{\infty} \frac{\{u\}}{u^{s+1}} \, du.
$$

[Note: $\{u\} = \theta - [u]$, so the last integral here is absolutely convergent for $\sigma > 0$.]

**Proof:** Noting that for $\sigma > 1$ one has

$$
\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{1 \leq n \leq x} n^{-s} + \sum_{n > x} n^{-s}, \quad (3.1)
$$

we may apply Riemann-Stieltjes integration to rewrite the final sum as

$$
\sum_{n > x} n^{-s} = \int_{x^+}^{\infty} u^{-s} \, d\{u\} = \int_{x^+}^{\infty} u^{-s} \, du - \int_{x^+}^{\infty} u^{-s} \, d\{u\}
$$

where

$$
\sum_{n \leq u} 1 = \frac{x^{1-s}}{s-1} + \frac{x^s}{s} - \int_{x}^{\infty} \frac{\{u\}}{u^{s+1}} \, du. \quad (3.2)
$$

The formula claimed in the theorem follows for $\sigma > 1$ and $s \neq 1$ by combining (3.1) and (3.2).

Since the integral $\int_{x}^{\infty} \frac{\{u\}}{u^{s+1}} \, du$ is convergent for $\sigma > 0$,
and uniformly for \( \sigma > \delta > 0 \), and the integrand is an analytic function of \( s \), it follows that the integral is analytic for \( \sigma > 0 \). Thus, by the uniqueness of analytic continuation, the claimed formula for \( \xi(s) \) holds for \( \sigma > 0 \).

**Corollary 3.6** The Riemann Zeta function has a simple pole at \( s = 1 \) with residue 1, but is otherwise analytic for \( \sigma > 0 \).

**Proof.** Apply Theorem 3.5 with \( x = 1 \) to obtain

\[
\xi(s) = 1 + \frac{1}{\sigma - 1} - s \int_{1}^{\infty} \frac{\xi(u)}{u^{s+1}} \, du \quad (\sigma > 0)
\]

\[
= \frac{s}{\sigma - 1} - s \int_{1}^{\infty} \frac{\xi(u)}{u^{s+1}} \, du \quad (\sigma > 0).
\]

(3.3)

Integrate by parts to analytically continue to \( \sigma > -1, \sigma > -2, \ldots \) (use Euler–Maclaurin summation formula).

**Corollary 3.7.** Let \( C_0 \) denote Euler's constant (often called \( \gamma \)), defined by

\[
C_0 = \lim_{x \to \infty} \left( \sum_{1 \leq n \leq x} \frac{1}{n} - \log x \right).
\]

Then the Laurent expansion of \( \xi(s) \) about \( s = 1 \) is given by

\[
\xi(s) = \frac{1}{\sigma - 1} + \sum_{k=0}^{\infty} a_k (s-1)^k,
\]

in which \( a_k = C_0 \).

**Proof.** We first contemplate Euler's constant \( C_0 \). One has

\[
\sum_{1 \leq n \leq x} \frac{1}{n} = \int_{1}^{x} u^{-1} \, d \left[ u \right] = \int_{1-}^{x} u^{-1} \, du - \int_{1-}^{1} u^{-1} \, d \left[ u \right]
\]

\[
= \log x + 1 - \left[ \frac{x^{3/2}}{\sqrt{x}} \right] - \int_{1}^{x} \frac{u^{3/2}}{u^{2}} \, du.
\]
\[ \sum_{1 \leq n \leq x} \frac{1}{n} = \log x + c_0 + O\left(\frac{1}{x}\right), \quad (3.4) \]

where

\[ c_0 = 1 - \int_1^\infty \left(1 - \frac{x}{u} \right) \frac{\text{d}u}{u^2}. \]

By comparing the definition of \( C_0 \) with (3.4), we see that \( c_0 = C_0 \).

Also, by comparison with (3.3), we find that

\[ S(s) = \frac{1}{s-1} + \left(1 - \int_1^\infty \frac{x u^3}{u^2} \text{d}u\right) + O\left(\log(1-s)\right) \]

as \( s \to 1 \).

This confirms the claim concerning the Laurent series for \( S(s) \).

\[ \]

8.4. **Arithmetic functions** — mean values and elementary estimates.

Given an arithmetic function \( f: \mathbb{N} \to \mathbb{C} \), we are interested in

\[ \text{this section in obtaining statistical information — average value, } \]

max/min value, variance, etc. This is, of course, a topic of great

importance in its own right. However, if we are to understand the

convergence of a Dirichlet series

\[ \sum_{n=1}^{\infty} f(n) n^{-s}, \]

then the order of growth of the partial sums \( \sum_{n \leq x} f(n) \) plays a

critical role in determining the absence of convergence \( \sigma_c \). This

is important for the discussion of the last two sections.

When arithmetic functions \( f \) and \( g \) are connected via

the relation

\[ f(n) = \sum_{d \mid n} g(d), \quad (4.1) \]

then one has a particularly simple approach to computing the
mean value \[ \frac{1}{N} \sum_{n=1}^{N} f(n). \]

Thus, one has
\[
\sum_{n=1}^{N} f(n) = \sum_{n=1}^{N} \sum_{d \mid n} g(d) = \sum_{1 \leq d \leq N} g(d) \sum_{1 \leq n \leq N} \frac{1}{d} \\
= \sum_{1 \leq d \leq N} g(d) \left\lfloor \frac{N}{d} \right\rfloor.
\]

Since \( \left\lfloor \frac{N}{d} \right\rfloor = \frac{N}{d} + o(1) \), we deduce that
\[
\frac{1}{N} \sum_{n=1}^{N} f(n) = \sum_{1 \leq d \leq N} \frac{g(d)}{d} + O \left( \sum_{1 \leq d \leq N} \log(d) \right).
\]

Often times one finds that the first sum converges, or at least is relatively well-behaved, and the second is also well-controlled.

Note that given \( f : \mathbb{N} \to \mathbb{C} \), one may obtain a function \( g \) satisfying (4.1) via Möbius inversion:
\[
f(n) = \sum_{d \mid n} g(d) \iff g(n) = \sum_{d \mid n} \mu(d) f(n/d).
\]

---

**Example 4.1** Let \( \sigma(n) = \sum_{d \mid n} d \). Then one has
\[
\sum_{n \leq x} \frac{\sigma(n)}{n} = \sum_{n \leq x} \sum_{d \mid n} \frac{1}{d} = \sum_{n \leq x} \frac{1}{d} \sum_{1 \leq m \leq x/d} 1 = \sum_{1 \leq d \leq x} \frac{1}{d} \left\lfloor \frac{x}{d} \right\rfloor.
\]
\[
\sum_{1 \leq d \leq x} \frac{1}{d^2} + O\left( \sum_{d \geq x} \frac{1}{d^2} \right)
\]
\[
= x \left( \sum_{d=1}^{\infty} \frac{1}{d^2} + O\left( \sum_{d \geq x} \frac{1}{d^2} \right) \right) + O\left( \log x \right)
\]
\[
= x \cdot 5(2) + O\left( \log x \right).
\]

So
\[
\frac{1}{x} \sum_{n \leq x} \frac{\sigma(n)}{n} = \frac{\pi^2}{6} + O\left( \frac{\log x}{x} \right).
\]

---

**Example 4.2.** Compute the average value of \( \frac{n}{\varphi(n)} \), where \( \varphi(n) \) denotes Euler's totient.

[Thus \( \varphi(n) = \prod_{p \mid n} (p^k - p^{k-1}) = \prod_{p \mid n} (1 - \frac{1}{p}) \).]

We first seek a function \( g : \mathbb{N} \to \mathbb{C} \) with
\[
\sum_{d \mid n} g(d) = \frac{n}{\varphi(n)}.
\]

But by Möbius inversion, one has
\[
g(n) = \sum_{d \mid n} \mu(d) \frac{n/d}{\varphi(n/d)}.
\]

This looks complicated, but since \( \mu(n) \) and \( n/\varphi(n) \) are both multiplicative functions, it suffices to consider prime power values of \( n \). (Ex. if \( f \) is multiplicative, so is \( \sum f(d) \); if \( a, b \) multiplicative, then \( ab \) is \( \sum a(d) b(n/d) \).

But
\[ g(n) = \prod_{p \mid n} g(p^k) = \frac{\mu(n)^2}{\phi(n)} \]

We next deduce that

\[
\sum_{1 \leq n \leq x} \frac{n}{\phi(n)} = \sum_{1 \leq n \leq x} \sum_{d \mid n} g(d)
\]

\[ = \sum_{1 \leq d \leq x} \frac{\mu(d)^2}{\phi(d)} \sum_{1 \leq m \leq x/d} 1 \]

\[ = x \sum_{1 \leq d \leq x} \frac{\mu(d)^2}{d \phi(d)} + \mathcal{O}\left( \sum_{1 \leq d \leq x} \frac{1}{\phi(d)} \right) \]

\[ = x \sum_{d=1}^{\infty} \frac{\mu(d)^2}{d \phi(d)} + \mathcal{O}\left( x^{\varepsilon} \right) \]

where \( \phi(d) \gg d^{1-\varepsilon} \), any \( \varepsilon > 0 \).

The infinite sum here is absolutely convergent with Euler product

\[
\prod_{p} \left( 1 + \frac{\mu(p)^2}{p(p-1)} + \frac{\mu(p^2)^2}{p^2 \cdot p(p-1)} + \cdots \right) = \prod_{p} \left( 1 + \frac{1}{p(p-1)} \right) \]

\[ = \prod_{p} \left( \frac{p(p-1)}{p^2 - p + 1} \right)^{-1} = \prod_{p} \left( \frac{p^2 - 1}{p^2 - 1} \cdot \frac{p(p-1)(p+1)}{p(p-1)(p+1)} \right)^{-1} \]
\[ \prod_p (1 - p^{-s})^{-1} = \frac{s(2) s(3)}{s(6)} \]

Thus we conclude that

\[ \frac{1}{x} \sum_{1 \leq n \leq x} \frac{n}{\phi(n)} = \frac{s(2) s(3)}{s(6)} + O(x^{-1}). \]

Variants of the basic idea:

(a) Powered convolutions of shape \( f(n) = \sum_{d^k \mid n} g(d) \) (\( k \geq 1 \)).

Example. (exercise) Show that \( \mu(n)^2 = \sum_{d^2 \mid n} \mu(d) \), and hence deduce that

\[ \sum_{1 \leq n \leq x} \frac{1}{n \text{ square-free}} = \frac{6}{\pi^2} x + O(x^{1/2}). \]

(b) Beyond Möbius inversion: rewrite the convolution sum in the shape

\[ f(n) = \sum_{d \mid n} a(d) b(n/d), \]

for suitable arithmetic functions \( a, b \). Then

\[ \sum_{1 \leq n \leq x} f(n) = \sum_{1 \leq d \leq x} a(d) \sum_{1 \leq m \leq x/d} b(m) \]

\[ = O(B(x/d)), \text{ say} \]

need a good estimate here.
Beyond Möbius inversion, II. The "bilinear" structures can be used to improve error terms – idea is that if \( n = kl \), then either \( k \leq \sqrt{n} \) or \( l \leq \sqrt{n} \).

More generally, suppose that \( 1 \leq y \leq x \),

\[
f(n) = \sum_{d \mid n} a(d) b(n/d) = \sum_{kl=n} a(k) b(l),
\]

and write

\[
A(x) = \sum_{1 \leq n \leq x} a(n) \quad \text{and} \quad B(z) = \sum_{1 \leq n \leq z} b(n).
\]

Then

\[
\sum_{1 \leq n \leq x} f(n) = \sum_{1 \leq kl \leq x} a(k) b(l)
\]

\[
= \sum_{1 \leq k \leq y} \sum_{1 \leq l \leq x/k} a(k) b(l) + \sum_{1 \leq k \leq x/y} \sum_{1 \leq l \leq x/y} a(k) b(l)
\]

\[
- \sum_{1 \leq k \leq y} \sum_{1 \leq l \leq x/y} a(k) b(l)
\]

includes the overcounting.

\[
= \sum_{1 \leq k \leq y} a(k) B(x/k) + \sum_{1 \leq k \leq x/y} b(k) A(x/k) - A(y) B(x/y)
\]

Example. Observe that naively, we have

\[
\sum_{1 \leq n \leq x} \tau(n) = \sum_{1 \leq d \leq x} \sum_{1 \leq m \leq x/d} 1 = \sum_{1 \leq d \leq x} \left\lfloor \frac{x}{d} \right\rfloor = x \sum_{1 \leq d \leq x} \frac{1}{d} + O(x)
\]

\[
= x \log x + O(x).
\]
Meanwhile, using the decomposition described in (c), we have

\[ \sum_{1 \leq n \leq x} \tau(n) = \sum_{1 \leq k \leq x} \left\lfloor \frac{x}{k} \right\rfloor + \sum_{1 \leq k \leq y} \left\lfloor \frac{x}{ky} \right\rfloor \]

\[ = x \sum_{1 \leq k \leq y} \frac{1}{k} + x \sum_{1 \leq l \leq \frac{x}{y}} - y \frac{x}{y} + o(y + \frac{x}{y}) \]

\[ = x \log y + C_0 + o\left(\frac{1}{y}\right) + x \left( \log \left( \frac{x}{y} \right) + C_0 + o\left(\frac{y}{x}\right) \right) \]

\[ - x + o\left( y + \frac{x}{y} \right) \]

\[ = x \log x + (2C_0 - 1)x + o\left( y + \frac{x}{y} \right). \]

The error term here is \( O\left(\frac{\sqrt{x}}{x^2}\right) \) if we take \( y = \sqrt{x} \).

\[ \text{8.5. Prime number theorems: the work of Chebyshev and Mertens.} \]

Some notation:

\[ \pi(x) = \sum_{p \leq x} 1, \quad \Theta(x) = \sum_{p \leq x} \log p, \quad \Psi(x) = \sum_{n \leq x} \Lambda(n). \]

**Theorem 5.1.** When \( x \geq 2 \), one has \( \Psi(x) \asymp x \).

**Proof.** We apply a common strategy in analytic number theory, that of majorising the characteristic function of a set. Since

\[ \sum_{d \mid m} \mu(d) = \begin{cases} 1, & \text{when } m = 1, \\ 0, & \text{otherwise}, \end{cases} \]

it is apparent that

\[ \Psi(x) = \sum_{1 \leq n \leq x} \Lambda(n) = \sum_{1 \leq n \leq x} \left( \sum_{d \mid n} \mu(d) \right) \Lambda(n) \]
It transpires that the Möbius function is difficult to handle, but we may majorize the characteristic function

$$1 = \sum_{1 \leq m \leq y} \sum_{d|m} \mu(d)$$

by substituting for \( \mu(d) \) an arithmetic function having appropriate properties. A particularly simple approach is to take

$$a(d) = \begin{cases} 1, & \text{when } d = 1, \\ -2, & \text{when } d = 2, \\ 0, & \text{when } d \geq 3. \end{cases}$$

We then have

$$\sum_{1 \leq m \leq y} \sum_{d|m} a(d) = \sum_{1 \leq d \leq y} a(d) \sum_{1 \leq k \leq y/d} 1 = \sum_{1 \leq d \leq y} a(d) \left\lfloor \frac{y}{d} \right\rfloor$$

$$= \lfloor y \rfloor - 2 \lfloor \frac{y}{2} \rfloor \in \{0, 1\}.$$

We deduce that

$$\sum_{1 \leq n \leq x} \Lambda(n) \left( \left\lfloor \frac{x}{n} \right\rfloor - 2 \left\lfloor \frac{x}{2n} \right\rfloor \right) \leq \psi(x).$$

Since for \( x \geq n > x/2 \), the parenthetical expression is equal to 1, it follows that

$$\psi(x) - \psi(x/2) \leq \sum_{1 \leq n \leq x} \Lambda(n) \sum_{1 \leq m \leq x/n} a(d) \leq \psi(x)$$

$$\sum_{1 \leq k \leq m \leq x} \Lambda(n) a(d) = \sum_{1 \leq d \leq x} a(d) \sum_{n | d} \Lambda(n)$$

$$\sum_{1 \leq d \leq x} a(d) \log d = \sum_{1 \leq d \leq x} a(d) \left( \sum_{1 \leq l \leq d} \log l \right)$$
By Riemann-Stieltjes integration, one finds that
\[ \sum_{1 \leq l \leq y} \log l = \int_{1}^{y} \log u \, d\left\{ \sum_{d \leq x} \frac{1}{d} \right\} = y \log y - y + O(\log y). \]

Thus
\[ \sum_{1 \leq l \leq x} \frac{\sum_{1 \leq d \leq x/l} \log l}{d} = \left( x \log x - x + O(\log x) \right)\left( \frac{1}{2} \log \left( \frac{x}{2} \right) - \frac{x}{2} + O(\log x) \right) = x (\log 2) + O(\log x). \]

Consequently, we find that
\[ \psi(x) \geq x (\log 2) + O(\log x) \]
and
\[ \psi(x) - \psi(x/2) \leq x (\log 2) + O(\log x). \]

By summing over dyadic intervals, we conclude that
\[ \psi(x) \leq (2 \log 2) \log x + O((\log x)^2), \]
and thus
\[ (\log 2) x + O(\log x) \leq \psi(x) \leq (2 \log 2) \log x + O((\log x)^2). \]

There are more efficient choices for the sequence \( (a(d))_{d \in \mathbb{N}} \).

Thus, one may take \( a(1) = a(30) = 1 \) and \( a(7) = a(3) = a(5) = -1 \), with \( a(d) = 0 \) otherwise. This gives
\[ (0.9212) x + O(\log x) \leq \psi(x) \leq (1.1056) x + O((\log x)^2). \]

[Even the weak version of the Prime Number Theorem embodied in Corollary 5.2 yields precise information concerning various sums and products involving \( 1/p \).]
Corollary 5.2. When \( x \geq 2 \), one has
\[
\Theta(x) = \psi(x) + O(x^{1/2}) \quad \text{and} \quad \pi(x) = \frac{\psi(x)}{\log x} + O\left(\frac{x}{(\log x)^2}\right).
\]
Moreover, one has \( \Theta(x) \sim x \) and \( \pi(x) \sim x / \log x \).

Proof. By the definitions of \( \psi(x) \) and \( \Theta(x) \), it follows that
\[
\psi(x) = \sum_{p^k \leq x} \log p = \sum_{k=1}^{\infty} \Theta(x^{1/k}),
\]
but for each \( y \in \mathbb{R}_{\geq 2} \), one has
\[
\Theta(y) \leq \psi(y) \leq y,
\]
and thus we deduce that
\[
\psi(x) - \Theta(x) = \sum_{k \geq 2} \Theta(x^{1/k}) \ll \sqrt[2]{x} + x^{1/3} \log x \ll x^{3/4}.
\]
This establishes our first claim concerning \( \Theta(x) \) and \( \psi(x) \). Since \( \psi(x) \sim x \), moreover, we see also that \( \Theta(x) = \psi(x) + O(x^{1/2}) \sim x \).

We apply Riemann–Stieltjes integration to obtain the result on \( \pi(x) \). Thus, we have
\[
\pi(x) = \int_2^x (\log u)^{-1} \, d \Theta(u) = \frac{\Theta(x)}{\log x} + \int_2^x \frac{\Theta(u)}{u (\log u)^2} \, du
\]
\[
= \frac{\Theta(x)}{\log x} + O\left( \int_2^x \frac{du}{(\log u)^2} \right) \times \frac{x}{\log x} + O\left( \frac{x}{(\log x)^2} \right).
\]
Hence \( \pi(x) \sim x / \log x \). Also, since \( \Theta(x) - \psi(x) = O(x^{1/2}) \), it follows also that \( \Pi(x) = \frac{\psi(x)}{\log x} + O\left( \frac{x}{(\log x)^2} \right) \).
Theorem 5.3. When $x \geq 2$, one has

(a) $\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1)$ and $\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1)$;

(b) $\int_1^x \frac{\psi(u)}{u^3} \, du = \log x + O(1)$;

(c) $\sum_{p \leq x} \frac{1}{p} = \log \log x + C_0 - \sum_{k=2}^{\infty} \frac{1}{k \log^k x} + O\left(\frac{1}{\log x}\right)$,

(d) $\prod_{p \leq x} (1 - \frac{1}{p})^{-1} = e^{C_0} \log x + O(1)$ \hspace{1cm} (Mertens' Theorem).

Proof: (a) We have $\log n = \sum_{d|n} \Lambda(d)$, and hence

$$\sum_{1 \leq m \leq x} \frac{\Lambda(d)}{d} \sum_{1 \leq d \leq x} \frac{1}{d} = x \sum_{1 \leq d \leq x} \frac{\Lambda(d)}{d} + O(x) = x \log x + O(x),$$

Thus

$$\sum_{1 \leq d \leq x} \frac{\Lambda(d)}{d} = \frac{x \log x + O(x)}{x} = \log x + O(1).$$

Moreover, we have

$$\sum_{1 \leq d \leq x} \frac{\Lambda(d)}{d} = \sum_{p \leq x} \frac{\log p}{p} + \sum_{k=2}^{\infty} \sum_{p^k \leq x} \frac{\log p}{p^k}$$

$$= \sum_{p \leq x} \frac{\log p}{p} + \sum_{p \leq x} \frac{\log p}{p(p-1)} = \sum_{p \leq x} \frac{\log p + O(1)}{p}.$$

(b) We have

$$\sum_{1 \leq d \leq x} \frac{\Lambda(d)}{d} = \int_2^x u^{-1} d\psi(u) = \frac{\psi(u)}{u} \bigg|_{2}^{x} + \int_2^x \frac{\psi(u)}{u^2} \, du.$$
The proof of parts (c) and (d) in the precise form stated is a little complicated.

(c) Define the functions \( L(x) \) and \( R(x) \) by putting

\[
L(x) = \sum_{p \leq x} \frac{\log p}{p} \quad \text{and} \quad R(x) = L(x) - \log x,
\]

so that according to part (a), we have \( R(x) \ll 1 \). Then by Riemann integration,

\[
\sum_{p \leq x} \frac{1}{p} = \int_{2}^{x} (\log u)^{-1} \, dL(u) = \int_{2}^{x} \frac{d(\log u)}{\log u} + \int_{2}^{x} \frac{dR(u)}{\log u}
\]

\[
= \log \log u \bigg|_{2}^{x} + \frac{R(u)}{\log u} \bigg|_{2}^{x} + \int_{2}^{x} \frac{R(u)}{u(\log u)^2} \, du
\]

\[
\uparrow \quad \text{absolutely convergent as} \quad x \to \infty.
\]

\[
= \log \log x - \log \log 2 + \frac{R(x)}{\log x} + 1 + \int_{2}^{\infty} \frac{R(u)}{u(\log u)^2} \, du
\]

\[
R(2) = -\log 2
\]

\[
- \int_{x}^{\infty} \frac{R(u)}{u(\log u)^2} \, du
\]

Write \( b = 1 - \log \log 2 + \int_{2}^{\infty} \frac{R(u)}{u(\log u)^2} \, du \), and recall that
\[ R(n) \ll 1. \text{ Then we deduce that } \]
\[
\sum_{p \leq x} \frac{1}{p} = \log \log x + b + O\left(\frac{1}{\log x}\right).
\]

Note that we have yet to confirm that \( b = C_0 - \sum_{p} \sum_{k=2}^{\infty} \frac{1}{kp^k} \)
as task to which we shall shortly return.

(d). Note that

\[
\log \left( \prod_{p \leq x} (1 - \frac{1}{p})^{-1} \right) = -\sum_{p \leq x} \log (1 - \frac{1}{p})
\]

\[= \sum_{p \leq x} \frac{1}{p} + \sum_{p \leq x} \sum_{k=2}^{\infty} \frac{1}{kp^k},\]

Making use of our conclusion in part (c), we deduce that

\[
\log \left( \prod_{p \leq x} (1 - \frac{1}{p})^{-1} \right) = \log \log x + c + \sum_{p > x} \sum_{k=2}^{\infty} \frac{1}{kp^k} + O\left(\frac{1}{\log x}\right),
\]

\[\sum_{n > x} \frac{1}{n^s} = O\left(\frac{1}{x}\right)\]

where

\[c = b + \sum_{p} \sum_{k=2}^{\infty} \frac{1}{kp^k}.\]

Hence

\[\prod_{p \leq x} (1 - \frac{1}{p})^{-1} = (\text{e}^c \log x)(1 + O\left(\frac{1}{\log x}\right)) = \text{e}^c \log x + O(1).\]

Here, we have yet to confirm that \( c = C_0 \). This we confirm by comparing \( \sum_{n \leq \log x} \frac{1}{n} \) with \( \sum_{p \leq x} \frac{1}{p} \).
We begin by observing that

$$
\sum_{\genfrac{}{}{0pt}{}{1 \leq n \leq x}{p^k \leq x}} \frac{\Lambda(n)}{n \log n} = \sum_{p \leq x} \sum_{k=1}^{\infty} \frac{\Lambda(p^k)}{p^k \log(p^k)} - \sum_{\begin{array}{c}
p \leq x \\
p^k > x
\end{array}} \frac{\Lambda(p^k)}{p^k \log(p^k)}
$$

$$
= -\sum_{p \leq x} \log \left(1 - \frac{1}{p}\right) + O\left(\frac{1}{\log x} \sum_{p \leq x} \sum_{k \geq 2} \frac{\log p}{p^k}\right)
$$

$$
= \log \left(\prod_{p \leq x} (1 - \frac{1}{p})^{-1}\right) + O\left(\frac{1}{\log(x)}\right).
$$

Hence we deduce that

$$
\sum_{\genfrac{}{}{0 pt}{}{1 \leq n \leq x}{n \log n}} \frac{\Lambda(n)}{n \log n} = \log \log x + c + O\left(\frac{1}{\log(x)}\right)
$$

$$
= \sum_{h \leq \log x} \frac{1}{n} + (c - C_0) + O\left(\frac{1}{\log(x)}\right), \quad \text{(using (3.4)).}
$$

We next introduce the auxiliary quantities, for $\delta > 0$,

$$
I_1(\delta) = \delta \int_{1^+}^{\infty} x^{1-\delta} \sum_{1 \leq n \leq x} \frac{\Lambda(n)}{n \log n} \, dx,
$$

$$
I_2(\delta) = \delta \int_{1^+}^{\infty} x^{1-\delta} \sum_{1 \leq n \leq \log x} \frac{1}{n} \, dx,
$$

$$
I_3(\delta) = \delta \int_{1^+}^{\infty} x^{1-\delta} \, dx = 1.
$$

Then it follows from (5.1) that

$$
I_1(\delta) = I_2(\delta) + c - C_0 + I_4(\delta),
$$
Where

\[ I_4(\delta) \ll \delta \int_{1+}^{\infty} \frac{x^{-1-\delta}}{\log(2x)} \, dx \ll \delta + \delta \int_{2}^{e^{\frac{1}{\delta}}} \frac{dx}{x \log x} \]

\[ + \delta^2 \int_{e^{\frac{1}{\delta}}}^{\infty} x^{-1-\delta} \, dx \ll \delta \log \left( \frac{1}{\delta} \right). \]

But by Theorem 2.4.3(ii)

\[ I_1(\delta) = \sum_{n=1}^{\infty} a_n n^{-\delta}, \quad \text{where} \quad a_n = \Lambda(n)/(n \log n), \]

whence

\[ I_1(\delta) = \log 5(1+\delta) = \log \left( \frac{1}{\delta} \right) + O(\delta) \quad \text{as} \quad \delta \to 0^+ \]

(using \( \delta(\delta) = \frac{1}{\delta} + c_0 + O(1) \)). Also,

\[ I_2(\delta) = \delta \sum_{n=1}^{\infty} \frac{1}{n} \int_{1}^{\infty} x^{-1-\delta} \, dx = \sum_{n=1}^{\infty} \frac{1}{n} e^{-\delta n} \]

\[ = \log (1 - e^{-\delta}) = -\log (\delta + O(\delta^2)) = \log \left( \frac{1}{\delta} \right) + O(\delta) \quad \text{as} \quad \delta \to 0^+. \]

We therefore conclude that as \( \delta \to 0^+ \), one has

\[ \log \left( \frac{1}{\delta} \right) = \log \left( \frac{1}{\delta} \right) + c - c_0 + O \left( \delta \log \left( \frac{1}{\delta} \right) \right), \]

whence

\[ c = c_0. \]

We may now conclude that

\[ \prod_{p \leq x} (1 - \frac{1}{p})^{-1} = e^{c_0 \log x + O(1)}, \]

and that

\[ b = c_0 - \sum_{p} \sum_{k=2}^{\infty} \frac{1}{kp^k}. \]
Corollary 5.4. One has

\[ 1 \geq \liminf_{x \to \infty} \frac{\Psi(x)}{\pi(x) \log x} \quad \text{and} \quad \limsup_{x \to \infty} \frac{\Psi(x)}{\pi(x) \log x} \geq 1. \]

**Proof.** By Corollary 5.2 we may focus on establishing that

\[ \liminf_{x \to \infty} \frac{\Psi(x)}{\log x} \leq 1 \quad \text{and} \quad \limsup_{x \to \infty} \frac{\Psi(x)}{\log x} \geq 1. \]

If, however, one were to have \( \limsup_{x \to \infty} \frac{\Psi(x)}{\log x} = a \), then given \( \varepsilon > 0 \) one must have \( \frac{\Psi(x)}{\log x} \leq (a + \varepsilon) \log x \) for all large enough \( x \), say for \( x \geq x_0 \). But then

\[
\int_{1}^{x} \psi(u) u^{-2} \, du \leq \int_{1}^{x_0} \psi(u) u^{-2} \, du + (a + \varepsilon) \int_{x_0}^{x} u^{-1} \, du
\]

\[
\leq (a + \varepsilon) \log x + o(1),
\]

whence

\[
\int_{1}^{x} \frac{\psi(u)}{u^2} \, du \leq (a + o(1)) \log x.
\]

But we have \( \int_{1}^{x} \frac{\psi(u)}{u^2} \, du = \log x + O(1) \), so necessarily

\[ a \geq 1. \]

The argument adapts easily to confirm that \( \liminf_{u \to \infty} \frac{\psi(u)}{u} \leq 1 \).

86. Bounds for arithmetic functions.

The relatively crude version of the Prime Number Theorem that we have established thus far permits fairly precise bounds to be derived for arithmetic functions of interest.
Theorem 6.1

When \( n \geq 3 \), one has

\[
\phi(n) \geq \frac{n}{\log n} \left( e^{-\gamma} + O \left( \frac{1}{\log \log n} \right) \right)
\]

and

\[
1 \leq \omega(n) \leq \frac{\log n}{\log n} \left( 1 + O \left( \frac{1}{\log \log n} \right) \right).
\]

Proof: Denote the \( k \)-th largest prime by \( p_k \), so that \( p_1 = 2 \), \( p_2 = 3 \), etc., and put \( m = \omega(n) \). Then once

\[
\frac{\phi(n)}{n} = \prod_{p \mid n} \left( 1 - \frac{1}{p} \right) \geq \prod_{i=1}^{m} \left( 1 - \frac{1}{p_i} \right),
\]

we find from Mertens' theorem that

\[
\frac{\phi(n)}{n} \geq \frac{e^{-\gamma}}{\log p_m} + O(1).
\]

However, we have \( p_1 \cdots p_m \leq n \), whence

\[
\sum_{p \leq p_m} \log p \leq \log n \Rightarrow \log p_m \leq \log \log n + O(1).
\]

Hence

\[
\phi(n) \geq \frac{e^{-\gamma}}{\log \log n} \left( \log p_m \right).
\]

\( \square \)

Likewise, we have \( \omega(n) \leq m \), where \( m \) is the largest natural number with \( p_1 \cdots p_m \leq n \). Then we have

\[
\sum_{p \leq p_m} \log p = \log n^{\Theta} \Rightarrow \log p_m \leq \log \log n^{\Theta} + O(1).
\]
Also,  
\[ m = \pi(p_m) = \Theta(p_m) + O\left(\frac{p_m}{\log p_m^2}\right) \]
\[ = \frac{\log n^*}{\log \log n^* + O(1))} + O\left(\frac{\log n^*}{(\log \log n^*)^2}\right) \]
\[ \leq \frac{\log n}{\log \log n} \left(1 + O\left(\frac{1}{\log \log n}\right)\right) \cdot \frac{1}{\log \log n}. \]

Even though \( \omega(n) \) may be as large as \( \log n / \log \log n \), it is typically much smaller.

**Theorem 6.2.** (Turan) For \( x \geq 3 \), one has
\[ \sum_{n \leq x} (\omega(n) - \log \log n)^2 \ll x \log \log x. \tag{6.1} \]

Thus, for almost all integers \( n \), one has \( \omega(n) = \log \log n + O(\log \log n) \).

**Proof.** We first show that (6.1) holds with \( \log \log n \) replaced by \( \log \log x \). Indeed, one has
\[ \sum_{n \leq x} (\omega(n) - \log \log x)^2 = S_2 - 2(\log \log x) S_1 + [x] (\log \log x)^2 . \tag{6.7} \]

Where
\[ S_j = \sum_{n \leq x} \omega(n)^j \quad (j = 1, 2). \]

But
\[ S_1 = \sum_{n \leq x} \omega(n) = \sum_{n \leq x} \sum_{\text{prime } p} \left\lfloor \frac{x}{p} \right\rfloor \]
\[ = x \sum_{p \leq x} \frac{1}{p} + O \left( \sum_{p \leq x} 1 \right) = x (\log \log x + b) + O \left( \frac{x}{\log x} \right). \]
Also,
\[ S_2 = \sum_{n \leq x} \left( \sum_{p_1 \leq x} \left( \frac{1}{p_1} \right) \sum_{p_2 \leq x} \left( \frac{1}{p_2} \right) \right) = \sum_{p_1 \leq x} \sum_{p_2 \leq x} \sum_{\text{gcd}(p_1, p_2) = 1} 1. \]

The contribution to \( S_2 \) from the terms with \( p_1 = p_2 \) is
\[ \sum_{p \leq x} \left[ \frac{x}{p} \right] = x \log \log x + O(x). \]

Meanwhile, the corresponding contribution with \( p_1 \neq p_2 \) is
\[ \sum_{p_1 \leq x} \sum_{p_2 \leq x} \frac{x}{p_1 p_2} \leq x \left( \sum_{p \leq x} \frac{1}{p} \right)^2 = x (\log \log x)^2 + O(x \log \log x). \]

Hence
\[ S_2 \leq x (\log \log x)^2 + O(x \log \log x), \]

By substituting into (6.2), we conclude that
\[ \sum_{n \leq x} \left( \omega(n) - \log \log x \right)^2 \leq x (\log \log x)^2 + O(x \log \log x) - 2(x \log \log x) \cdot (\log \log x) \cdot \log \log x \]
\[ + x (\log \log x)^2 + O(x \log \log x) \]
\[ \leq x \log \log x. \]

Finally, we have
\[ \sum_{3 \leq n \leq x} \left( \log \log n - \log \log x \right)^2 \leq \int_{3^-}^{x} (\log \log u - \log \log x)^2 \, du \]
\[ = \left[ \log \log u - \log \log x \right]_{3^-}^{x} - 2 \int_{3^-}^{x} \frac{(\log \log u - \log \log x)}{\log u} \, du \]
\[ = -2 \int_{3^-}^{x} \frac{\log \log u}{\log u} \, du + 2 \int_{3^-}^{x} \frac{\log \log x}{\log u} \, du + O \left( \int_{3^-}^{x} \frac{1}{\log u} \, du \right) \]
\[ \leq x \log \log x. \]

Thus it follows by the triangle inequality that
\[ \left( \sum_{n \leq x} \left( \frac{\omega(n)}{n} \log n \right)^2 \right)^{1/2} \leq (x \log \log x)^{1/2} + (x \log \log x) \leq (x \log \log x)^{1/2}. \]
§ 7. Additive and multiplicative characters.

Our goal in the next several lectures will be the proof of Dirichlet's theorem:

\textbf{Theorem 7.1} (Dirichlet, 1837) Suppose that \( a \in \mathbb{Z} \), \( q \in \mathbb{N} \) and \((aq) = 1\). Then there are infinitely many primes \( p \) with \( p \equiv a \) (mod \( q \)).

The detection of the congruence class \( a \mod q \) requires a consideration of Fourier analysis associated with the additive group \( \mathbb{Z}/q\mathbb{Z} \) and multiplicative group \((\mathbb{Z}/q\mathbb{Z})^*\). This we discuss in the present section.

Additive character. First we recall the theory associated with the additive group of residues \( \mathbb{Z}/q\mathbb{Z} \). Write \( e(\theta) := e^{2\pi i \theta} \), and, when \( a \in \mathbb{Z} \) and \( q \in \mathbb{N} \), put \( e_q(a) = e(a/q) \).

\textbf{Orthogonality (Ex.)}: One has

\[ q^{-1} \sum_{a=1}^{q} e(a h / q) = \begin{cases} 1, & \text{if } h = 0 \pmod{q}, \\ 0, & \text{if } h \neq 0 \pmod{q}. \end{cases} \]

When \( q \nmid h \), have

\[ \sum_{a=1}^{q} e(a h / q) = e(h/q)(e(qh/q)-1)/(e(q)-1) \]

This permits us to define Fourier expansions via the finite Fourier transform of a function \( f : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C} \):

\[ \hat{f}(k) := q^{-1} \sum_{h=1}^{q} f(h) e(-kh/q). \tag{7.1} \]

We claim that given a function \( f : \mathbb{Z} \to \mathbb{C} \), periodic with period \( q \), the formula (7.1) delivers the expansion
\[ f(n) = \sum_{k=1}^{q} \hat{f}(k) e\left(\frac{kn}{q}\right). \quad (7.2) \]

Proof of claim: By orthogonality,
\[
\sum_{k=1}^{q} \hat{f}(k) e\left(\frac{kn}{q}\right) = q^{-1} \sum_{k=1}^{q} \left( \sum_{m=1}^{q} f(m) e\left(-\frac{km}{q}\right) \right) e\left(\frac{kn}{q}\right)
\]
\[
= \sum_{m=1}^{q} \left( q^{-1} \sum_{k=1}^{q} e\left(\frac{k(n-m)}{q}\right) \right) f(m)
\]
\[
\begin{cases} = 0 & \text{when } n \not\equiv m \pmod{q} \\ = 1 & \text{when } n \equiv m \pmod{q} \end{cases}
\]
\[ = f(n). \square \]

Analogues of Parseval/Plancherel: One has (Ex.)
\[
\sum_{k=1}^{q} \left| \hat{f}(k) \right|^2 = q^{-1} \sum_{n=1}^{q} \left| f(n) \right|^2. \quad (7.3)
\]

Notice that the characters \( e\left(\frac{kn}{q}\right) \) are indeed additive for \( k \in \mathbb{Z} \), in the sense that \( e\left(\frac{kn}{q}\right) e\left(\frac{kz}{q}\right) = e\left(\frac{(k+kz)n}{q}\right) \).

Multiplicative characters.

We now turn to consider the multiplicative group \((\mathbb{Z}/q\mathbb{Z})^*\), and more generally functions \( f: \mathbb{Z} \to \mathbb{C} \) having the property of being periodic with period \( q \), and being multiplicative.

Definition 7.1: A **Dirichlet character** is a totally multiplicative function \( \chi: \mathbb{Z} \to \mathbb{C} \) having the properties:

(i) \( \chi(n) = 0 \) if and only if \( (n,q) > 1 \); and

(ii) \( \chi \) is periodic with period \( q \).
Theorem 7.2. Suppose that \( f: \mathbb{N} \to \mathbb{C} \) is multiplicative, and satisfies the conditions:

(i) \( f(n) = 0 \) whenever \( (n, q) > 1 \); and

(ii) \( f \) is periodic with period \( q \).

Then \( f \) is a Dirichlet character modulo \( q \).

Proof. It suffices to show that \( f \) is totally multiplicative with the property that \( f(n) \neq 0 \) whenever \( (n, q) = 1 \).

First, when \( (mn, q) > 1 \), then either \( (m, q) > 1 \) or \( (n, q) > 1 \), and hence \( f(m)f(n) = 0 = f(mn) \). Suppose then that \( (mn, q) = 1 \), when \( (m, q) = (n, q) = 1 \). There exists \( k \in \mathbb{Z} \) having the property that \( n + kq \equiv 1 \pmod{m} \), so that \( (m, n + kq) = 1 \). Then the multiplicative property of \( f \) ensures that

\[
\begin{align*}
  f(m(n + kq)) &= f(m)f(n + kq) \\
                 &\equiv f(mn) & \text{(periodicity)} \\
                 &\equiv f(m)f(n) & \text{(periodicity)}.
\end{align*}
\]

Then \( f \) is totally multiplicative.

In order to check that \( f(n) \neq 0 \) whenever \( (n, q) = 1 \), observe that \( f(n)^{\phi(q)} = f(n^{\phi(q)}) = f(1) = 1 \), whence \( f(n) \) is a \( \phi(q) \)-th root of unity and hence non-zero.

Notice that we can always think of a Dirichlet character \( \chi \) modulo \( q \) as being a map from \( (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^* \).

Since \( \chi(mn) = \chi(m)\chi(n) \) for all \( m \) and \( n \), moreover, one sees
that \( \chi: (\mathbb{Z}/q\mathbb{Z})^* \rightarrow \mathbb{C}^* \) is a group homomorphism.

- The identity element is the principal character \( \chi_0 \) defined by taking \( \chi_0(n) = 1 \) whenever \((n, q) = 1\).
- The inverse of a character \( \chi \) is the conjugate \( \overline{\chi} \) defined by taking \( \overline{\chi}(n) = \overline{\chi(n)} \) for \((n, q) \neq 1\).

In order to see the latter, one observes that the \( \chi \) values of a character are roots of unity. Indeed, as in the conclusion of the proof of Theorem 7.2, we see that when \( \chi \) is a character modulo \( q \), then \( \chi(n) = 1 \) whenever \((n, q) = 1\).

In order to understand the structure of the group of characters, it is useful to start with some basic examples.

**Example 7.3.** Let \( q \) be a prime number, say \( p \), and consider the character \( \chi \) modulo \( p \).

Note that \((\mathbb{Z}/p\mathbb{Z})^*\) is a cyclic group of order \( \phi(p) = p-1 \) having a generator \( g \) (a primitive root). Thus, given an integer \( n \) with \((n, p) = 1\), there is an integer \( u = \text{ind}_n \) having the property that \( n \equiv g^u \pmod{p} \). We then have

\[ \chi(n) = \chi(g^u) = \chi(g)^u. \]

So the character \( \chi \) is determined by its value at a single primitive root \( g \) modulo \( p \). This value, moreover, is a \((p-1)\)-th
root of unity.

It follows that there are precisely \((p-1)\) characters modulo \(p\), namely
\[
\chi_k : (\mathbb{Z}/p\mathbb{Z})^* \rightarrow C^*
\]
defined by taking
\[
\chi_k(n) = e \left( \frac{kn \ind(n)}{p-1} \right) \quad (0 \leq k \leq p-2).
\]
Since the values \(\chi_k(g) = e \left( \frac{k}{p-1} \right)\) are distinct for
\(0 \leq k \leq p-2\), these Dirichlet characters are all distinct. Moreover,
the character \(\chi_0(\cdot) = 1\) is indeed the principal character.

Notice that the group of characters \(\{\chi_0, \ldots, \chi_{p-2}\} = \{\chi_k^* : 0 \leq k \leq p-2\}\) is isomorphic to \((\mathbb{Z}/p\mathbb{Z})^*\).

---

**Example 7.4.** Let \(q\) be an odd prime power, say \(p^h\), or else be 2 or 4, and consider the character \(\chi mod q\).

In this situation \((\mathbb{Z}/p^h\mathbb{Z})^*\) (either \(p\) odd, or \(p=2\) and \(h=1,2\)) is again cyclic of order \(\phi(p^h)\), having a generator \(g\) (a primitive root). Indeed, if \(g\) is a primitive root modulo \(p^2\), then it is primitive modulo \(p^h\) for \(h \geq 3\), when \(p\) is odd. In such circumstances, a discussion paralleling
that of Example 7.3 shows that there are precisely
\[
\phi(p^h) = p^h - p^{h-1}
\]
characters modulo \(p^h\), namely
\( \chi_k : (\mathbb{Z}/p^k \mathbb{Z})^* \rightarrow \mathbb{C}^* \),
defined by taking
\[
\chi_k(n) = e\left( k \, \text{ind}_{z_i}(n) / \varphi(p^k) \right) \quad (0 \leq k \leq \varphi(p^k)-1).
\]
Also, writing \( \hat{G} \) for the group of characters modulo \( p^k \), one sees that \( \hat{G} = \langle \chi_i \rangle \cong \mathbb{C}_{\varphi(p^k)} \cong (\mathbb{Z}/p^k \mathbb{Z})^* \).

\[\text{Example 7.5.}\]
Let \( q = 2^k \) with \( h \geq 3 \), and consider the character \( \chi \) modulo \( 2^h \).

In this case \( (\mathbb{Z}/2^h \mathbb{Z})^* \) is generated by the elements \(-1\) and \( 5 \), and is isomorphic to \( C_2 \times C_2^{h-2} \). Given an integer \( n \) with \( (n, 2^h) = 1 \), there is a pair of integers \( x, y \) and \( n \equiv (-1)^y 5^x \pmod{2^h} \).

We then have
\[
\chi(n) = \chi((-1)^y 5^x) = \chi(-1)^y \chi(5)^x.
\]
So the character \( \chi \) is determined by its values at \(-1\) and \( 5 \).

Moreover, one has \( \chi(5)^{2^{h-2}} = 1 \) and \( \chi(-1)^2 = 1 \).

It follows that there are precisely \( 2^{h-1} = \varphi(2^h) \) characters modulo \( 2^h \), namely
\[
\chi_{x,k} : (\mathbb{Z}/2^h \mathbb{Z})^* \rightarrow \mathbb{C}^*,
\]
defined by taking
\[
\chi_{x,k}(n) = e\left( \frac{k \, \text{ind}_{-1}(n)}{2} + \frac{x \, \text{ind}_s(n)}{2^{h-2}} \right).
\]
In this case, the group of characters
\[ \{ \chi_{x,k} : x \in \{0,1\}, 0 \leq k \leq 2^{h-2} - 1 \} = \langle \chi_{0,1}, \chi_{1,0} \rangle \]
\[ \cong C_2 \times C_2^{h-2} \cong \left( \mathbb{Z}/2^h \mathbb{Z} \right)^*. \]

---

**Example 7.6.** Let \( q \in \mathbb{N} \) with \( q > 1 \), and consider the character \( \chi \) modulo \( q \).

By the Chinese Remainder Theorem, one finds that
\[ \left( \mathbb{Z}/q \mathbb{Z} \right)^* \cong \bigotimes_{p^* \mid q} \left( \mathbb{Z}/p^* \mathbb{Z} \right)^*, \]
and so \( \left( \mathbb{Z}/q \mathbb{Z} \right)^* \) is generated by suitable representatives of primitive roots \( g_{p^*} \), for \( p^* \mid q \), with \( -1 \) and \( 5 \) if \( 8 \mid q \). It follows that there are precisely \( \varphi(q) \) characters modulo \( q \), namely
\[ \chi_{k} : \left( \mathbb{Z}/q \mathbb{Z} \right)^* \rightarrow \mathbb{C}^* \]
defined by taking
\[ \chi_{k}(n) = e \left( \sum_{p^* \mid q} \frac{k \cdot \text{ind}_{p^*}(n)}{\varphi(p^*)} \right), \text{ when } k \neq q, \]
and
\[ \chi_{\overline{k},k}(n) = e \left( \frac{\text{ind}_{\overline{k}}(n)}{2} + \frac{\text{ind}_{5}(n)}{2^{h-2}} + \sum_{p^* \mid q \text{ odd}} \frac{k \cdot \text{ind}_{p^*}(n)}{\varphi(p^*)} \right), \]
when \( 2^h \mid q \) with \( h \geq 3 \), where
\[ k_0 \in \{0,1\}, 0 \leq k_1 \leq 2^{h-2} - 1, \text{ and} \]
\[ 0 \leq k_{p^*} \leq \varphi(p^*) - 1 \text{ (p odd).} \]

Notice that \( \{ \chi_{l,k} \} \cong \left( \mathbb{Z}/q \mathbb{Z} \right)^* \) in this case also.
Theorem 7.7. The multiplicative group \((\mathbb{Z}/q\mathbb{Z})^*\) of reduced residue classes \((n \mod q)\) has \(\varphi(q)\) Dirichlet characters. If \(\chi\) is such a character, then

\[
\sum_{\substack{n=1 \\ (n,q)=1}}^q \chi(n) = \begin{cases} 
\varphi(q), & \text{when } \chi = \chi_0, \\
0, & \text{when } \chi \neq \chi_0.
\end{cases}
\]

Moreover, when \((n,q)=1\), one has

\[
\sum_{\chi \in \chi(q)} \chi(n) = \begin{cases} 
\varphi(q), & \text{when } n \equiv 1 \pmod{q}, \\
0, & \text{otherwise}.
\end{cases}
\]

Here, we write \(\chi(q)\) for the group of \(\varphi(q)\) Dirichlet characters modulo \(q\).

Proof. When \(\chi = \chi_0\), the first assertion is plain. Suppose then that \(\chi \neq \chi_0\). There exists an integer \(m\) with \((m,q)=1\) and \(\chi(m) \neq 1\). But then, by noting that multiplication by \(m\) permutes the reduced residues modulo \(q\), we deduce that

\[
\sum_{n=1}^q \chi(n) = \sum_{n=1}^q \chi(mn) = \chi(m) \sum_{n=1}^q \chi(n),
\]

whereas

\[
(\chi(m) - 1) \sum_{\substack{n=1 \\ (n,q)=1}}^q \chi(n) = 0 \implies \sum_{\substack{n=1 \\ (n,q)=1}}^q \chi(n) = 0.
\]

For the second assertion, when \(n \equiv 1 \pmod{q}\) one has

\[
\sum_{\chi \in \chi(q)} \chi(n) = \sum_{\chi \in \chi(q)} \chi(1) = \text{card} \(\chi(q)\) = \varphi(q).\]
Also, when \((n,q) = 1\) and \(n \not\equiv 1 \pmod{q}\), there is a character \(\chi_1\) modulo \(q\) with \(\chi_1(n) \not= 1\) (by our characterization of Dirichlet characters). Thus

\[
\sum_{\chi \in \chi(q)} \chi(n) = \sum_{\chi \in \chi(q)} (\chi \chi_1)(n) = \chi_1(n) \sum_{\chi \in \chi(q)} \chi(n)
\]

use group structure of \(\chi(q)\)

\[
\Rightarrow (\chi_1(n) - 1) \sum_{\chi \in \chi(q)} \chi(n) = 0 \quad \Rightarrow \sum_{\chi \in \chi(q)} \chi(n) = 0, \quad \square
\]

Another consequence of our characterization of characters.

**Theorem 7.8.** (i) If \(\chi_i\) is a character modulo \(q_i\) for \(i=1,2\), then \(\chi_3 : \mathbb{Z} \to \mathbb{C}\) is a character modulo \(q\), where \(q = [q_1,q_2]\) and \(\chi_3(n) = \chi_1(n) \chi_2(n)\) for \(n \in \mathbb{Z}\).

(ii) If \(q = q_1 q_2\) with \((q_1,q_2) = 1\), and \(\chi\) is a character modulo \(q\), then there exist unique characters \(\chi_i \pmod{q_i}\) \((i=1,2)\) such that \(\chi(n) = \chi_1(n) \chi_2(n)\) for all \(n\).

**Exercise:** Analogues of Parseval / Plancherel:

(i) \[
\frac{1}{\phi(q)} \sum_{\chi \in \chi(q)} \left| \sum_{n=1}^{q} c_n \chi(n) \right|^2 = \sum_{\substack{n=1 \cr (n,q) = 1}}^{q} |c_n|^2 \quad (c_n \in \mathbb{C})
\]

(ii) \[
\frac{1}{\phi(q)} \sum_{n=1}^{q} \left| \sum_{\chi \in \chi(q)} c_{\chi} \chi(n) \right|^2 = \sum_{\chi \in \chi(q)} |c_{\chi}|^2 \quad (c_{\chi} \in \mathbb{C}),
\]
§ 8. Dirichlet $L$-functions and Dirichlet's theorem on primes.

The Dirichlet $L$-function $L(s, \chi)$ associated with the character $\chi$ modulo $q$ is defined for $\sigma > 1$ by the Dirichlet series

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}. \quad (8.1)$$

This series is, of course, very much analogous to the series defining $\zeta(s)$, and we can develop the associated theory, and consequences for prime number sums, in a manner analogous to that employed above.

First we consider questions of convergence in (8.1), and the associated Euler product. Plainly, the series (8.1) is absolutely convergent for $\sigma > 1$, since

$$\sum_{n=1}^{\infty} |\chi(n) n^{-s}| = \sum_{n=1}^{\infty} n^{-\sigma} < \infty.$$

When $\chi \equiv \chi_0$, we have

$$\sum_{1 \leq n \leq q} \chi(n) = k \sum_{n=1}^{q} \chi(n) = 0,$$

when

$$\left| \sum_{1 \leq n \leq x} \chi(n) \right| \leq q. \quad (8.2)$$

Thus Theorem 2.4 implies that the series (8.1) converges for $\sigma > 0$.

The absolute convergence of (8.1) for $\sigma > 1$ ensures that one has the Euler product

$$L(s, \chi) = \prod_{p} (1 - \chi(p) p^{-s})^{-1} \quad (\sigma > 1). \quad (8.2)$$
Hence, in particular, one sees that

\[ L(s, \chi_0) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\log n} \chi(n) n^{-s} = \frac{\zeta(s)}{\prod_{p} (1 - p^{-s})}, \quad (\sigma > 1). \] 

This relation shows that in fact \( L(s, \chi_0) \) is analytic for \( \sigma > 0 \) except for a simple pole at \( s = 1 \).

As in our previous discussion for \( \zeta(s) \), information for prime sums may be inferred by examining the logarithmic derivative of \( L(s, \chi) \). We summarize such deliberations in the next theorem.

**Theorem 8.1.** If \( \chi \neq \chi_0 \), then \( L(s, \chi) \) is analytic for \( \sigma > 0 \).

Meanwhile, when \( \sigma > 0 \) the function \( L(s, \chi_0) \) is analytic except for a simple pole at \( s = 1 \) with residue \( \varphi(q)/q \). In either case, when \( \sigma > 1 \), one has

\[ \log L(s, \chi) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} \chi(n) n^{-s}, \]

and

\[ \frac{L'}{L}(s, \chi) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} \chi(n) n^{-s}. \]

**Proof.** The only claim to check concerns the residue of \( L(s, \chi_0) \) at \( s = 1 \). For this we have

\[ \lim_{s \to 1} \left( (s-1) \zeta(s) \prod_{p \mid q} (1 - p^{-s}) \right) = \prod_{p \mid q} (1 - p^{-1}) = \varphi(q) / q. \]

So far as Dirichlet's theorem on primes \( p \equiv a \pmod{q} \) is concerned, we are interested in the sum

\[ \sum_{1 \leq n \leq x, \ n \equiv a \pmod{q}} \Lambda(n) \text{ as an analogue of } \psi(x) = \sum_{1 \leq n \leq x} \Lambda(n). \]
However, the congruence class \( n \equiv a \pmod{q} \) is easily singled out via the identity (valid whenever \((q,a) = 1\))

\[
\frac{1}{\phi(q)} \sum_{\chi \in \chi(q)} \chi(a) \chi(n) = \begin{cases} 
1, & \text{when } n \equiv a \pmod{q}, \\
0, & \text{when } n \not\equiv a \pmod{q}.
\end{cases}
\]

\[
\left( \sum_{\chi \in \chi(q)} \chi(a^{-1}n) \right) = \begin{cases} 
1, & \text{when } a^{-1}n \equiv 1 \pmod{q}, \\
0, & \text{when } a^{-1}n \not\equiv 1 \pmod{q}.
\end{cases}
\]

Thus we have

\[
\sum_{\substack{n = 1 \\
\Lambda(n) \equiv a \pmod{q} \\, \text{and} \, n \not\equiv 0 \pmod{q} \, \text{and} \, n \not\equiv a \pmod{q}}} \frac{\Lambda(n)}{\phi(q)} n^{-s} = \frac{1}{\phi(q)} \sum_{n=1}^{\infty} \Lambda(n) n^{-s} \sum_{\chi \in \chi(q)} \chi(a) \chi(n)
\]

\[
= \frac{1}{\phi(q)} \sum_{\chi \in \chi(q)} \chi(a) \cdot \left( -\frac{L'}{L} (s, \chi) \right) \quad (s > 1).
\]

We can now discern a strategy for proving that there are infinitely many primes \( p \) with \( p \equiv a \pmod{q} \) when \((q,a) = 1\). It is tempting to believe that when \( \chi \neq \chi_0 \), the Dirichlet series

\[
-\frac{L'}{L} (s, \chi) = \sum_{n=1}^{\infty} \frac{\Lambda(n) \chi(n)}{n^s}
\]

converges for \( s > 0 \). Certainly, for \( s > 1 \), the series is analytic (since the same is true of \( L(s, \chi) \) and hence also \( \log L(s, \chi) \)) pointed that \( L(s, \chi_0) \neq 0 \). Then pointed that \( L(1, \chi) \neq 0 \), one has

\[
\lim_{s \to 1^+} \frac{L'}{L} (s, \chi) = \frac{L'}{L} (1, \chi) < \infty.
\]

Moreover, the function \( L(s, \chi_0) \) has a simple pole at \( s = 1 \), and hence \(-\frac{L'}{L} (s, \chi_0)\) also has a simple pole at \( s = 1 \) with residue 1. Indeed, by examining the Laurent series for \( L(s, \chi_0) \)
about \( s = 1 \), namely
\[
L(s, \chi_0) = \frac{\varphi(q)/q}{s-1} + a_0 + \sum_{i \geq 1} a_i (s-1)^i,
\]
we see that
\[
\log L(s, \chi_0) = \log \left( \frac{1}{s-1} \right) + \log \left( \frac{\varphi(q)/q}{1} \right) + O(s-1),
\]
whence
\[
- \frac{L'(s, \chi_0)}{L} = \frac{1}{s-1} + b_0 + \sum_{i \geq 1} b_i (s-1)^i.
\]
Hence (8.4) yields the formula
\[
\sum_{\substack{n=1 \\
 \text{n \equiv a (mod q)}}} \Lambda(n) n^{-s} = \frac{1}{\varphi(q)} (s-1)^{-1} + c_0 + \sum_{i \geq 1} c_i (s-1)^i,
\]
as \( s \to 1+ \).

It remains to prove that \( L(1, \chi) \neq 0 \) for \( \chi \neq \chi_0 \).

Theorem 8.2. (Dirichlet). When \( \chi \) is a character modulo \( q \) with \( \chi \neq \chi_0 \), one has \( L(1, \chi) \neq 0 \).

Proof. The argument required to establish that \( L(1, \chi) \neq 0 \)
\((\chi \neq \chi_0)\) divides naturally into two halves, and involves some lemmas; the proof of which we will provide en passant. The division of cases is natural in hindsight, and a common theme in the subject — namely that of clarifying \( \chi \) as real or complex.

Case 1: \( \chi \) complex. A character is called real if all of its values are real, and is otherwise called complex. Note that if \( \chi \) is a complex character, then the character \( \overline{\chi} \)
is a distinct complex character. Thus the complex characters come in conjugate pairs that share many of the same properties.

Observe first that, in view of Theorem 8.1, one has

$$\log \left( \prod_{\chi \in \chi(q)} L(s, \chi) \right) = \sum_{\chi \in \chi(q)} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} \chi(n)n^{-s}, \quad (s > 1)$$

whence

$$\prod_{\chi \in \chi(q)} L(s, \chi) = \exp \left( \phi(q) \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-s} \right) \quad (n \equiv 1 \pmod{q})$$

By taking $s = \sigma > 1$, therefore, we find that the last sum is a non-negative real number, whence

$$\prod_{\chi \in \chi(q)} L(s, \chi) \geq 1. \quad (8.5)$$

Consider the Laurent expansions of $L(s, \chi)$ about $s = 1$ for $\chi \in \chi(q)$. The function $L(s, \chi_0)$ is analytic for $s > 0$ except for a simple pole at $s = 1$ with residue $\phi(q)/q$, so

$$L(s, \chi_0) = \frac{\phi(q)/q}{(s-1)} + c_0 + c_1(s-1) + \cdots$$

for suitable $c_0, c_1, \ldots \in \mathbb{C}$. Meanwhile, when $\chi \neq \chi_0$, the function $L(s, \chi)$ is analytic for $s > 0$, whence

$$L(s, \chi) = c_0(\chi) + c_1(\chi)(s-1) + \cdots$$

for suitable $c_i(\chi) \in \mathbb{C}$. Then we see via (8.5) that one can have $L(1, \chi) = 0$ for at most one character $\chi \neq \chi_0$, for otherwise the product on the right hand side of (8.5) would have a zero at $s = 1$. Moreover, any zero of $L(s, \chi)$ must be simple.

Suppose that $\chi$ is a complex character for which $L(1, \chi) = 0$. 
Then $\chi$ is a character with $\chi \neq \chi_0$ and $L(1, \chi) = 0$, contradicting our observation that there is at most one character $\chi$ with $\chi \neq \chi_0$ for which $L(1, \chi) = 0$. This assertion that $L(1, \chi) = 0$ appears obvious enough, but requires some justification. One can see this by brute-force by examining the partial sums of $L(\sigma, \chi)$ as $\sigma \to 1+$, which shows that

$$
\lim_{\sigma \to 1+} L(\sigma, \chi) = \lim_{\sigma \to 1+} L(\sigma, \chi) = \lim_{\sigma \to 1+} L(\sigma, \chi) = 0.
$$

A more rigorous approach would be to apply the Schwarz reflection principle to show that $L(s, \chi) = L(\overline{s}, \chi)$ (hint: consider $L(s, \chi) \pm L(s, \overline{\chi})$).

It remains to show that $L(1, \chi) \neq 0$ for real characters $\chi$.

Case 2: $\chi$ real and $\chi \neq \chi_0$. If $\chi$ is a real character, then $\chi^2 = \chi_0$. A character is called quadratic if it has order 2 in the character group, whence $\chi^2 = \chi_0$ but $\chi \neq \chi_0$. Such characters are tricky to handle owing to their solitary life-style. Rather than deal with $\chi$ directly, we derive an auxiliary quantity exhibiting more amenable positivity. This is motivated by the observation that since when $L(1, \chi) = 0$,

$$
L(s, \chi) = c_0(\chi)(s-1) + c_2(\chi)(s-1)^2 + \cdots
$$

and

$$
\zeta(s) = \frac{1}{s-1} + c_0 + \cdots,
$$

we have

$$
\zeta(s) L(s, \chi) = c_1(\chi) + b_1(s-1) + b_2(s-1)^2 + \cdots
$$

for suitable $b_i \in \mathbb{C}$, wherein $c_1(\chi) \neq 0$ (we don't need to know this, but else we contradict (8.5)).
One has
\[ S(s) L(s, \chi) = \sum_{n=1}^{\infty} r(n)n^{-s} \quad (s>1), \]
where
\[ r(n) = \sum_{d|n} \chi(d). \]
The function \( r(n) \) is multiplicative, and one has
\[ r(p^h) = \begin{cases} 
1 & \text{when } p \nmid q, \\
h+1 & \text{when } \chi(p) = +1, \\
1 & \text{when } \chi(p) = -1 \text{ and } 2h, \\
0 & \text{when } \chi(p) = -1 \text{ and } 2h. 
\end{cases} \]

[Note here that when \( \chi(p) = -1 \), one has
\[ r(p^{2l}) = \sum_{m=0}^{2l} \chi(p)^m = (1-1) + \ldots + (1-1) + 1 = +1, \]
and
\[ r(p^{2l-1}) = \sum_{m=0}^{2l-1} \chi(p)^m = (1-1) + \ldots + (1-1) = 0. \]
]

Thus we see that \( r(n) \geq 0 \) for all \( n \in \mathbb{N} \), and moreover that \( r(n^2) \geq 1 \) for all \( n \in \mathbb{N} \).

Let \( \sigma_c \) denote the abscissa of convergence of the Dirichlet series \( \sum_{n=1}^{\infty} r(n)n^{-s} \). Then since
\[ \sum_{n=1}^{\infty} r(n)n^{-1/2} \geq \sum_{n=1}^{\infty} r(n^2)n^{-1} \geq \sum_{n=1}^{\infty} n^{-1} = +\infty, \]
it is apparent that \( 1 \leq \sigma_c \leq \frac{1}{2} \). However, when \( L(1, \chi) = 0 \), the function \( S(s) L(s, \chi) \) is analytic for \( s > 0 \), including
the point \( s = 1 \) in view of (8.6). This analyticity at \( s = \sigma_c \) is untenable (i.e., leads to a contradiction), in view of the following lemma of Landau — and the desired conclusion \( L(1, \chi) \neq 0 \) follows at once. \( \|
\)

**Lemma 8.3** (Landau) Let \( \alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s} \) be a Dirichlet series whose abscissa of convergence \( \sigma_c \) is finite. Then provided that \( a_n \geq 0 \) for all \( n \in \mathbb{N} \), the point \( s = \sigma_c \) is a singularity of the function \( \alpha(s) \).

**Proof.** By replacing \( a_n \) by \( a_n n^{-\sigma_c} \), we may suppose that the abscissa of convergence of \( \alpha(s) \) is \( \sigma_c = 0 \). If \( \alpha(s) \) were not analytic at \( s = 0 \), then it would possess a singularity there, and the proof would be complete. We may therefore suppose to the contrary that \( \alpha(s) \) is analytic at \( s = 0 \), and hence also analytic in the domain

\[
\mathcal{D} = \{ s \in \mathbb{C} : \sigma > 0 \text{ or } |s| < 1 \},
\]

provided that \( s \) is sufficiently small. Our strategy now is to show that the series \( \sum_{n=1}^{\infty} a_n n^{-s} \) converges to \( \alpha(s) \) for a real number \( \sigma < 0 \), and this we achieve by considering a power series expansion for \( \alpha(s) \) about \( s = 1 \).

Since \( s = 1 \) is well inside the half-plane of convergence of \( \alpha(s) \), we may apply a Taylor expansion to obtain
\[ \alpha(s) = \sum_{k=0}^{\infty} c_k (s-1)^k, \quad (8.7) \]

where
\[ c_k = \frac{\alpha^{(k)}(1)}{k!} = \frac{1}{k!} \sum_{n=1}^{\infty} a_n (-\log n)^k n^{-1}. \]

The radius of convergence of (8.7) is the distance of 1 to the nearest singularity of \( \alpha(s) \).

Since \( \alpha(s) \) is analytic in \( \mathbb{D} \), and the nearest points not in \( \mathbb{D} \) are \( \pm i \delta \), we see that the radius of convergence of the series (8.7) is at least \( \sqrt{1+\delta^2} = 1+\delta' \), say. Thus, when \( |s-1| < 1+\delta' \), one has
\[ \alpha(s) = \sum_{k=0}^{\infty} \frac{(1-s)^k}{k!} \sum_{n=1}^{\infty} a_n (-\log n)^k n^{-1}. \]

If \( s=\sigma < 1 \), then all the terms are non-negative, and we may rearrange the summation for \( -\delta' < \sigma < 1 \) to deduce that
\[ \alpha(\sigma) = \sum_{n=1}^{\infty} a_n n^{-1} \sum_{k=0}^{\infty} \frac{(1-\sigma)^k (-\log n)^k}{k!}. \]
\[ = \sum_{n=1}^{\infty} a_n n^{-1} \exp \left( (1-\sigma) \log n \right) = \sum_{n=1}^{\infty} a_n n^{-\sigma}. \]

Thus \( \alpha(s) \) converges at \( s = -\delta' / 2 \), contrary to our assumption that \( \alpha(s) \) has absence of convergence \( \sigma_c = 0 \). We are therefore forced to conclude that \( \alpha(s) \) is not analytic at \( s = 0 \), but instead has a singularity at \( s = 0 \).
As an immediate corollary of Theorem 8.2, we deduce Dirichlet's theorem.

**Corollary 8.4.** (Dirichlet). If $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$, then one has

$$
\sum_{p \equiv a \ (\text{mod} \ q)} \frac{\log p}{p} = +\infty.
$$

In particular, there are infinitely many primes $p$ with $p \equiv a \ (\text{mod} \ q)$.

**Proof.** Recall from (8.4) that when $(a, q) = 1$ and $\sigma > 1$, one has

$$
\sum_{n=1}^{\infty} \Lambda(n) n^{-s} = -\frac{1}{\phi(q)} \sum_{\chi \in \chi(q)} \frac{\Lambda^\star(n)}{L'(s, \chi)}.
$$

Analytic for $\sigma > 0$ except at zeros of $L(s, \chi)$, when $\chi \neq \chi_0$

$$
= \frac{1}{\phi(q)(s-1)} + O_q(1) \quad \text{as} \quad s \to 1+.
$$

Then

$$
\lim_{s \to 1+} \sum_{n=1}^{\infty} \Lambda(n) n^{-s} = +\infty,
$$

and indeed

$$
\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} = +\infty.
$$

Moreover, the contribution to this sum from prime powers is

$$
\sum_{p^k \equiv a \ (\text{mod} \ q)} \frac{\log p}{p^k} \leq \sum_{p \equiv a \ (\text{mod} \ q)} \frac{\log p}{p} \sum_{k=2}^{\infty} p^{-k} = \sum_{p \equiv a \ (\text{mod} \ q)} \frac{\log p}{p(p-1)} < \infty,
$$

whence

$$
\sum_{p \equiv a \ (\text{mod} \ q)} \frac{\log p}{p} = +\infty.
$$
We record some consequences of this circle of ideas in the next theorem.

**Theorem 8.5.** When \( a \in \mathbb{Z} \) and \( q \in \mathbb{N} \) satisfy \((a,q)=1\), and \( x \geq 2\), one has:

(a) \[
\sum_{\substack{n \leq x \\ n \equiv a \pmod{q}}} \Lambda(n) n^{-1} = \frac{1}{\varphi(q)} \log x + O_q(1) \quad \text{and} \quad \sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} \frac{\log p}{p} = \frac{1}{\varphi(q)} \log x + O_q(1),
\]

(b) \[
\sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} \frac{1}{p} = \frac{1}{\varphi(q)} \log \log x + b(q,a) + O_q \left( \frac{1}{\log x} \right),
\]

where \( b(q,a) = \frac{1}{\varphi(q)} \left( C_0 + \sum_{pq \leq x} \log \left( 1 - \frac{1}{p} \right) + \sum_{x \neq \chi_0} \overline{\chi}(a) \log L(1, \chi) \right) - \sum_{p^k \equiv a \pmod{q}} \frac{1}{k} \right), \quad k > 1.
\]

(c) \[
\prod_{\substack{p \leq x \\ p \equiv a \pmod{q}}} \left( 1 - \frac{1}{p} \right)^{-1} = c(q,a) \left( \log x \right)^{1/\varphi(q)} \left( 1 + O_q \left( \frac{1}{\log x} \right) \right),
\]

where \( c(q,a) = \left( e^{C_0} \varphi(q) \prod_{x \neq \chi_0} \overline{\chi}(a) \prod_{p} \left( 1 - \frac{1}{p} \right)^{-\frac{x(p)}{p}(1 - \frac{\chi(p)}{p})} \right). \)

**Proof.** We will leave this as an exercise for the enthusiastic, as a consequence of the following lemma.

**Lemma 8.6.** Suppose that \( \chi \) is a non-principal \( \chi \)-character. Then for \( x \geq 2 \), one has
(a) \[ \sum_{n \leq x} \frac{\chi(n) \Lambda(n)}{n} \ll 1 \quad \text{and} \quad \sum_{p \leq x} \frac{\chi(p) \log p}{p} \ll 1 ; \]

(b) \[ \sum_{p \leq x} \frac{\chi(p)}{p} = b(x) + O_x \left( \frac{1}{\log x} \right) ; \]
where \[ b(x) = \log L(1, \chi) - \sum_{p^k (k \geq 1)} \frac{\chi(p^k)}{k p^k} ; \]

(c) \[ \prod_{p \leq x} \left( 1 - \frac{\chi(p)}{p} \right)^{-1} = L(1, \chi) + O_x \left( \frac{1}{\log x} \right). \]

Note: One can apply these notations, and the orthogonality of characters, to detect the reduced residue class a modulo q, to obtain the conclusions of Theorem 8.5. The missing character is \( \chi_0 \), and \( \chi_0(n) = \begin{cases} 1, & \text{when } (q, n) = 1 ; \\ 0, & \text{when } (q, n) > 1 . \end{cases} \)

such as \[ \sum_{n \leq x} \frac{\chi_0(n) \Lambda(n)}{n} \] are easily estimated by means of corresponding estimates for all the primes. Thus, for example,

\[ \sum_{n \leq x} \frac{\chi_0(n) \Lambda(n)}{n} = \sum_{n \leq x} \frac{\Lambda(n)}{n} - \sum_{p^k \leq x} \frac{\log p}{k p^k} = \sum_{n \leq x} \frac{\Lambda(n)}{n} + O_q(1), \]

where

\[ \sum_{n \leq x} \frac{\chi_0(n) \Lambda(n)}{n} = \log x + O_q(1). \]

Proof of Lemma 8.6. We establish part (a) using ideas from the corresponding proof of Theorem 5.3. One has

\[ \sum_{n \leq x} \frac{\chi(n) \log n}{n} = \sum_{n \leq x} \frac{\chi(n)}{n} \sum_{d \mid n} \Lambda(d) = \sum_{d \leq x} \frac{\Lambda(d) \chi(d)}{d} \sum_{m \leq x/d} \frac{\chi(m)}{m} . \]
The last sum is fairly easily estimated. One has
\[ \sum_{m \leq y} \frac{\lambda(m)}{m} = L(1, \chi) - \sum_{m > y} \frac{\lambda(m)}{m}. \]

To estimate the tail, put \( S(x) = \sum_{n \leq x} \lambda(n) \). Then
\[ \sum_{m > y} \frac{\lambda(m)}{m} = \int_{y}^{\infty} u^{-1} ds(u) = -S(y) + \int_{y}^{\infty} S(u) u^{-2} du \ll y^{-1}. \]

Recall here that \( |S(y)| \ll y = O(y) \). Hence one sees that
\[ \sum_{n \leq x} \frac{\lambda(n) \log n}{n} = \sum_{d \leq x} \frac{\Lambda(d) \chi(d)}{d} \left( L(1, \chi) + O_{\chi} \left( \frac{1}{\log d} \right) \right) \]
\[ = L(1, \chi) \sum_{d \leq x} \frac{\Lambda(d) \chi(d)}{d} + O_{\chi} \left( \frac{1}{x} \sum_{d \leq x} \frac{\Lambda(d)}{d} \right) = O(x). \]

Observe next that \( L(1, \chi) \neq 0 \), so we have
\[ \sum_{d \leq x} \frac{\Lambda(d) \chi(d)}{d} = \frac{1}{L(1, \chi)} \sum_{n \leq x} \frac{\lambda(n) \log n}{n} + O_{\chi}(1) \]
\[ = \frac{1}{L(1, \chi)} \left( -L'(1, \chi) - \sum_{n > x} \frac{\lambda(n) \log n}{n} \right) + O(1). \]

Again using \( L - S \) integration, we have
\[ \sum_{n > x} \frac{\lambda(n) \log n}{n} = \int_{x}^{\infty} \frac{\log u}{u} dS(u) \]
\[ = -S(x) \log x - \int_{x}^{\infty} S(u) (1 - \log u) u^{-2} du \ll \frac{\log x}{x}. \]

Thus
\[ \sum_{d \leq x} \frac{\Lambda(d) \chi(d)}{d} = -\frac{L'(1, \chi)}{L(1, \chi)} + O(1) \ll 1. \]
The remaining conclusions follow much as in the argument of Theorem 5.3. 

89. Dirichlet series and Mellin transforms.

Consider a Dirichlet series \( \alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s} \) with abscissa of convergence \( \sigma_c \), and consider also the partial sum of coefficients
\[ A(x) = \sum_{1 \leq n \leq x} a_n. \]
We saw in Theorem 2.4 that whenever \( \sigma > \max \{ 0, \sigma_c \} \), one has
\[ \alpha(s) = S \int_{1}^{\infty} A(x) x^{-s-1} \, dx. \quad (9.1) \]

This is a special example of a Mellin transform.

**Definition 9.1.** Suppose that \( f: \mathbb{R} \to \mathbb{C} \). Then the **Mellin transform** \( F(s) \) of the function \( f(x) \) is defined by
\[ F(s) = \int_{0}^{\infty} f(x) x^{s-1} \, dx. \]

By analogy with the theory of Fourier transforms and their inverses, it transpires that under appropriate conditions, one may reverse this transformation.

**Definition 9.2.** When \( F: \mathbb{C} \to \mathbb{C} \), we define the **inverse Mellin transform** \( f(x) \) of the function \( F(s) \) by putting
\[ f(x) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} F(s) x^{-s} \, ds \]
(when \( \sigma_0 \) is suitably large).

The special case of the inverse Mellin transform of interest to us
is embodied in Perron's formula.

**Theorem 9.1 (Perron's formula).** Let $\sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series with abscissa of convergence $\sigma_c$. Then whenever $\sigma_0 > \max \{ \sigma_c, \sigma_1 \}$ and $x > 0$, one has

$$\sum_{n \leq x} a_n = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\alpha(s) x^s}{s} \, ds.$$

Here, in the summation over $a_n$, in the situation that $x \in \mathbb{N}$, the final term is counted with weight $1/2$.

Thus

$$\sum_{n \leq x} a_n = a_1 + \ldots + a_{x-1} + \frac{1}{2} a_x \quad \text{when } x \in \mathbb{N}.$$

**Proof.** The key observation is that, by the calculus of residues, one has

$$\frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \frac{y^s}{s} \, ds = \begin{cases} 1, & \text{when } y > 1, \\ 0, & \text{when } 0 < y < 1, \end{cases}$$

all provided that $\sigma_0 > 0$. In order to see this, note that we may consider the following contour integrals and take the limit as $R \to \infty$:
When \( y > 1 \), we find that with \( \cos (2\pi \alpha) = \sigma_0 \), one has that the circular segment of the contour contributes

\[
\frac{1}{2\pi i} \int_{\alpha}^{1-\alpha} \frac{y \text{Re}(\theta)}{\text{Re}(\theta)} \, d(\text{Re}(\theta)) = \int_{\alpha}^{1-\alpha} y \text{Re}(\theta) \, d\theta.
\]

But \( R \sin (\frac{\pi}{2} - 2\pi \alpha) = \sigma_0 \), so that \( \alpha = \frac{1}{4} + O(1/R) \). Then

\[
\left| \int_{\alpha}^{1-\alpha} y \text{Re}(\theta) \, d\theta \right| \leq \int y \sigma_0 \, d\theta + \int_{1-\alpha}^{1-\alpha} y \sqrt{\log R} \, d\theta
\]

for a suitable \( C > 0 \), whence

\[
\left| \int_{\alpha}^{1-\alpha} y \text{Re}(\theta) \, d\theta \right| \ll y, \sigma_0, \sqrt{\log R} / R \rightarrow 0 \text{ as } R \rightarrow \infty.
\]

But the function \( y^s / s \) is analytic within the contour except for a simple pole at \( s = 0 \), whence

\[
\frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \frac{y^s}{s} \, ds = \lim_{R \rightarrow \infty} \frac{1}{2\pi i} \int_{C_R} \frac{y^s}{s} \, ds = \lim_{s \rightarrow 0} y^s = 1.
\]

Meanwhile, when \( 0 < y < 1 \), a similar argument shows that the circular segments of the contour contribute

\[
- \frac{1}{2\pi i} \int_{-\alpha}^{\alpha} \frac{y \text{Re}(\theta)}{\text{Re}(\theta)} \, d(\text{Re}(\theta)) = - \int_{-\alpha}^{\alpha} y \text{Re}(\theta) \, d\theta
\]

\[
\ll \int y \sigma_0 \, d\theta + \int_{1-\alpha}^{1-\alpha} y \sqrt{\log R} \, d\theta
\]

\[
\ll y, \sigma_0, \sqrt{\log R} / R \rightarrow 0 \text{ as } R \rightarrow \infty.
\]

In this instance, the function \( y^s / s \) is analytic within the contour (no poles!), whence

\[
\frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \frac{y^s}{s} \, ds = \lim_{R \rightarrow \infty} \frac{1}{2\pi i} \int_{C_R} \frac{y^s}{s} \, ds = 0.
\]
We add to this the Cauchy principal value of the corresponding integral when \( y = 1 \) and \( \sigma_0 > 0 \):

\[
\lim_{T \to \infty} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{ds}{s} = \frac{1}{2}
\]

Note that as \( T \to \infty \), the integral gives

\[
\frac{1}{2\pi i} \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{d\left(\text{Re}(\theta)\right)}{\text{Re}(\theta)} = \frac{1}{2}
\]

We now return to the proof of Perron's formula. The idea is to truncate the series \( \alpha(s) \). Thus, if we take \( N \) larger than \( 2k+2 \) and \( \sigma > \max \{ \sigma_0, \sigma_c \} \), we have

\[
\alpha(s) = \sum_{n \leq N} a_n n^{-s} + \sum_{n > N} a_n n^{-s} = \alpha_1(s) + \alpha_2(s), \text{ say.}
\]

But then, when \( \sigma_0 > \max \{ \sigma_0, \sigma_c \} \), we find that

\[
\sum_{n \leq x} a_n = \sum_{n \leq x} a_n \lim_{T \to \infty} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{(x/n)^s}{s} ds = \lim_{T \to \infty} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha_1(s) \frac{x^s}{s} ds.
\]

It remains to handle \( \alpha_2(s) \), the tail of the Dirichlet series. By Riemann–Stieltjes integration, we see that for \( \sigma > \max \{ \sigma_0, \sigma_c \} \),

\[
\alpha_2(s) = \int_N^\infty u^{-s} d\left( A(u) - A(N) \right) = s \int_N^\infty (A(u) - A(N)) u^{-s-1} du.
\]

But whenever \( \theta > \max \{ \sigma_0, \sigma_c \} \), we have \( A(u) - A(N) \ll u^\theta \), whence for \( \sigma > \theta \),

\[
\alpha_2(s) \ll 1/s! \int_N^\infty u^{\theta-\sigma-1} du \ll \left( \frac{1/s!}{\sigma-\theta} \right) N^{\sigma-\theta}
\]
We estimate \( \int_{\sigma_0-iT}^{\sigma_0+iT} \frac{\alpha_1(s) x^s}{s} \, ds \) for large \( T \) by considering the contour:

\[
\begin{array}{c}
\text{Im} \\
\sigma_0+iT \\
\text{Im} \\
\sigma_0-iT \\
\sigma_0 \\
\text{Re} \\
T-iT \\
\text{Re} \\
T+iT
\end{array}
\]

By the residue theorem, we have

\[
\int_{\sigma_0-iT}^{\sigma_0+iT} \frac{\alpha_2(s) x^s}{s} \, ds = - \int_{T-iT}^{T+iT} \frac{\alpha_2(s) x^s}{s} \, ds + \int_{\sigma_0-iT}^{\sigma_0+iT} \frac{\alpha_2(s) x^s}{s} \, ds
\]

\[
- \int_{T+iT}^{T-iT} \frac{\alpha_2(s) x^s}{s} \, ds.
\]

We have

\[
\int_{\sigma_0-iT}^{\sigma_0+iT} \frac{\alpha_2(s) x^s}{s} \, ds \ll \frac{N^\theta}{\sigma_0-\theta} \int_{\sigma_0}^{\infty} \left( \frac{x}{N} \right)^{\sigma_0} \, d\sigma_0 \ll \frac{N^\theta}{\sigma_0-\theta} \frac{(x/N)^{\sigma_0}}{\log(N/x)}
\]

and

\[
\int_{T-iT}^{T+iT} \frac{\alpha_2(s) x^s}{s} \, ds \ll \frac{N^\theta}{T-\theta} \int_{-T}^{T} x^T \, d\gamma \ll N^\theta (x/N)^T
\]

\[
\ll N^\theta (x/N)^{\sigma_0}.
\]

Thus

\[
\int_{\sigma_0-iT}^{\sigma_0+iT} \frac{\alpha_2(s) x^s}{s} \, ds \ll \frac{N^\theta (x/N)^{\sigma_0}}{\sigma_0-\theta} \ll x^{\sigma_0} N^{\theta-\sigma_0},
\]

provided that \( \sigma_0 > \theta > \max\{0, \sigma_0\} \).

On combining the contributions from \( \alpha_1(s) \) and \( \alpha_2(s) \), we conclude that

\[
\lim_{T \to \infty} \sup_{n \in \mathbb{N}} \left| \sum_{n \leq x} \alpha_n - \frac{1}{2\pi i} \int_{\sigma_0-iT}^{\sigma_0+iT} \frac{\alpha(s) x^s}{s} \, ds \right| \ll x^{\sigma_0} N^{\theta-\sigma_0}.
\]
This relation holds for all large $N$, so on taking $N \to \infty$ we find that
\[
\sum_{n \leq x} a_n = \lim_{T \to \infty} \frac{1}{\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s) \frac{x^s}{s} \, ds.
\]

In applications we require a quantitative version of this result useful in obtaining asymptotic relations. This makes use of the sine integral
\[
si(x) = -\int_x^\infty \frac{\sin u}{u} \, du.
\]
By integrating by parts, we see that when $x \geq 1$, one has
\[
si(x) \ll \min \{x, 1/x^2\}.
\]
Moreover, on evaluating the integral (an exercise in contour integrals), one has
\[
si(x) + si(-x) = -\int_{-\infty}^\infty \frac{\sin u}{u} \, du = -\pi.
\]

**Theorem 9.2.** Suppose that $\sigma_0 > \max \{\sigma_0, \sigma_0^2\}$ and $x > 0$. Then
\[
\sum_{n \leq x} a_n = \frac{1}{\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s) \frac{x^s}{s} \, ds + R(T),
\]
\[
\text{where}
\]
\[
R(T) = \frac{1}{\pi} \sum_{\frac{x}{2} < n < x} a_n \sin \left( T \log \frac{x}{n} \right) - \frac{1}{\pi} \sum_{x < n \leq 2x} a_n \sin \left( T \log \frac{n}{x} \right)
\]
\[
+ O \left( \frac{4^{\sigma_0} + x^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_0}} \right).
\]

We note, as we shall show, that the "error term" $R(T)$ may be estimated more simply as
\[
R(T) \ll \sum_{\frac{x}{2} < n < 2x} |a_n| \min \left\{ 1, \frac{x}{T|x-n|} \right\} + \frac{4^{\sigma_0} + x^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma_0}}.
\]
Proof: Since \( \alpha(s) \) is absolutely convergent on \([\sigma_0 - iT, \sigma_0 + iT]\), we find that

\[
\frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(s) \frac{x^s}{s} \, ds = \sum_{n} a_n \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \left( \frac{x}{n} \right)^s \, ds.
\]

It therefore follows that the desired conclusion is a consequence of the formula (when \( \sigma_0 > 0 \)),

\[
\frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} y^s \frac{ds}{s} = \begin{cases} 
1 + O\left( y^{\sigma_0}/T \right), & \text{when } y \geq 2, \\
1 + \frac{1}{\pi} \text{si}(T \log y) + O\left(2^{\sigma_0}/T\right), & \text{when } 1 \leq y \leq 2, \\
-\frac{1}{\pi} \text{si}\left(T \log\left(\frac{1}{y}\right)\right) + O\left(2^{\sigma_0}/T\right), & \text{when } \frac{1}{2} \leq y \leq 1, \\
O\left( y^{\sigma_0}/T \right), & \text{when } y \leq \frac{1}{2}.
\end{cases}
\]

This we now set about proving, and divide into cases:

(i) Suppose that \( y \geq 2 \).

By the calculus of residues, we have

\[
\frac{1}{2\pi i} \int_{\gamma} y^s \frac{ds}{s} = \lim_{s \to 0} y^s = 1
\]

up

simple pole at

\( s = 0 \)

On the other hand, we have

\[
\int_{-\infty + iT}^{\sigma_0 + iT} y^s \frac{ds}{s} = \int_{-\infty}^{\sigma_0} y^{\sigma \pm iT} \frac{d\sigma}{\sigma \pm iT} \ll \frac{1}{T} \int_{-\infty}^{\sigma_0} y^\sigma \frac{d\sigma}{T \log y}
\]

\[
\ll \frac{y^{\sigma_0}}{T}, \quad (\text{since } \log y \geq 2)
\]

Thus

\[
\frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} y^s \frac{ds}{s} = \frac{1}{2\pi i} \left( \int_{-\infty - iT}^{\sigma_0 - iT} - \int_{-\infty + iT}^{\sigma_0 + iT} + \int_{\sigma_0 + iT}^{\sigma_0 - iT} - \int_{\sigma_0 - iT}^{\sigma_0 + iT} y^s \frac{ds}{s} \right)
\]

\[
= 1 + O\left( y^{\sigma_0}/T \right). \square
\]
(iii) Suppose that $1 \leq y \leq 2$.

In this case

$$\frac{1}{2\pi i} \int_{\gamma} \frac{e^{yS}}{\bar{S}} \, ds = 0,$$

yet

$$\int_{\sigma_0 \pm iT} y^S \, ds < \frac{1}{T} \int_{0}^{\sigma_0} y^S \, ds \leq \frac{1}{T} \int_{0}^{\sigma_0} 2^S \, ds < \frac{2}{T},$$

while the integral over the semi-circular arc of radius $\varepsilon$ (sufficiently small) contributes something asymptotic to

$$\frac{1}{2\pi i} \int_{\gamma} \frac{d(\varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} = -\frac{1}{2} \quad \text{as } \varepsilon \to 0.$$

Then, since

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \left( \int_{\varepsilon}^{iT} + \int_{-iT}^{\varepsilon} \right) \frac{y^S \, ds}{\bar{S}} = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_{\varepsilon}^{iT} (y^S - y^{T-iT}) \, dt$$

$$= \frac{1}{T} \int_{\sigma_0}^{T \log y} \frac{\sin V}{V} \, dV = \frac{1}{\pi} (-\sin(0) + \sin(T \log y))$$

$$= \frac{1}{2} + \frac{1}{\pi} \sin(T \log y),$$

then we deduce that

$$\frac{1}{2\pi i} \int_{\sigma_0 \pm iT} y^S \, ds = \left( \frac{1}{2} + \frac{1}{\pi} \sin(T \log y) \right) - \left( -\frac{1}{2} \right) + O\left( \frac{2^{\sigma_0}}{T} \right). \quad \Box.$$

(iii) Suppose that $\frac{1}{2} \leq y \leq 1$.

The treatment is the same, but note that

$$\sin(T \log \left( \frac{1}{y} \right)) = \sin(T \log y)$$

and

$$-\pi - \sin(T \log \left( \frac{1}{y} \right)). \quad \Box$$

(iv) Suppose that $y \leq \frac{1}{2}$. We take our contour to the right, noting that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{e^{yS}}{\bar{S}} \, ds = 0,$$
\[ \int_{\sigma_0 - iT}^{\sigma_0 + iT} y^s \frac{ds}{s} = \int_{-\infty}^{\infty} y^{\sigma_0 + iT} \frac{d\sigma}{\sigma + iT} \ll \frac{1}{\log \log \log \log \log \log n}. \]

So that
\[ \frac{1}{\log \log \log \log \log \log n} \ll y^{\sigma_0 / T}. \quad Q.E.D. \]

In order to obtain the consequent simplification, note that

\[ \tilde{\sigma} (T \mid \log n / x) \ll \min \left\{ 1, \frac{1}{T \log \log \log \log \log n} \right\} \]

But \( \frac{n}{x} = 1 + \frac{n-x}{x} \), hence \( \log \left| \frac{n}{x} \right| \approx \left| \frac{n-x}{x} \right| \) for \( -\frac{1}{2} \leq \left| \frac{n-x}{x} \right| \leq 1 \).

Thus
\[ \tilde{\sigma} (T \mid \log n / x) \ll \min \left\{ 1, \frac{x}{T \left| x-n \right|} \right\} \quad \text{when} \quad \frac{x}{2} \leq n \leq 2x. \]

It therefore follows that
\[ R \ll \sum_{\frac{x}{2} \leq n \leq 2x} \left| a_n \right| \min \left\{ 1, \frac{x}{T \left| x-n \right|} \right\} + \frac{4^{\sigma_0} + x^{\eta_0}}{T} \sum_{n=1}^{\infty} \left| a_n \right|. \]

There are weighted versions of these formulæ. Thus, for example, if one has weighted sums of the shape
\[ A_w(x) = \sum_{n=1}^{\infty} a_n w(n/x), \]

and
\[ K(s) = \int_0^\infty w(x) x^{s-1} \, dx, \quad \text{(Mellin transform of } w) \]

then we expect (under appropriate conditions)—that
\[ \alpha(s) K(s) = \int_0^\infty A_w(x) x^{-s-1} \, dx \]

and
\[ A_w(x) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \alpha(s) K(s) x^s \, ds. \]
§10. A zero-free region for \( \zeta(s) \).

Our goal in the next section is the proof of the Prime Number Theorem in the shape

\[
\pi(x) = \int_2^x \frac{dt}{\log t} + O\left( x e^{-c \sqrt{\log x}} \right) \sim \frac{x}{\log x},
\]

for a suitable \( c > 0 \). We have seen a pathway towards this conclusion, since \( \pi(x) \) is intimately related to \( \Psi(x) \), and \( \Psi(x) \) is a partial sum of \( \Lambda(n) \), and hence given in Perron's formula in terms of

\[
-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.
\]

A key question then arises — what are the poles of \( \frac{\zeta'}{\zeta}(s) \)? We know that there is a pole at \( s = 1 \), but there may also be poles at the zeros of \( \zeta(s) \). Thus we spend time seeking a zero-free region for \( \zeta(s) \). This entails some review of basic complex analysis.

Our first objective is a relation between the number of zeros of an analytic function in a domain, and its rate of growth. This idea generalises the fact that a polynomial of degree \( k \) (giving like \( 1z^k \)) has at most \( k \) zeros.

Lemma 10.1. (Jensen's Inequality). Suppose that \( f(z) \) is analytic in a domain containing the disc \( |z| \leq R \). Suppose also that \( |f(z)| \leq M \) in this disc, and that \( f(0) \neq 0 \). Then, whenever \( 0 < r < R \), the number of zeros of \( f(z) \) in the disc \( |z| \leq r \) is at most

\[
\log \left( \frac{M}{|f(0)|} \right) \cdot \log \left( \frac{R}{r} \right).
\]

Proof: Let the zeros of \( f(z) \) in the disc \( |z| \leq R \) be
Here, the $k$-th factor in the product has a pole at $z = z_k$, and when $|z| = R$ one has
\[
\frac{R^2 - z z_k}{R(z - z_k)} = \left| \frac{z(z - \overline{z_k})}{R(z - z_k)} \right| = 1.
\]
Then we see that $g(z)$ is analytic in the disc $|z| \leq R$, and when $|z| = R$ one has
\[
|g(z)| = |f(z)| \leq M.
\]
Thus, by the maximum modulus principle, we see that
\[
|f(0)| \prod_{k=1}^{n} \frac{R}{|z_k|} = |g(0)| \leq M.
\]
But $1/z_k \leq R$ for every $k$, and when $1/z_k \leq r$ one has further,
\[
R/|z_k| \geq R/r.
\]
Then the number, say $N$, of zeros of $f(z)$ in the disc $|z| \leq r$, satisfies
\[
M > |f(0)| \left( \frac{R}{r} \right)^N,
\]
whence
\[
N \leq \frac{\log \left( \frac{M}{|f(0)|} \right)}{\log \left( \frac{R}{r} \right)}.
\]
[Ex. Check what happens if one has "infinitely" many zeros of $f$ in $|z| \leq R$.]

The conclusion of Lemma 10.1 is only useful if we are able to provide a bound for $|f(z)|$ throughout the disc $|z| \leq R$. Very often it is simpler to bound the real value of such a function, and so it is useful to infer bounds for $|f(z)|$ from corresponding bounds for $\Re (f(z))$. 

\[\text{Figure 10.1} \]
Lemma 10.2 (The Borel–Cauchy theorem). Suppose that $h(z)$ is analytic in a domain containing the disc $|z| \leq R$. Suppose also that $h(0) = 0$, and that $\text{Re}(h(z)) \leq M$ for $|z| \leq R$. Then, whenever $|z| = r < R$, one has

$$|h(z)| \leq \frac{2Mr}{R-r} \quad \text{and} \quad |h'(z)| \leq \frac{2MR}{(R-r)^2}.$$

Proof. Our strategy is to expand $h(z)$ and $h'(z)$ as Taylor series about $0$, and to this end we require estimates for the Taylor series coefficients $h^{(k)}(0)/k!$. Observe that

$$0 = h(0) = \frac{1}{2\pi i} \oint_{|z|=R} h(z) \frac{dz}{z} = \int_0^1 h(\text{Re}(\theta)) \, d\theta.$$

Also, when $k>0$, we have

$$\int_0^1 h(\text{Re}(\theta)) e(k\theta) \, d\theta = \frac{R^{-k}}{2\pi i} \oint_{|z|=R} h(z) z^{-k-1} \, dz = 0 \quad (10.1)$$

and

$$\int_0^1 h(\text{Re}(\theta)) e(-k\theta) \, d\theta = \frac{R^k}{2\pi i} \oint_{|z|=R} h(z) z^{-k-1} \, dz = R^k \frac{h^{(k)}(0)}{k!} \quad (10.2).$$

Combining (10.1) and (10.2) when $k>0$, we see that

$$\int_0^1 h(\text{Re}(\theta)) \left( 1 + \infty (2\pi (k\theta + \phi)) \right) \, d\theta = \int_0^1 h(\text{Re}(\theta)) \left( 1 + \frac{e(k(\theta + \phi) + \phi)}{2} \right) \, d\theta$$

$$= \frac{e^{-\phi}}{2} \cdot \frac{R^k h^{(k)}(0)}{k!}.$$

Thus, since $\text{Re}(h(z)) \leq M$, we see that

$$\text{Re} \left( \frac{e^{-\phi}}{2} \cdot \frac{R^k h^{(k)}(0)}{k!} \right) \leq \int_0^1 M \left( 1 + \infty (2\pi (k\theta + \phi)) \right) \, d\theta.$$
We choose \( \phi \) so that \( e(-\phi) h^u(0) = h^{u_1}(0) \), and hence infer that

\[
\frac{|h^u(0)|}{k!} \leq \frac{2M}{R^k} \int_0^1 \left( 1 + \cos(2\pi (k\theta + \phi)) \right) \, d\theta = \frac{2M}{R^k}.
\]

We may now apply this upper bound in a Taylor expansion to deduce that whenever \( |z| \leq r < R \), one has

\[
|h(z)| \leq \sum_{k=1}^\infty \frac{|h^u(0)|}{k!} r^k \leq 2M \sum_{k=1}^\infty \left( \frac{r}{R} \right)^k = \frac{2Mr}{R-r}
\]

and

\[
|h'(z)| \leq \sum_{k=1}^\infty \frac{|h^u(0)|}{k!} r^k k^{-1} \leq 2M \sum_{k=1}^\infty \frac{k (\frac{r}{R})^{k-1}}{R} = \frac{2Mr}{(R-r)^2}.
\]

Finally, in our discussion of general results, we come to a formula for the logarithmic derivative of an analytic function in terms of local zeros.

**Lemma 10.3.** Suppose that \( f(z) \) is analytic in a domain containing the disc \( |z| \leq 1 \). Suppose also that \( |f(z)| \leq M \) in this disc, and \( f(0) \neq 0 \). Then, whenever \( r \) and \( R \) are fixed with \( 0 < r < R < 1 \), and \( |z| \leq r \), one has

\[
\frac{f'(z)}{f(z)} = \sum_{k=1}^N \frac{1}{z-z_k} + O_r \left( \log \left( \frac{M}{|f(0)|} \right) \right),
\]

where the summation is over all zeros of \( f \) for which \( |z| \leq R \).

**Proof.** If necessary by slightly increasing the value of \( R \), we may suppose that \( f(z) \) has no zeros on the circle \( |z| = R \). Note that this assumes that \( f \) has a finite number of zeros in \( |z| \leq R+\epsilon \),
When \( \varepsilon \) is small, which follows from Lemma 10.1. We define \( g(\varepsilon) \) by the Blaschke product
\[
g(\varepsilon) = \prod_{k=1}^{n} \left( \frac{R^2 - \varepsilon \overline{z}_k}{R(z - z_k)} \right).
\]

Note that, as a consequence of Lemma 10.1, one has
\[
n \leq \log \left( \frac{M/|f(\varepsilon)|}{\log \left( \frac{M}{|f(\varepsilon)|} \right)} \right) \leq \log \left( \frac{M}{|f(\varepsilon)|} \right) \quad \text{— (10.3)}
\]

Also, when \( |\varepsilon| = R \), one has
\[
|\frac{R^2 - \varepsilon \overline{z}_k}{R(z - z_k)}| = 1,
\]

whence for \( |\varepsilon| = R \) one has \( |g(\varepsilon)| = |f(\varepsilon)| \leq M \). It follows from the maximum modulus principle that \( |g(\varepsilon)| \leq M \) for \( |\varepsilon| \leq R \).

Observe next that
\[
|g(0)| = |f(0)| \prod_{k=1}^{n} \frac{R}{|z_k|} \geq |f(0)| > 0.
\]

Since \( g(\varepsilon) \) has no zeros in the disc \( |\varepsilon| \leq R \), we may put
\[
h(\varepsilon) = \log \left( \frac{g(\varepsilon)}{g(0)} \right). \quad \text{Then } h(0) = 0, \quad \text{and when } |\varepsilon| < R,
\]
\[
\text{Re} \left( h(\varepsilon) \right) = \log |g(\varepsilon)| - \log |g(0)| \leq \log M - \log |f(\varepsilon)|.
\]

We therefore deduce from the Borel-Carathéodory Lemma that when \( |\varepsilon| \leq R \), one has
\[
h'(\varepsilon) \ll \log \left( \frac{M}{|f(\varepsilon)|} \right).
\]

But
\[
h'(\varepsilon) = \frac{g'(\varepsilon)}{g(\varepsilon)} = \frac{f'(\varepsilon)}{f(\varepsilon)} - \sum_{k=1}^{n} \frac{1}{z - z_k} + \sum_{k=1}^{n} \frac{1}{z - R^2/z_k}.
\]

Thus,
\[
\frac{f'(z)}{f(z)} = \sum_{k=1}^{n} \frac{1}{z-z_k} + \sum_{k=1}^{n} \frac{1}{z-R_k^2/E_k} \ll \log \left( \frac{M}{|f(0)|} \right).
\]

But \( |R_k^2/E_k| > R \), so that when \( |z| \leq r \), one has \( |z-R_k^2/E_k| \geq R-r \), whence
\[
\sum_{k=1}^{n} \frac{1}{z-R_k^2/E_k} \leq \frac{n}{R-r} \ll \log \left( \frac{M}{|f(0)|} \right). \tag{10.3}
\]

We may therefore conclude that
\[
\frac{f'(z)}{f(z)} = \sum_{k=1}^{n} \frac{1}{z-z_k} + O \left( \log \left( \frac{M}{|f(0)|} \right) \right).
\]

At last, we may apply these ideas to obtain a formula for \( \frac{S'}{S} (s) \) in terms of zeros of \( S(s) \). In preparation for this we define a crude bound for \( S(s) \).

**Lemma 10.4.** Let \( \delta > 0 \) be fixed. Then one has
\[
S(s) = \frac{1}{s-1} + O(1)
\]
uniformly for \( s \) in the rectangle \( \delta \leq s \leq 2 \) and \( |t| \leq 1 \), and further
\[
S(s) \ll \left( 1 + 2^{1-\delta} \right) \min \left( \frac{1}{s-1}, \log 2 \right)
\]
uniformly for \( \delta \leq s \leq 2 \) and \( |t| \geq 1 \). (Here, we write \( t=|t|+i \).)

**Proof.** For the first assertion we recall that
\[
S(s) = \frac{S}{s-1} - s \int_{1}^{\infty} \frac{\psi(u)}{u^{s+1}} \, du \quad (s > 0) \quad (\text{see (8.3)}).
\]
The second term here is \( \ll 1 s ! \int_{1}^{\infty} u^{-1-\delta} \, du \ll 1/\delta \), and the desired conclusion follows.

Let us turn to the second assertion. We recall the conclusion of
Theorem 3.5, namely that when \( s \neq 1 \) and \( \sigma > 0 \), \( x > 0 \), one has

\[
\mathcal{F}(s) = \sum_{n \leq x} n^{-s} + \frac{x^{1-s}}{s-1} + \frac{\log x}{x^s} - s \int_{x}^{\infty} \frac{\xi u^s}{u^{s+1}} \, du.
\]

The integral here satisfies

\[
\int_{x}^{\infty} \frac{\xi u^s}{u^{s+1}} \, du \ll \int_{x}^{\infty} u^{-1-\sigma} \, du \ll x^{-\sigma}/\sigma,
\]

whence

\[
\mathcal{F}(s) = \sum_{n \leq x} n^{-s} + \frac{x^{1-s}}{s-1} + O \left( \frac{|\log x|}{x^s} \right)
\]

\[
= \sum_{n \leq x} n^{-s} + \frac{x^{1-s}}{s-1} + O \left( \frac{\tau}{x^{1-\sigma}} \right). \quad (10.4)
\]

We may suppose that \( x \geq 2 \), whence

\[
\sum_{n \leq x} n^{-s} \ll \sum_{n \leq x} n^{-\sigma} \ll 1 + \int_{1}^{x} u^{-\sigma} \, du \quad (\sigma > 0).
\]

When \( 0 < \sigma < 1 \) and \( 1 \leq u \leq x \), one has \( u^{-\sigma} \ll u^{-1} \),

and then \( \int_{1}^{x} u^{-\sigma} \, du \ll \int_{1}^{x} u^{-1} \, du = \log x \).

Meanwhile, when \( (\sigma - 1) > 1/\log x \), one has instead

\[
\int_{1}^{x} u^{-\sigma} \, du < \int_{1}^{\infty} u^{-\sigma} \, du = 1/(\sigma - 1),
\]

and when \( 0 \leq \sigma \leq 1 - 1/\log x \),

\[
\int_{1}^{x} u^{-\sigma} \, du = \frac{x^{1-\sigma} - 1}{1-\sigma} \ll \frac{x^{1-\sigma}}{1-\sigma}.
\]

Hence

\[
\sum_{n \leq x} n^{-s} \ll (1 + x^{1-\sigma}) \min \left\{ \frac{1}{|1-\sigma-1|}, \log x \right\}
\]

uniformly for \( 0 \leq \sigma \leq 2 \). On substituting this estimate into (10.4),

we conclude that with \( x = \tau \) one has

\[
\mathcal{F}(s) \ll (1 + \tau^{1-\sigma}) \min \left\{ \frac{1}{|1-\sigma-1|}, \log \tau \right\}.
\]
Lemma 10.5. Suppose that \(|t| \geq \frac{7}{8}\) and \(\frac{5}{6} \leq \sigma \leq 2\). Then one has
\[
\frac{5'}{5}(s) = \sum \frac{1}{\rho s - \rho} + O \left( \log \left( \frac{|1+t|+4}{t} \right) \right),
\]
where the summation is over all zeros \(\rho\) of \(5(s)\) satisfying\[|\rho - (\frac{3}{2} + it)| \leq \frac{5}{6}.\]

[Recall that \(s = \sigma + it\)].

Proof: We apply Lemma 10.3 to the function
\[
f(z) = 5 \left( \frac{z}{s} + \frac{5/2 + it}{s} \right),
\]
with \(R = 5/6\) and \(r = 2^{1/3}\). We have the absolutely convergent product formula
\[
f(0) = 5(\frac{3}{2} + it) = \prod_{\rho} \left( 1 - \rho - \frac{3}{2} - it \right)^{-1},
\]
whence
\[
|f(0)| \geq \prod_{\rho} \left( 1 - \rho - \frac{3}{2} - it \right)^{-1} > 0.
\]
Thus \(|f(0)| \gg 1\). Moreover, by Lemma 10.4, we have
\[
|f(z)| \ll \frac{1}{2} \log t \ll t \quad \text{for} \quad |z| \leq 1.
\]
We therefore conclude that
\[
\frac{f'(z)}{f(z)} = \sum \frac{1}{\rho (z + \frac{3}{2} + it) - \rho} + O \left( \log \frac{t}{T} \right),
\]
so that

$$\frac{5'}{5} (s) = \sum_{\rho} \frac{1}{s-\rho} + O(\log s).$$

How now do we establish a zero-free region for $\zeta(s)$? Suppose that $\zeta(s)$ has a zero very close to the line $\sigma = 1$, say with imaginary part $\gamma$. As a thought experiment, suppose this zero is at $s = 1 + i\gamma$.

Then with $\delta \to 0^+$, we would have

$$\frac{5'}{5} (1 + \delta + i\gamma) \sim \frac{m}{\delta},$$

where $m$ is the multiplicity of this zero. Notice here that

$$\Re \left( \frac{5'}{5} (1 + \delta + i\gamma) \right) = -\sum_{n=1}^{\infty} \Lambda(n) n^{-1-\gamma} \cos (\gamma \log n)$$

$$\leq \sum_{n=1}^{\infty} \Lambda(n) n^{-1-\delta} = -\frac{5'}{5} (1+\delta) \sim \frac{1}{\delta},$$

so necessarily $m \leq 1$. If this occurs with something close to equality, it follows that $\cos (\gamma \log n) \approx 1$ for most $n = p^h$.

Whence

$$p^{i\gamma} \approx 1$$

for most $p$. But then

$$p^{2i\gamma} \approx 1$$

for most $p$,

Whence

$$\frac{5'}{5} (1 + \delta + 2i\gamma) \sim -\sum_{n=1}^{\infty} \Lambda(n) n^{-1-\delta} \sim -\frac{1}{\delta}. \quad \text{(pole at } 1 + 2i\gamma)$$

This heuristic argument shows that the existence of a zero very close to the $1$-line implies the existence of a pole very close to the $1$-line distinct from that at $s=1$, yielding a contradiction.

We now set about making this heuristic rigorous, and
this involves relating the values of \(-\frac{s'}{s}(s)\) with \(s = \sigma + it\) to the value at \(s = \sigma + 2it\).

**Lemma 10.6.** When \(\sigma > 1\), one has

\[
\text{Re} \left( -\frac{3}{5} \frac{s'}{s}(\sigma) - 4 \frac{s'}{s}(\sigma + it) - \frac{s'}{s}(\sigma + 2it) \right) \geq 0.
\]

**Proof.** The left hand side here is

\[
\sum_{n=1}^{\infty} \Lambda(n) n^{-\sigma} \left( 3 + 4\cos(\theta \log n) + \infty (2i \log \log n) \right).
\]

But

\[
3 + 4\cos(\theta) + \infty (2i) = 2(1 + \cos \theta)^2 \geq 0
\]

for all \(\theta\), and so the desired conclusion follows.

**Theorem 10.7.** There is an absolute constant \(c > 0\) such that

\[
\delta(s) \neq 0 \quad \text{for} \quad \sigma > 1 - c / \log T.
\]

**Remark:** The sharpest known zero-free region is due to Korobov and Vinogradov (1958) separately:

\[
\sigma > 1 - c (\log T)^{-2/3} \log \log T^{-4/3}.
\]

**Riemann Hypothesis:** \(\sigma = \frac{1}{2}\).

**Proof (of Lemma 10.7):** When \(\sigma > 1\), one has

\[
\delta(s) = \prod_p \left( 1 - p^{-s} \right)^{-1} \geq \prod_p \left( 1 - p^{-\sigma} \right)^{-1} > 0,
\]

by absolute convergence of the product. Thus we may restrict attention to the situation with \(\sigma \leq 1\).
Next, from the formula
\[ s(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\text{e}^u}{u^{s+1}} \, du, \]
we find that
\[ |s(s) - \frac{s}{s-1}| \leq |s| \int_1^\infty \frac{du}{u^{s+1}} = \frac{|s|}{\alpha}, \quad (\alpha > 0). \]

Thus \( s(s) \neq 0 \) when \( \alpha > \Im s \). This condition is satisfied when
\[ \sigma^2 > \alpha^{-1} + t^2 \]
\[ \leftrightarrow \quad \sigma > (1 + t^2)^{1/2}. \]

Then \( s(s) \neq 0 \) for \( 1 + t^2 \leq \pi/8 \) and \( \frac{\theta}{\pi} \leq \sigma \leq 1 \). (\( \frac{1}{\pi} (\frac{\pi}{2})^{1/2} = 1 - \frac{15}{128} < \frac{3}{4} \)).

Suppose next that there is a zero \( \rho = \rho_0 + i\gamma_0 \) of \( s(s) \) with \( \frac{5}{6} \leq \rho_0 \leq 1 \) and \( \gamma_0 \geq \frac{\pi}{8} \). Since \( \text{Re}(\rho) \leq 1 \) for all zeros \( \rho \) of \( s(s) \), we have
\[ \text{Re} \left( \frac{1}{s-\rho} \right) > 0 \quad \text{whenever} \quad \sigma > 1. \]

We apply Lemma 10.5 with \( s = 1 + \delta + i\gamma_0 \) to see that for small positive values of \( \delta \), and for a suitable \( c_1 > 0 \), one has
\[ -\text{Re} \left( \frac{s'}{s} \left( 1 + \delta + i\gamma_0 \right) \right) \leq -\frac{1}{1+\delta-\rho_0} + c_1 \log (1+\gamma_0+4). \]

Similarly, but now with \( s = 1 + \delta + 2i\gamma_0 \), we obtain
\[ -\text{Re} \left( \frac{s'}{s} \left( 1 + \delta + 2i\gamma_0 \right) \right) \leq c_1 \log (1+\gamma_0+4). \]

Finally, we have
\[ -\frac{s'}{s} = \frac{1}{\delta} + O(1). \]

Hence, on combining these estimates via Lemma 10.6, we discern that for a suitable \( c_2 > 0 \), one has
\[
\frac{3}{\delta} - \frac{4}{1+\delta-\beta_0} + c_2(\log(1 \gamma_\delta + 4)) \geq 0
\]

where \( \delta = \frac{1}{2c_2 \log(1 + \gamma_\delta)} \) we obtain

\[
7c_2 \log(1 + \gamma_\delta) \geq 4 / (1 + \delta - \beta_0).
\]

This, in turn, implies that

\[
1 + \delta - \beta_0 = 1 + \frac{1}{2c_2 \log(1 + \gamma_\delta)} - \beta_0 \geq \frac{4}{7c_2 \log(1 + \gamma_\delta)}
\]

so that

\[
1 - \beta_0 \geq \frac{1}{14c_2 \log(1 + \gamma_\delta)} = \frac{1}{14c_2 \log \tau}.
\]

This completes the proof of the theorem.

When it comes to applying Perron's formula to give explicit versions of the prime number theorem, it is useful to have available bounds for \( \frac{1}{\zeta(s)} \), \( \frac{\zeta'}{\zeta}(s) \), and \( \log \zeta(s) \) in the vicinity of the line \( \sigma = 1 \).

**Theorem 10.8.** Suppose that \( c > 0 \) is an absolute constant with the property that \( \frac{1}{\zeta(s)} \neq 0 \) for \( \sigma \geq 1 - c / \log \tau \). Then whenever \( \sigma \geq 1 - c / (2 \log \tau) \) and \( 1 + t \geq 7/8 \), one has

\[
\frac{1}{\zeta(s)} \ll \log \tau, \quad \frac{\zeta'}{\zeta}(s) \ll \log \tau, \quad \frac{1}{\zeta(s)} \ll \log \tau,
\]

and

\[
|\log \zeta(s)| \leq \log \log \tau + O(1).
\]

**Proof.** The first bound in (10.5) follows from Lemma 10.4, since for \( \sigma \geq 1 - \frac{1}{2} c / \log \tau \), one has...
\[ s(s) \ll (1 + T^{1 - \sigma}) \min \left\{ \frac{1}{10 - 11}, \log T^3 \right\}, \quad \text{for } |t| \leq 1 \]

whilst

\[ s(s) = \frac{1}{s - 1} + O(1) \ll \log T, \quad \text{for } \frac{3}{8} \leq |t| \leq 1. \]

We next consider the second estimate claimed in (10.5). When \( \sigma > 1 \), one has

\[ \left| \frac{s'(s)}{s} \right| \leq \sum_{n=1}^{\infty} \Lambda(n) n^{-\sigma} = -\frac{s'(s)}{s} (\sigma) \ll \frac{1}{\sigma - 1}, \]

(using the Laurent series for \( s(\sigma) \)). Thus, whenever \( \sigma > 1 + 1/\log T \), it follows immediately that \( \frac{s'(s)}{s} \ll \log T \). We now extrapolate from this bound for \( \frac{s'(s)}{s} \) when \( s = 1 + 1/\log T + it \) to obtain bounds for \( \frac{s'(s)}{s} \) inside the line \( \sigma = 1 \). Observe that

\[ \frac{s'(s)}{s} \ll \frac{\log T}{s} \]

and further, from Lemma 10.5, whenever \( 1 + 1/\log T \leq \sigma \leq 1 + 1/\log T \),

\[ \frac{s'(s)}{s} = \sum_{\rho} \frac{1}{s - \rho} + O\left( \log T \right), \quad \text{where } |\rho - (\frac{1}{2} + it)| \leq \frac{5}{6}. \]

In particular, one has

\[ \log T \gg \frac{s'(s)}{s} = \sum_{\rho} \frac{1}{s - \rho} + O\left( \log T \right) \Rightarrow \Re\left( \sum_{\rho} \frac{1}{s - \rho} \right) \ll \log T. \]

Given any \( s = \sigma + it \) with \( 1 - \frac{1}{2} c/\log t \leq \sigma \leq 1 + 1/\log t \), one therefore sees that

\[ \frac{s'(s)}{s} - \frac{s'(s_1)}{s_1} = \sum_{\rho} \left( \frac{1}{s - \rho} - \frac{1}{s_1 - \rho} \right) + O\left( \log T \right). \]

But

\[ \frac{1}{s - \rho} - \frac{1}{s_1 - \rho} = \frac{s_1 - s}{(s - \rho)(s_1 - \rho)} \ll \frac{1}{|s_1 - \rho|^2 \log T} \]
Note here that since $\zeta(s) \neq 0$ for $s > 1 - \varepsilon / \log T$, then
\[ \Re(p) \leq 1 - \varepsilon / \log T \]
for all zeros $p$ counted in the sum, whence
\[ \Re(s - p) \gg 1 / \log T \]
at the same time as one has
\[ |s - p| \ll 1 / \log T. \]
This justifies the assertion that $|s - p| \ll 1 / \log T$ employed above. Since also $|s_1 - p| \gg 1 / \log T$, one concludes that
\[ \frac{1}{s - p} - \frac{1}{s_1 - p} \ll \frac{1}{|s_1 - p|} \ll \Re\left(\frac{1}{s_1 - p}\right), \]
whence
\[ \sum_p \left( \frac{1}{s - p} - \frac{1}{s_1 - p} \right) \ll \Re\left(\sum_p \frac{1}{s_1 - p}\right) \ll \log T. \]

We therefore conclude that
\[ \frac{5'}{5}(s) = \frac{5'}{5}(s_1) + O\left(\log T\right) \ll \log T. \]

The third estimate of (10.5) (and indeed the first) can be obtained from (10.6), since
\[ \log(1/15) = -\Re(\log 5) \leq -\log \log T + O(1) \]
implies
\[ 1/15(s_1) \ll \log T. \]
Thus we may concentrate once m the proof of (10.6). Here we proceed similarly in moving from $\sigma = 1 + 1 / \log T$ to the left of the line $\sigma = 1$. Observe that when $\sigma > 1$, one has
\[ \left| \log \zeta(s) \right| \leq \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-\sigma} = \log \zeta(0). \]
But
\[ \zeta(0) = \frac{s}{s-1} - s \int_1^{\infty} \frac{\zeta(u)}{u^{s+1}} du < \frac{s}{s-1} \left( \text{by (3.13)} \right), \]
d so when $\sigma = 1 + 1 / \log T$ one has
\[ \log(\zeta(0)) < \log \left(\log T\right). \]

It follows that
\[ \left| \log \zeta(s) \right| \ll \log \log T \text{ uniformly when } s_1 = 1 + 1 / \log T + it. \]
In order to move from $s_i$ to the point $s = \sigma + it$, with $1 - \frac{1}{2}c / \log t \leq \sigma \leq 1 + 1/\log t$, we observe that

$$\log S(s) - \log S(s_i) = \int_{s_i}^{s} \frac{\xi'(z)}{\xi(z)} \, dz$$

$$\ll \left| s - s_i \right| \sup_{z \in [s_i, s]} \left| \frac{\xi'(z)}{\xi(z)} \right|$$

$$\ll \frac{1}{\log t} \cdot \log t \quad \text{(using (10.5))}.$$

But then $\left| \log S(s) \right| = \left| \log S(s_i) \right| + O(1) \leq \log \log t + O(1)$. \hfill \Box

§11. The Prime Number Theorem.

We are now equipped to apply Perron's formula to obtain an asymptotic formula for

$$\psi(x) = \sum_{n \leq x} \Lambda(n)$$

via bounds for $\frac{\xi'}{\xi}(s)$.

**Theorem 11.1.** There is a positive constant $c$ having the property that

$$\psi(x) = x + O \left( x \exp \left( -c \sqrt{\log x} \right) \right),$$

$$\Theta(x) = x + O \left( x \exp \left( -c \sqrt{\log x} \right) \right),$$

and

$$\pi(x) = \text{li}(x) + O \left( x \exp \left( -c \sqrt{\log x} \right) \right),$$

uniformly for $x \geq 2$. [Here $\text{li}(x) = \int_2^x \frac{dt}{\log t}$ is the logarithmic integral.]

Proof. Recall that

$$-\frac{\xi'}{\xi}(s) = \sum_{n=1}^{\infty} \Lambda(n) n^{-s}.$$

Thus, it follows from the quantitative version of Perron's formula (Theorem 9.2) that for $\sigma > 1$,

$$\psi(x) = -\frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\xi'(s)}{\xi(s)} \frac{x^s}{s} \, ds + R(T),$$

where

$$R(T) = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\xi'(s)}{\xi(s)} \frac{x^s}{s} \, ds + O(1).$$
Where
\[ R(T) \approx \left( \sum_{\frac{X}{2} < n < 2X} \Lambda(n) \min \left\{ 1, \frac{x}{T|1-x-n|} \right\} \right) + \left( \frac{4x}{T} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1/2}} \right). \]

In an 4 of Problem Sheet 4, one shows that with \(2 \leq T \leq x\) and \(\sigma_0 = 1 + \frac{1}{\log x}\), one has \(R(T) \ll \frac{x}{T} \left( \log x \right)^2\), whence
\[ \Psi(x) = -\frac{1}{2\pi i} \int_{\sigma_0-iT}^{\sigma_0+iT} \frac{\frac{x^s}{s}}{\Gamma(s)} ds + O\left( \frac{x}{T} \left( \log x \right)^2 \right). \]

\[ \sigma_0 = 1 + \frac{1}{\log x}, \quad \sigma_1 = 1 - \frac{b}{\log T}. \]

We now turn our attention towards the calculation of the integral in this formula for \(\Psi(x)\). We close the contour \([\sigma_0-iT, \sigma_0+iT]\) by using a vertical contour that avoids including any of the zeros of \(\hat{\psi}(s)\). We know from Theorem 10.7 that there is a positive constant \(b > 0\) such that
\( S(s) \) has no zeros \( s \) with \( \sigma > 1 - \frac{2b}{\log T} \). We put \( \sigma_1 = 1 - \frac{b}{\log T} \), and close the contour as indicated in the diagram. It follows that

\(-\frac{s'}{s}(s)\) has only a simple pole at \( s = 1 \) with residue 1, and otherwise is analytic within \( C \). Hence

\[
\frac{1}{2\pi i} \oint \left( -\frac{s'}{s}(s) \right) \frac{x^s}{s} ds = \lim_{s \to 1} \frac{x^s}{s} = x.
\]

Also, one has

\[
\int_{\sigma - iT}^{\sigma + iT} \left( -\frac{s'}{s}(s) \right) \frac{x^s}{s} ds \ll \log T \cdot \frac{x^{\sigma_0}}{T} \left( \sigma_0 - \sigma_1 \right) \ll \frac{x}{T} \quad (\text{since } x \gg T)
\]

and

\[
\int_{\sigma - iT}^{\sigma + iT} \left( -\frac{s'}{s}(s) \right) \frac{x^s}{s} ds \ll \log T \cdot \frac{x^{\sigma_1}}{} \int_{-T}^{T} \frac{dt}{1 + |t|} + x^{\sigma_1} \int_{-1}^{1} \frac{dt}{|s_1 + it - 1|} \ll x^{\sigma_1} \left( \log T \right)^2 + \frac{x^{\sigma_1}}{1 - \sigma_1}
\]

Thus, we conclude that

\[
\frac{1}{2\pi i} \int_{\sigma - iT}^{\sigma + iT} \left( -\frac{s'}{s}(s) \right) \frac{x^s}{s} ds = x + O \left( \frac{x}{T} + x^{\sigma_1} \left( \log T \right)^2 \right),
\]

whence

\[
\psi(x) = x + O \left( x(\log x)^2 \left( \frac{1}{T} + x^{-b/\log T} \right) \right).
\]

We have now only to optimize the size of the error term here, and this is achieved by choosing \( T \) so that

\[
T \gg x^{b/\log T} \leftrightarrow \log T \approx b \log \frac{x}{\log T}.
\]

Then we may take \( \log T = \sqrt{\log x} \), or equivalently,

\[
T = \exp \left( \sqrt{\log x} \right).
\]
Then we have
\[ x (\log x)^2 \left( \frac{1}{y} + x^{-b/\log x} \right) \ll x (\log x)^2 \left( \exp (-\sqrt{\log x}) + \exp (-b \sqrt{\log x}) \right), \]
for a suitable \( c > 0 \). This proves that
\[ \psi(x) = x + O \left( x \exp (-c \sqrt{\log x}) \right). \]

We have shown (cf. Problems 3, En 1) that
\[ \Theta(x) = \psi(x) + O \left( \sqrt{x} \right), \]
and thus
\[ \theta(x) = x + O \left( x \exp (-c \sqrt{\log x}) \right). \]

Finally, by Riemann-Stieltjes integration, we have
\[ \pi(x) = \int_{2^{-}}^{x} \frac{1}{\log u} \, d\Theta(u) = \int_{2^{-}}^{x} \frac{du}{\log u} + \int_{2^{-}}^{x} \frac{d(\Theta(u)-u)}{\log u} \]
\[ = \text{li}(x) + \left[ \frac{\Theta(u)-u}{\log u} \right]_{2^{-}}^{x} + \int_{2^{-}}^{x} \frac{\Theta(u)-u}{u (\log u)^2} \, du, \]
where
\[ \pi(x) - \text{li}(x) \ll x \exp (-c \sqrt{\log x}) + \int_{2^{-}}^{x} \exp (-c \sqrt{\log u}) \, du \ll x \exp (-c \sqrt{\log x}). \]

Conversely
\[ \int_{t}^{2^{1}} \exp (-c \sqrt{\log u}) \, du \quad \text{for} \quad t = 2^{1}. \]

Hence
\[ \pi(x) = \text{li}(x) + O \left( x \exp (-c \sqrt{\log x}) \right). \]

Exercise: Prove that for a suitable \( c > 0 \), one has
\[ \sum_{n \leq x} \mu(n) \ll x \exp (-c \sqrt{\log x}). \]

World Record:
- Karatsuba-Vinogradov, 1958: \( \pi(x) = \text{li}(x) + O(x \exp (-c \log x \log \log x)^{1/5}) \)
- Riemann Hypothesis: \( \pi(x) = \text{li}(x) + O \left( x^{\frac{1}{2}} \log x \right) \).
§12. The distribution of smooth numbers. (a.k.a. "pseudonyxology")

Integers \( n \in \mathbb{N}_1 \), all of whose prime factors are small are frequently called "smooth numbers" (occasionally "friable numbers"). They play a prominent role in factorization algorithms, hence cryptography, as well as being of interest in their own right. They are also used in the circle method (Waring's problem, etc).

**Definition 12.1.** Let

\[
S(x, y) = \{ n \in \mathbb{Z} \cap [1, x] : \text{pln} \rightarrow \text{p prime } \Rightarrow p \leq y \},
\]

\[
\psi(xy) = \text{card } (S(x, y)).
\]

We first investigate the situation in which \( y \) is a power of \( x \) in size. Here, we have in mind (in particular) the scenario where \( y = x^\eta \) with \( \eta > 0 \) very small.

It transpires that a special function, called the Dickman function \( \rho(n) \), plays a distinguished role in the subject. This function \( \rho : [0, \infty] \rightarrow \mathbb{R} \) is defined to be the unique continuous function satisfying the differential-delay equation

\[
u \rho'(u) = -\rho(u-1) \quad (u > 1),
\]

subject to

\[
\rho(u) = 1 \quad (0 \leq u \leq 1).
\]

It is useful to note before proceeding further that since

\[
\rho'(u) = -\frac{\rho(u-1)}{u} \quad (u > 1),
\]

we see that whenever \( 1 \leq n \leq \rho \), one has
\[ \rho(u) - \rho(u) = - \int_u^v \frac{\rho(t)}{t} \, dt. \] 

(12.1)

In order to get a sense for how \( \rho(u) \) behaves, observe that when \( u > 1 \), one has

\[ (u \rho(u))' = u \rho'(u) + \rho(u) = \rho(u) - \rho(u-1), \]

whence

\[ u \rho(u) = \int_{u-1}^u \rho(v) \, dv + C, \]

for \( u \neq 1 \), whence

\[ C = 1 \cdot \rho(1) - \int_0^1 \rho(v) \, dv = 1 - 1 = 0. \]

One can now proceed by induction to show that \( \rho(u) > 0 \) for all \( u > 0 \) from the relation

\[ \rho(u) = \frac{1}{u} \int_{u-1}^u \rho(v) \, dv. \]

It therefore follows from the relation \( \rho'(u) = -\frac{1}{u} \rho(u-1) \) \( (u > 1) \) that \( \rho(u) \) is decreasing for \( u > 1 \), whence by induction

\[ \rho(u) \leq \frac{1}{u!} = \frac{1}{\Gamma(u+1)}, \quad (n \in \mathbb{N}) \]

A more careful analysis shows that for \( u > 0 \), one has

\[ \frac{1}{2 \Gamma(2u+1)} \leq \rho(u) \leq \frac{1}{\Gamma(u+1)}. \]

Thus, when \( u \) is large, we have \( \rho(u) \leq u^{-u} e^{u} \) roughly speaking.

**Theorem 12.2.** (Dickman, 1930). For each \( 0 \leq u \leq 1 \), one has

\[ \Psi(x, x^{1/u}) = \rho(u) x + O \left( \frac{x}{\log x} \right) \quad (0 \leq u \leq 1, x \geq 2). \]

**Proof.** We apply a variation of the inclusion-exclusion principle, sometimes referred to as Buchstab's identity. This is easy to follow in the initial stages of the arguments. First, trivially, when \( 0 \leq u \leq 1 \), one
\[ \psi(x, y) = x \sum_{1 \leq n \leq x} \sum_{\substack{y < p \leq x \atop \text{prime } p}} \frac{1}{p} \]

\[ = x \log \log \log x + O \left( \frac{x}{ \log x} \right) + O \left( \frac{x}{ \log \log x} \right). \]

**But when** \( 1 < u \leq 2 \), one has

\[ \rho(u) - \rho(1) = - \int_1^u \frac{\rho(t) \log t}{t} \, dt = - \int_1^u \frac{dt}{t} = - \log u, \]

whence \( \rho(u) = 1 - \log u \). Thus

\[ \psi(x, y) = x \rho(u) + O \left( \frac{x}{ \log x} \right), \]

when \( y = x^{1/u} \) with \( 1 < u \leq 2 \).
We now proceed by induction on \( U \), supposing that the conclusion of the theorem holds for \( 0 \leq u \leq U \). This has been established with \( U = 2 \) as the basis of the induction, and we now seek to establish the conclusion for \( 0 \leq u \leq U + 1 \). Suppose then that

when \( 0 \leq u \leq U \), one has

\[
\psi(x, x^{1/u}) = \rho(u) x + O \left( \frac{x}{\log x} \right).
\]

Given any integer \( n \) with \( 1 \leq n \leq x \), let \( P(n) \) denote the largest prime factor of \( n \). Then one has

\[
\psi(x, y) = 1 + \sum_{p \leq y} \text{card} \{ 2 \leq n \leq x : P(n) = p \} = 1 + \sum_{p \leq y} \psi(x/p, p).
\]

Thus, when \( y = x^{1/u} \) with \( U < u \leq U + 1 \), one sees that

\[
\psi(x, y) - \psi(x, x^{1/u}) = \sum_{p \leq y} \psi(x/p, p) - \sum_{p \leq x^{1/u}} \psi(x/p, p)
\]

\[
= \sum_{y < p \leq x^{1/u}} \psi(x/p, p).
\]

We may now apply the inductive hypothesis, since \( x/p = p^u \), with

\[
u_p = \frac{\log (x/p)}{\log p} = \frac{\log x - \log p}{\log p} \leq u - 1 \leq U.
\]

Thus,

\[
\sum_{y < p \leq x^{1/u}} \psi(x/p, p) = \sum_{y < p \leq x^{1/u}} \left( \rho \left( \frac{\log x}{\log p} - 1 \right) \frac{x}{p} + O \left( \frac{x}{\log x} \frac{\sum_{y < p \leq x^{1/u}}}{p} \right) \right)
\]

\[
= x \sum_{y < p \leq x^{1/u}} \frac{\rho \left( \frac{\log x}{\log p} - 1 \right)}{p} + O \left( \frac{x}{\log x} \frac{\sum_{y < p \leq x^{1/u}}}{p} \right)
\]

\[+ O \left( \frac{x}{\log x} \frac{\sum_{y < p \leq x^{1/u}}}{p} \right).
\]
Since \[ \sum_{y < p \leq x^{1/5}} \frac{1}{p} = \log \left( \frac{x \log x}{\log y} \right) = \log \left( \frac{u}{u} \right) = 0(1) \],

we estimate the main term by applying Riemann - Stieltjes integration. Put

A(z) = \sum_{p \leq z} \frac{1}{p} \text{ so by Mertens' theorem we have } A(z) = \log \log z + C + o(1),

where \( r(z) = O \left( \frac{1}{\log z} \right) \). Thus,

\[ \sum_{y < p \leq x^{1/5}} \frac{\rho \left( \frac{\log x}{\log y} \right) - 1}{p} = \int_{y}^{x^{1/5}} \rho \left( \frac{\log x}{\log y} \right) - 1 \, dA(z) \]

\[ = \int_{y}^{x^{1/5}} \rho \left( \frac{\log x}{\log y} \right) - 1 \, d \log \log y + \int_{y}^{x^{1/5}} \frac{\rho \left( \frac{\log x}{\log y} - 1 \right)}{y} \, dr(z) \]

\[ = \int_{u}^{U} \rho(t-1) \frac{dt}{t} + \left[ \rho \left( \frac{\log x}{\log z} - 1 \right) \right]_{y}^{x^{1/5}} - \int_{y}^{x^{1/5}} \frac{\rho \left( \frac{\log x}{\log y} \right) - 1}{y} \, d r(z) \]

\[ (t = \frac{\log x}{\log y}) \]

\[ \frac{dt}{t} = - \frac{\log x \, dz}{z \left( \log z \right)^2 \, (\log x / \log z)} = d \left( \log \log z \right) \]

(12.11)

\[ = \rho(U) - \rho(u) + O \left( \frac{1}{\log x} \left( 1 + \int_{X^{1/5}}^{x^{1/5}} d \rho \left( \frac{\log x}{\log y} - 1 \right) \right) \right) \]

\[ = \rho(U) - \rho(u) + O \left( \frac{1}{\log x} \right). \]

We conclude that

\[ \psi(x, y) - \psi(x, x^{1/5}) = -x (\rho(U) - \rho(u)) + O \left( x / \log x \right), \]

\[ = x \rho(u) + O \left( x / \log x \right), \]

since \( \psi(x, x^{1/5}) = x \rho(U) + O \left( x / \log x \right) \).

This confirms the inductive hypothesis when \( U \leq u \leq U+1 \), and so the conclusion of the theorem follows by induction. //

The behaviour of \( \psi(x, y) \) is very different when \( y \) is
extremely small, and this is of interest for cryptographic purposes. One can obtain asymptotic information here, but to illustrate ideas we obtain a lower bound useful in many applications.

**Theorem 12.3.** Suppose that \( y \leq x \leq y \). Then one has

\[ \psi(x, y) \gg \frac{x}{y} \exp \left( - u \log \log x + \frac{u}{2} \right), \]

where \( u = \frac{\log x}{\log y} \).

**Corollary 12.4.** One has \( \psi(x, (\log x)^a) \gg x^{1 - \frac{1}{a} + o(1)} \) for \( a > 1 \).

[In fact, also, \( \psi(x, (\log x)^a) \ll x^{1 - \frac{1}{a} + o(1)} \).]

**Proof.** From Theorem 12.3, one has

\[ \psi(x, (\log x)^a) \gg x (\log x)^{1-a} \exp \left( - \frac{\log x}{a \log \log x} \log \log x + \frac{1}{2} \frac{\log x}{a \log \log x} \right) \]

\[ = x^{1 - \frac{1}{a}} \exp \left( \frac{1}{2} \frac{\log x}{a \log \log x} - a \log \log x \right) = x^{1 - \frac{1}{a} + o(1)}. \]

**Corollary 12.5.** When \( c > 0 \) is fixed, one has

\[ \psi(x, \exp(c \sqrt{\log x \log \log x})) \gg x \exp \left( - (c + \frac{1}{c} + o(1)) \sqrt{\log x \log \log x} \right). \]

**Proof.** From Theorem 12.3, one has

\[ \psi(x, \exp(c \sqrt{\log x \log \log x})) \gg x \exp \left( - c \sqrt{\log x \log \log x} - \frac{\log x}{c \sqrt{\log x \log \log x}} \log \log x \right. \]

\[ + \left. \frac{1}{2} \frac{\log x}{c \sqrt{\log x \log \log x}} \right) \]

\[ = x \exp \left( - (c + \frac{1}{c}) \sqrt{\log x \log \log x} + \frac{1}{2c} \frac{\log x}{\sqrt{\log x \log \log x}} \right). \]
Proof. (cf. Theorem 12.3). Write \( r = \pi(y) \), and let the first \( r \) prime numbers be \( p_1, \ldots, p_r \). Then if \( n \in S(x,y) \), one has

\[
\log n = a_1 \log p_1 + \cdots + a_r \log p_r,
\]

for some non-negative integers \( a_1, \ldots, a_r \) with \( a_1 \leq \log n / \log p_1 \). Each choice of \( a \) with \( a_1 + \cdots + a_r \leq \log n / \log p_r \) gives a unique element of \( S(x,y) \). Thus

\[
\psi(x,y) \geq \# \{ a_i \in \mathbb{Z}_{\geq 0} : a_1 + \cdots + a_r \leq \frac{\log n}{\log p_i} \in \mathbb{Z} \}.
\]

\[
= \# \{ a_i \in \mathbb{Z}_{\geq 0} : a_0 + \cdots + a_r = \frac{\log n}{\log p_r} \}
\]

\[
= \text{coeff. of } x^r \text{ in } \left( \sum_{a=0}^{\infty} \frac{x^a}{a!} \right)^r = (1 - e^{-x})^{-r} = \sum_{k=0}^{\infty} \binom{r+k}{k} \frac{x^k}{k!}
\]

By Stirling's formula, we obtain

\[
\psi(x,y) \asymp \binom{r+k}{k} \left( \frac{r+k}{r} \right)^r \frac{1}{\sqrt{k}} \quad \text{since } r \gg k.
\]

Put \( z = \frac{y}{\log y} = \sqrt{\frac{1}{\log x}} \). Then we see that

\[
\psi(x,y) \gg \left( 1 + \frac{y}{\log y} \right)^k \left( 1 + \frac{k \log y}{y} \right)^{\frac{1}{\sqrt{k}}} \gg (z (1 + \frac{1}{z})^2)^k.
\]

We have that \( z (1 + \frac{1}{z})^2 = \exp(\log(1 + \frac{1}{z}) + \log z) \) is an increasing function of \( z \) for \( z \gg 1 \), whence

\[
\psi(x,y) \gg (W(1 + \frac{1}{W})^W)^k \gg (W(1 + \frac{1}{W})^W)^{u_1},
\]

where \( W = \frac{y}{u \log y} \), and where \( u = \frac{\log x}{\log y} \). But \( W \leq \frac{1}{\sqrt{k}} \),

whence

\[
\psi(x,y) \gg \frac{1}{y} \left( \frac{y}{u \log y} \right)^u \left( 1 + \frac{u \log y}{y} \right)^{u \log y}
\]
$$\frac{1}{y} e^{\frac{u \log x}{\log y}} \exp \left( - u \log \log x + \frac{u}{\log y} \log \left( 1 + \frac{\log x}{y} \right) \right) = \frac{x}{y} \exp \left( - u \log \log x + \frac{1}{u} \frac{\log x}{\log y} \right).$$

We now turn to some applications and open problems concerning smooth numbers. First an application. Suppose that \( q \) is large, and that \( \chi \) is a non-principal character modulo \( q \). Then we have

$$\sum_{1 \leq n \leq q} \chi(n) = 0,$$

so that \( \chi(n) \neq 1 \) for some integer \( n \) with \( 1 \leq n \leq q \). But we might expect that \( \chi(n) \neq 1 \) for a much smaller value of \( q \) — this value of \( n \) is a “witness” to \( \chi \) being non-principal. To illustrate ideas, consider a large prime number \( q \).

Suppose that \( \chi(n) = 1 \) for all integers \( n \) with \( 1 \leq n \leq y \). Then in particular, one has \( \chi(m) = 1 \) for all prime numbers \( 1 \leq m \leq y \), whence \( \chi(m) = 1 \) for \( m \in S(q, \pi) \). We deduce that

$$0 = \sum_{1 \leq n \leq q} \chi(n) = \sum_{m \in S(q, \pi)} \chi(m) + \sum_{1 \leq n \leq q} \chi(n) \quad \text{with} \quad n \notin S(q, \pi)$$

$$\geq \psi(q, \pi) - (q - 1 - \psi(q, \pi)) = 2 \psi(q, \pi) - (q - 1).$$

Hence

$$\psi(q, \pi) \leq \frac{1}{2} (q - 1) \quad \Rightarrow \quad \frac{\psi(q, \pi)}{q} \leq \frac{1}{2} + O\left( \frac{1}{q} \right).$$

We now may apply Theorem 12.2, noting that when \( y = \frac{q}{2} \), one has

$$\rho \left( \frac{\log q}{\log y} \right) = \rho \left( \frac{\log q}{\log \frac{q}{2}} \right) = 1 - \log \frac{q}{2} = \frac{1}{2}.$$
Thus, the relation $\frac{\psi(q, y)}{q} \leq \frac{1}{2} + O(C_q)$ ensures that $y \leq q^{\frac{1}{8} + o(1)}$. Thus necessarily, one finds that $\chi(\pi) \neq 1$ for some prime $\pi$ with $\pi \leq q^{\frac{1}{8} + o(1)}$.

**Theorem 12.6.** When $p$ is a large prime number and $\chi$ is a non-principal character modulo $p$, there is a prime number $\pi$ with $\pi \leq p^{\frac{1}{8} + o(1)}$ for which $\chi(\pi) \neq 1$.

We'll see later that this exponent can be improved — Vinogradov conjectured that in fact the bound of Theorem 12.6 can be replaced by $\pi \leq p^{\varepsilon}$ for any $\varepsilon > 0$.

The Twin Prime Conjecture — that there are infinitely many primes $p$ for which $p + 2$ is also prime — remains wide open. On the other hand, the Twin Smooth-number Conjecture is a theorem:

**Theorem 12.7.** There are infinitely many integers $n$ for which $n$ and $n + 1$ are both $n^{\varepsilon}$-smooth, for any fixed $\varepsilon > 0$.

**Proof.** We take $n + 1 = m^k$ and $n = m^{k+1}$ with a suitable choice for $k$, and with $m = 2$, say. It follows that $n + 1$ is $n^{\varepsilon}$-smooth, and we may factor $n$ as a product of cyclotomic polynomials

\[ n = \prod_{d|\Phi_k(n)} \Phi_d(n) \]

where $\deg \Phi_d(t) = \phi(d)$.

The largest prime dividing $n$ has size $n^{\varepsilon}$, where
\[ \theta = \frac{\log \Phi_2(m)}{\log m^k} \leq \frac{\phi(k)}{k} = \frac{\pi}{\pi k^p}. \]

Then if we take \( k = \frac{\pi}{p \log y} \),

\[ \frac{\phi(k)}{k} = \frac{\pi}{p \log y} \left( 1 - \frac{1}{p} \right) \leq \frac{1}{\log y}, \]

whence

\[ \theta \leq \frac{1}{\log y} \leq \frac{1}{\log \log k} \leq \frac{1}{\log \log \log n}. \]

Since \( \theta < \varepsilon \) whenever \( n \) is sufficiently large, it follows that \( n \) and \( n+1 \) are both \( n^\varepsilon \)-smooth.

Similar arguments show that one can find integers \( n \) infinitely often with

\[ n, n+1, \ldots, n+t(n) \quad \text{all } n^\varepsilon \text{-smooth}, \]

where

\[ t(n) \geq \frac{\log \log \log n}{\log \log \log \log n} \] (Balog & H.)

Also, given \( \phi(n) = a t^2 + b t + c \in \mathbb{Z}[t] \), it follows that there are infinitely many integers \( n \) so that \( \phi(n) \) is \( n^\varepsilon \)-smooth (Balog, Frei, Matrin & H.). But no such result is known for cubic or higher degree polynomials.
§13. The functional equation for \(\zeta(s)\).

We saw in Corollary 2.6 that the Riemann zeta function is analytic in \(\Re(s) > 0\) except for a simple pole at \(s = 1\) with residue 1. We now seek to extend this continuation to over the whole complex plane. This requires new ideas, and leads to an elegant formulation in what is known as the functional equation.

**Theorem 13.1**

The function

\[
\zeta(s) = \frac{1}{2} s(s-1) \frac{s}{\pi^2} \Gamma(\frac{s}{2}) \pi^{-s/2}
\]

is entire, and one has

\[
\zeta(s) = \zeta(1-s) \quad \text{for all } s \in \mathbb{C} \quad \text{(the functional equation)}
\]

The proof of this assertion requires some preparation — perhaps the most elegant route goes via the application of theta functions. There is a kind of reciprocity for theta functions that follows via Poisson summation.

**Theorem 13.2.** Let \(\alpha \in \mathbb{R}\) and \(z \in \mathbb{C}\) satisfy \(\Re(z) > 0\). Then one has

\[
\sum_{n = -\infty}^{\infty} e^{-\pi (n+\alpha)^2 z} = z^{-\frac{1}{2}} \sum_{k = -\infty}^{\infty} e(k\alpha) e^{-\pi k^2 / z}  \quad (13.1)
\]

Moreover, under the same conditions,

\[
\sum_{n = -\infty}^{\infty} (n+\alpha) e^{-\pi (n+\alpha)^2 z} = -i z^{-3/2} \sum_{k = -\infty}^{\infty} k e(k\alpha) e^{-\pi k^2 / z} \quad (13.2)
\]

in which the branch of \(z^{3/2}\) is determined by taking \(1^{3/2} = 1\).

*Proof.* Write \(f(u) = e^{-\pi u^2 z}\), so that the left hand side of (13.1) can be written as

\[
\sum_{n \in \mathbb{Z}} f(n+\alpha).
\]
Notice how that since $\text{Re}(z) > 0$, this sum is absolutely convergent for each fixed value of $x$, and also uniformly convergent for $x$ in any compact set. The Fourier transform of $f$ is given by

$$\mathcal{F}(t) := \int_{\mathbb{R}} f(u) e(-tu) \, du,$$

so that if $g(u) = f(u + a)$, we have

$$\mathcal{F}(t) = \int_{\mathbb{R}} f(u + a) e(-tu) \, du = \int_{\mathbb{R}} f(v) e(-t(v-a)) \, dv = e(ta) \mathcal{F}(t).$$

Moreover,

$$\mathcal{F}(t) = \int_{-\infty}^{\infty} \exp\left( -\pi u^2 z - 2\pi itu \right) \, du$$

$$= e^{-\pi t^2 z} \int_{-\infty}^{\infty} e^{-\pi (u + it/z)^2} \, du$$

$$= e^{-\pi t^2 z} \int_{-\infty}^{\infty} e^{-\pi \left( z^{-\frac{1}{2}} u + it/z z^{-\frac{1}{2}} \right)^2} \, du,$$

where

$$s = z^{-\frac{1}{2}} u + it/z z^{-\frac{1}{2}}.$$

We can move the contour $C$ defined by $s = z^{-\frac{1}{2}} u + it/z z^{-\frac{1}{2}}$ to the real line, since when $|u|$ is large, say $|u| > R$, we have $|e^{-\pi s^2}|$ very small,

$$|e^{-\pi s^2}| = e^{-\pi u^2 \text{Re}(z)} = O(e^{-R^2 \cdot \text{Re}(z)}).$$

Thus the contribution from this manoeuvre at $\infty$ is 0, and one obtains

$$\mathcal{F}(t) = z^{-\frac{1}{2}} e^{-\pi t^2 z} \int_{-\infty}^{\infty} e^{-\pi s^2} \, ds = z^{-\frac{1}{2}} e^{-\pi t^2 z}.$$

We may conclude thus far that

$$\sum_{k \in \mathbb{Z}} \hat{g}(k) = \sum_{k \in \mathbb{Z}} e(k\alpha) \mathcal{F}(k) = \sum_{k \in \mathbb{Z}} e(k\alpha) \cdot z^{-\frac{1}{2}} e^{-\pi k^2 z}.$$
Since \( \text{Re}(z) > 0 \), the Fourier transforms \( \hat{g}(k) = e^{kx} \hat{f}(k) \) are rapidly decaying as \( |k| \to \infty \), and thus by Poisson summation we have

\[
\sum_{k \in \mathbb{Z}} \hat{g}(k) = \sum_{n \in \mathbb{Z}} g(n) = \sum_{n \in \mathbb{Z}} f(n + \alpha) = \sum_{n \in \mathbb{Z}} e^{-\pi (n + \alpha)^2 z}.
\]

Thus,

\[
\sum_{n \in \mathbb{Z}} e^{-\pi (n + \alpha)^2 z} = z^{-1/2} \sum_{k \in \mathbb{Z}} e(k \alpha) e^{-\pi k^2 / z},
\]

In order to obtain the relation (13.2), we may differentiate (13.1) with respect to \( \alpha \) noting that the series are uniformly convergent for \( \alpha \in \text{compact sets} \). Thus

\[
\frac{d}{d\alpha} \left( \sum_{n=-\infty}^{\infty} \Theta \left( -\pi (n + \alpha)^2 z \right) \right) = \frac{d}{d\alpha} \left( z^{-1/2} \sum_{k=-\infty}^{\infty} e(k\alpha) \exp \left( -\pi k^2 / z \right) \right)
\]

\[
\downarrow
\]

\[
\sum_{n=-\infty}^{\infty} -2\pi (n + \alpha) z \Theta \left( -\pi (n + \alpha)^2 z \right) = z^{-1/2} \sum_{k=-\infty}^{\infty} \alpha k e(k\alpha) \exp \left( -\pi k^2 / z \right).
\]

Define \( \Gamma(s, \alpha) = \int_{\alpha}^{\infty} e^{-w} w^{s-1} \, dw \) (incomplete gamma function).

**Theorem 13.3.** Suppose that \( s \in \mathbb{C} \setminus [0, 1]^\uparrow \) and \( \text{Re}(z) > 0 \). Then one has

\[
\zeta(s) \Gamma(s/2, \pi z) = \pi^{-s/2} \sum_{n=1}^{\infty} \Gamma \left( \frac{s}{2}, \pi n^2 z \right) n^{-s} + \pi^{(s-\frac{1}{2})/2} \sum_{n=1}^{\infty} \Gamma \left( \frac{s-\frac{1}{2}}{2}, \pi n^2 z \right) n^{s-1} + \frac{\pi^{(s-\frac{1}{2})/2}}{\sin \pi s}.
\]

**Proof.** Recall that when \( s > 0 \), one has

\[
\Gamma(s) = \int_{0}^{\infty} x^{s-1} e^{-x} \, dx.
\]
Then by the change of variable \( x = \pi n^2 u \), we see that
\[
\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-x} x^{s/2 - 1} \, dx = n^s \pi^{s/2} \int_0^\infty e^{-\pi n^2 u} u^{s/2 - 1} \, du.
\]
We therefore deduce that when \( \sigma > 1 \) (required to obtain absolute conv. of \( \sum \))
\[
\sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 u} u^{s/2 - 1} \, du = \int_0^\infty u^{s/2 - 1} \left( \sum_{n=1}^\infty \frac{e^{-\pi n^2 u}}{u} \right) \, du
\]

\[
\sum_{n=1}^\infty n^{-s} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) = \zeta(s) \pi^{-s/2} \Gamma\left(\frac{s}{2}\right).
\]

We next adjust the integral on the right hand side so as to rewrite it in terms of incomplete gamma functions (with improved convergence properties). When
\[\text{Re}(z) > 0,\]
we may move the path of integration \([0, \infty)\) to \([0, z] \cup [z, \infty)\). Then on integrating
\[\Theta_1(u) = \sum_{n=1}^\infty e^{-\pi n^2 u},\]
we see that
\[\int_0^\infty \Theta_1(u) u^{s/2 - 1} \, du = \int_0^z \Theta_1(u) u^{s/2 - 1} \, du + \int_z^\infty \Theta_1(u) u^{s/2 - 1} \, du.
\]

\[I_1 \quad I_2\]

Now, reversing course, we see that
\[I_2 = \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 u} u^{s/2 - 1} \, du \quad \text{with} \quad u = x/(n^2 \pi),
\]
\[= \sum_{n=1}^\infty n^{-s} \pi^{-s/2} \int_0^\infty e^{-x} x^{s/2 - 1} \, dx \quad \text{with} \quad x = \pi n^2 z,
\]
\[= \pi^{-s/2} \sum_{n=1}^\infty n^{-s} \Gamma\left(\frac{s}{2}, \pi n^2 z\right),
\]

Thus far we have shown that
\[\zeta(s) \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) = \pi^{-s/2} \sum_{n=1}^\infty n^{-s} \Gamma\left(\frac{s}{2}, \pi n^2 z\right) + I_1.
\]
In order to analyze $I_1$, we put $\Theta(u) = 1 + 2\Theta_1(u)$, so

$$\Theta(u) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 u}.$$ 

Then

$$I_1 = \int_0^{\infty} \frac{1}{2} (\Theta(u) - 1) u^{s/2} - 1 \, du = \frac{1}{2} \int_0^{\infty} \Theta(u) u^{s/2} - 1 \, du - \frac{1}{2} \int_0^{\infty} u^{s/2} - 1 \, du.$$ 

But by Theorem 10.2, we have $\Theta(u) = u^{-1/2} \Theta(1/u)$. Thus

$$\frac{1}{2} \int_0^{\infty} \Theta(u) u^{s/2} - 1 \, du = \frac{1}{2} \int_0^{\infty} \Theta(1/u) u^{(s-3)/2} \, du$$

$$= \int_0^{\infty} \Theta(1/u) u^{(s-3)/2} \, du + \frac{1}{2} \int_0^{\infty} u^{(s-3)/2} \, du$$

$$= \int_0^{\infty} \Theta_1(v) v^{-(s+1)/2} \, dv$$

Then

$$I_1 + \frac{1}{2} \frac{1}{s-1} \frac{1}{s-1/2} = \sum_{n=1}^{\infty} \int_{1/2}^{\infty} e^{-\pi n^2 v} v^{-(s+1)/2} \, dv$$

$$= \sum_{n=1}^{\infty} \frac{1}{2} \frac{1}{s-1} \pi^{(s-1)/2} \int_{\pi n^2 / 2}^{\infty} e^{-x} x^{-(s+1)/2} \, dx$$

$$= \pi^{(s-1)/2} \sum_{n=1}^{\infty} n^{s-1} \Gamma((1-s)/2, \pi n^2 / 2).$$

Then we conclude that

$$\delta(s) \pi^{-s/2} \Gamma(s/2) = \pi^{-s/2} \sum_{n=1}^{\infty} n^{-s} \Gamma(s/2, \pi n^2 / 2)$$

$$+ \pi^{(s-1)/2} \sum_{n=1}^{\infty} n^{s-1} \Gamma((1-s)/2, \pi n^2 / 2)$$

$$- \frac{1}{s} \pi^{s/2} + \frac{1}{s-1} \frac{1}{s-1/2} \pi^{(s-1)/2}.$$  \quad (13.91)$$

Thus far, we have established this conclusion for $\text{Re}(s) > 1$. 

However, when \( \sigma < 0/13 \), the last two terms are entire. Also, for each \( z \) with \( \Re(z) > 0 \) and \( |z| > 1 \), one has

\[
\Gamma(s, a) = \int_{1}^{\infty} e^{-u} u^{s-1} du
= \int_{0}^{\infty} e^{-u-a} (u+a)^{s-1} du
\]

\[
\ll 1 \cdot 1^{s-1} \quad \text{uniformly for } |a| \leq c,
\]

\[
\ll 1 \cdot |a|^{s-1} \quad \text{fixed } c > 0.
\]

\[
n^{-s} \Gamma(s/2, \pi n^2/2) \ll r^{-s}, \quad (n^2)^{s/2} - 1 = n^{-2}.
\]

Then

\[
\sum_{n=1}^{\infty} n^{-s} \Gamma(s/2, \pi n^2/2)
\]

is uniformly convergent for \( s \in \) any compact set, whence (Weierstrass) entire. Likewise for

\[
\sum_{n=1}^{\infty} n^{s-1} \Gamma((1-s)/2, \pi n^2/2)
\]

since

\[
n^{-s} \Gamma((1-s)/2, \pi n^2/2) \ll r^{s-1}, \quad (n^2)^{1/2} - 1 \ll n^{-2}.
\]

Then the four terms on the right hand side of (13.1) are all entire, except possibly at \( s = 0 \) and \( 1 \).

By the uniqueness of analytic continuation, therefore, the identity (13.1) holds for all \( s \in \mathbb{C} \) except at \( s = 0 \) and \( 1 \).

---

Proof of Theorem 13.1: The expression on the right hand side in Theorem 13.3 is left unchanged if we interchange

\[
s \quad \text{with} \quad 1-s
\]

and

\[
z \quad \text{with} \quad 1/z.
\]

Thus the left hand side is also left unchanged by the same operations, and we have

\[
5(s) \Gamma(s/2) \pi^{-s/2} = 5(1-s) \Gamma((1-s)/2) \pi^{-(1-s)/2}.
\]
except possibly when \( \sigma \in \{0,1\} \). If we multiply by \( s(1-s) \), then the pole of \( \zeta(s) \) is cancelled and the identity is seen to hold even when \( \sigma \in \{0,1\} \).

**Corollary 13.4.** When \( s \neq 1 \), one has
\[
\zeta(s) = \frac{\zeta(1-s)}{2^s \pi^{s-1} \Gamma(1-s) \sin \left( \frac{1}{2} \pi s \right)}.
\]

**Proof:** We have \( \zeta(s) = \frac{\zeta(1-s)}{s(1-s)} \), and thus for \( s \neq 1 \) one has
\[
\frac{1}{2} s(s-1) \zeta(s) \Gamma \left( \frac{s}{2} \right) \pi^{-s/2} = \frac{1}{2} (1-s)(-s) \zeta(1-s) \Gamma \left( \frac{1-s}{2} \right) \pi^{1-s/2}
\]
\[
\Downarrow
\]
\[
\zeta(s) = \frac{\zeta(1-s)}{\Gamma \left( \frac{1}{2} \right) \pi^{1-s/2}}.
\]

This relation requires some careful consideration when \( s = 0 \) and when \( \Gamma \left( \frac{1}{2}(1-s) \right) \) is unbounded, but this can be justified via analyticity. Moreover, one has the classical relations
\[
\Gamma(s) \Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \quad \text{and} \quad \Gamma(s) \Gamma(s+\frac{1}{2}) = \sqrt{\pi} 2^{1-2s} \Gamma(2s).
\]

Thus
\[
\frac{1}{\Gamma \left( \frac{1}{2} s \right)} = \frac{\sin \left( \frac{1}{2} \pi s \right)}{\pi} \Gamma \left( 1 - \frac{1}{2} s \right)
\]
\[
\Rightarrow \quad \frac{\Gamma \left( \frac{1}{2} - \frac{1}{2} s \right)}{\Gamma \left( \frac{1}{2} s \right)} = \frac{\sin \left( \frac{1}{2} \pi s \right)}{\pi} \Gamma \left( 1 - \frac{1}{2} s \right) = \frac{\sin \left( \frac{1}{2} \pi s \right)}{\pi} 2^{s} \Gamma(1-s).
\]

Then we conclude that
\[
\zeta(s) = \frac{\zeta(1-s)}{2^s \pi^{s-1} \Gamma(1-s) \sin \left( \frac{1}{2} \pi s \right)}.
\]

Observe that putting \( s = -\sigma \), with \( \sigma > 0 \), we see that
\[
\zeta(-\sigma) = \zeta(\sigma+1) 2^{-\sigma} \pi^{-\sigma-1} \Gamma(\sigma+1) \sin \left( \frac{1}{2} \pi \sigma \right).
\]

We know already that \( \zeta(s) \neq 0 \) for \( \text{Re}(s) > 0 \), and thus it follows that \( \zeta(-\sigma) = 0 \) if and only if \( \sin \left( \frac{1}{2} \pi \sigma \right) = 0 \), which is to say when \( \sigma \) is an even integer.

**Corollary 13.5.** Other than the trivial zeros \(-2,-4,\ldots\), all zeros of \( \zeta(s) \) satisfy \( 0 < \text{Re}(s) < 1 \).
We also see that

\[ S(\sigma) = \lim_{\sigma \to 0} \frac{5(1-\sigma)}{2^\pi} \pi^{\sigma-1} \Gamma(1-\sigma) \sin \left( \frac{\pi \sigma}{2} \right) \]

\[ = \pi^{-1} \cdot \frac{1}{2} \pi \Gamma(1) \lim_{\sigma \to 0} -\frac{1}{\sigma} \cdot \sigma = -\frac{1}{2}. \]

It is of course conjectured that all non-trivial zeros \( \rho \) of \( S(\sigma) \) satisfy \( \Re(\rho) = \frac{1}{2} \). We have shown that \( S(\rho) = 0 \) implies that \( \rho < 1 - c/\log(1t1+4) \) for a suitable \( c > 0 \). It follows from Corollary 13.4 that if \( \rho \) is non-trivial, then in fact

\[ \rho \geq c/\log(1t1+4). \]


We have shown (Corollary 8.4 - Dirichlet's theorem) that when \( a \in \mathbb{Z} \) and \( q \in \mathbb{N} \) satisfy \( (a,q) = 1 \), then there are infinitely many prime numbers \( p \) with \( p \equiv a \pmod{q} \). Further,

\[ \sum_{\substack{p \leq x \atop p \equiv a \pmod{q}}} \frac{\log p}{p} = \frac{1}{\phi(q)} \log x + O_q(1). \]

This suggests an equidistribution of primes amongst the residue classes \( a \pmod{q} \) with \( (a,q) = 1 \), so that we expect that

\[ \pi(x; q, a) : = \sum_{\substack{p \leq x \atop p \equiv a \pmod{q}}} 1 \]

should satisfy

\[ \pi(x; q, a) \sim \frac{1}{\phi(q)} \cdot \frac{x}{\log x}, \quad \text{as} \ x \to \infty. \]

It is our objective now to prove this asymptotic formula in as much uniformity as we are able, adopting a strategy that parallels that
employed in our proof of the Prime Number Theorem

\[ \pi(x) = \text{li}(x) + O\left(x \exp\left(-c\sqrt{\log x}\right)\right) \quad (c > 0). \]

We begin by deriving a zero-free region. This entails some preliminary
examination of the behaviour of the L-functions \( L(s, \chi) \),
defined for \( \sigma > 1 \) in the relation

\[ L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \]

in which \( \chi \) is a Dirichlet character modulo \( q \).

We first seek an analogue of Lemma 10.5 with \( L(s, \chi) \) in place
of \( \zeta(s) \). This entails bounding \( L(s, \chi) \) (cf. Lemma 10.4).

**Lemma 14.1.** Let \( \chi \) be a non-principal character modulo \( q \), and
suppose that \( \sigma > 0 \) is fixed. Then one has

\[ L(s, \chi) \ll (1 + (q^{-2})^{1-\sigma}) \min\left\{ \frac{1}{10^{-11}}, \log(q^{-2}) \right\}, \]

uniformly for \( \sigma \leq \sigma \leq 2 \).

**Proof:** Write

\[ S(u, \chi) = \sum_{\chi(n) \leq u} \chi(n). \]

Then by Riemann-Stieltjes integration, one finds that

\[ L(s, \chi) = \sum_{n \leq x} \chi(n)n^{-s} + \int_{x^-}^{\infty} u^{-s} \, dS(u, \chi) \]

\[ = \sum_{n \leq x} \chi(n)n^{-s} + \left[ S(u, \chi)u^{-s} \right]_{x^-}^{\infty} - \int_{x^-}^{\infty} S(u, \chi) \, d u^{-s} \]

\[ = \sum_{n \leq x} \chi(n)n^{-s} - S(x, \chi)x^{-s} + s \int_{x^-}^{\infty} u^{-s-1} S(u, \chi) \, du. \]

Observe that by the orthogonality of non-principal characters, one has
\[ \sum_{n=m+1}^{m+q} \lambda(n) = 0 \quad \text{for all } m, \text{ where} \]
\[ |S(n, \chi)| \leq q. \]

Then if we take \( x = q^2 \), we find that the final two terms contribute
\[ \ll q \cdot (q^2)^{-\sigma} + q^2 \int_{x}^{\infty} u^{-\sigma-1} \, du \ll (q^2)^{1-\sigma}. \]

In the proof of Lemma 10.4 we showed that
\[ \sum_{n \leq x} n^{-s} \ll (1 + x^{1-\sigma}) \min \left\{ \frac{1}{1-\sigma}, \log x \right\}, \]
whence
\[ \sum_{n \leq x} \chi(n) n^{-s} \ll (1 + x^{1-\sigma}) \min \left\{ \frac{1}{1-\sigma}, \log x \right\} \ll (1 + (q^2)^{1-\sigma}) \min \left\{ \frac{1}{1-\sigma}, \log (q^2) \right\}. \]

Hence
\[ L(s, \chi) \ll (1 + (q^2)^{1-\sigma}) \min \left\{ \frac{1}{1-\sigma}, \log (q^2) \right\}. \]

Now we come to the analogue of Lemma 10.5. Write \( E_0(\chi) = \begin{cases} 1, & \text{when } \chi = \chi_0, \\ 0, & \text{when } \chi \neq \chi_0. \end{cases} \)

**Lemma 14.2.** Suppose that \( \chi \) is a character modulo \( q \). Then whenever \( \frac{5}{6} \leq \sigma \leq 2 \), one has
\[ \frac{-L'}{L} (s, \chi) = \frac{E_0(\chi)}{s-1} - \sum_{\rho} \frac{1}{s-\rho} + O \left( \log (q^2) \right), \]
where the sum is over all zeros \( \rho \) of \( L(s, \chi) \) with \( 1 - (\frac{3}{2} + it) \leq \frac{5}{6}. \)

**Proof.** Suppose first that \( \chi \neq \chi_0 \), so that \( E_0(\chi) = 0 \). In this situation we apply Lemma 10.3 with \( f(z) = L(z + (\frac{3}{2} + it), \chi), \) \( r = \frac{5}{6} \) and \( \sigma = \frac{2}{3} \). We have from Lemma 14.1 that when \( |2| \leq 1 \)
then
\[ f(z) \ll (1 + (q^2)^{1/2}) \log (q^2) \ll q^2 \quad (\text{Re}(z) \leq \frac{1}{4}), \]
and when \( \text{Re}(z) > \frac{1}{2} \) we have
\[ L(s, \chi) \ll \sum_{n \geq 1} n^{-s} \ll 1, \]
so the bound is trivial. Thus we can take \( M = O(q^2) \) in Lemma 10.3.
Moreover, one has
\[ |f(\alpha)| = \left| \frac{1}{5} \left( \frac{3}{2} + it, \chi \right) \right| = \prod_p \left( 1 - \chi(p) p^{-\frac{3}{2} + it} \right)^{-1} \]
\[ \geq \prod_p \left( 1 + p^{-\frac{3}{2}} \right)^{-1} \gg 1. \]

Then
\[ -\frac{L'}{L}(s, \chi) = -\sum_p \frac{1}{s - \rho} + O\left( \log \left( \frac{M}{\log \|10\|} \right) \right), \]
\[ \ll \log \|q\|. \]

where the summation is over zeros \( \rho \) of \( L(s, \chi) \) with \( |\rho - \left( \frac{3}{2} + it \right)| \leq \frac{3}{4} \).

Suppose next that \( \chi = \chi_0 \). Then we have
\[ L(s, \chi_0) = \frac{\zeta(s)}{\prod_{p \mid q} \left( 1 - p^{-s} \right)} \quad \text{for} \quad s > 1, \]
But the right hand side is analytic for \( s \in \mathbb{C} \) except for a simple pole at \( s = 1 \). The function \( L(s, \chi_0) \) is therefore given by the right hand side for all \( s \in \mathbb{C} \setminus \{1\} \) by uniqueness of analytic continuation. The zeros of \( L(s, \chi_0) \) are therefore given by the zeros of \( \zeta(s) \) together with the zeros of \( \prod_{p \mid q} \left( 1 - p^{-s} \right) \) (for \( p \mid q \)). The latter are given by
\[ s = \frac{2\pi i}{\log p} + k \quad (k \in \mathbb{Z}). \]
All such zeros satisfy
\[ \left| \frac{2\pi i}{\log p} + k - \left( \frac{3}{2} + it \right) \right| > \frac{3}{2}, \]
so that they are excluded from the sum over \( \rho \) in the statement of the lemma. Thus
\[ \frac{L'}{L}(s, \chi_0) = \frac{d}{ds} \left( \frac{\zeta(s)}{\prod_{p \mid q} \left( 1 - p^{-s} \right)} \right) \]
\[ \frac{\zeta(s)}{\prod_{p \mid q} \left( 1 - p^{-s} \right)} \]
\[ = \frac{\zeta'(s)}{\zeta(s)} + \sum_{p \mid q} \frac{\log p}{p^s - 1}. \]
Here, when $\sigma > \frac{5}{6}$, we have

$$\sum_{plq} \frac{\log p}{p^{\sigma-1}} \ll \sum_{plq} \frac{\log p}{p^\sigma} \ll \omega(q) \ll \log q.$$ 

Thus, making use of Lemma 10.15, we deduce that

$$- \frac{L'}{L}(s, \chi_0) = \frac{1}{s-1} - \sum_p \frac{1}{s-p} + O(\log T) + O(\log q)$$

$$= \frac{E_0(\chi_0)}{s-1} - \sum_p \frac{1}{s-p} + O(\log(qT)).$$

We also require an analogue of Lemma 10.16.

**Lemma 14.13.** When $\sigma > 1$, one has

$$\text{Re} \left( -3 \frac{L'}{L}(\sigma, \chi_0) - 4 \frac{L'}{L}(\sigma + it, \chi) - \frac{L'}{L}(\sigma + 2it, \chi^2) \right) \geq 0.$$ 

**Proof.** The left-hand side here is

$$\text{Re} \left( \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}} (3 + 4\chi(n)n^{-it} + \chi(n^2)n^{-2it}) \right).$$

When $(n, q) = 1$, one has

$$\chi(n)n^{-it} = \exp (-it \log n + 2\pi i \delta_n),$$

for a suitable real number $\delta_n$. Thus, for some $\Theta_n \in \mathbb{R}$, we have

$$\chi(n)n^{-it} = e^{i\Theta_n},$$

whence (14.1) may be rewritten as

$$\sum_{\substack{n=1 \\ (n,q)=1}}^{\infty} \frac{\Lambda(n)}{n^{\sigma}} (3 + 4\Theta_n + \cos 2\Theta_n) \geq 0.$$ 

The non-negativity of the left-hand side therefore follows at once.

**Theorem 14.4.** There is an absolute constant $c > 0$ with the following property:
(i) If \( \chi \) is not a quadratic character modulo \( q \), then \( L(s, \chi) \) has no zeros in the region
\[
\sigma > 1 - \frac{c}{\log(qt)}.
\]
(ii) If \( \chi \) is a quadratic character modulo \( q \), then \( L(s, \chi) \) has at most one zero in the region
\[
\sigma > 1 - \frac{c}{\log(qt)}.
\]
If such a zero exists, then it is necessarily real and satisfies \( \beta < 1 \).

**Proof.** When \( \chi = \chi_0 \), we have
\[
L(s, \chi_0) = S(s) \prod_{p \mid q} \left( 1 - p^{-s} \right),
\]
so that the zeros of \( L(s, \chi_0) \) are those of \( S(s) \) together with zeros of the factors \( 1 - p^{-s} \) (p19). There are no zeros of the former with
\[
\sigma > 1 - \frac{c_1}{\log t},
\]
for a suitable \( c_1 > 0 \). As for the latter factors, these have zeros precisely where
\[
\exp(s \log p) = 1 \quad \iff \quad s = \frac{2k\pi i}{\log p} \quad (k \in \mathbb{Z}),
\]
and again, none of these zeros satisfy \( \sigma > 1 - \frac{c}{\log (qt)} \) (they have real part 0).

Henceforth, we may suppose that \( \chi \) is non-principal. Here we observe first that in view of the Euler product
\[
L(s, \chi) = \prod_p \left( 1 - \frac{\chi(p)}{p^s} \right)^{-1} \quad (\sigma > 1),
\]
we have \( \Re\left( L(s, \chi) \right) \geq \prod_p (1 + p^{-s})^{-1} > 0 \quad (s > 1) \).

Thus \( L(s, \chi) \neq 0 \) for \( s > 1 \). We therefore seek to show that no zeros exist with real part small and to the left of the \( 1 \)-line. By way of deriving a contradiction, suppose that \( L(s, \chi) \) has a zero \( \rho = \rho_0 + i\gamma_0 \) with \( \gamma_0 \in \mathbb{R} \), and \( \frac{1}{2} \leq \rho_0 \leq 1 \). We divide into cases:

(i) Suppose that \( \chi \) is complex. Thus \( \chi(n) \) is not real-valued.

We seek to apply Lemma 14.3, this asserting that

\[
\Re\left( -3 \frac{L'}{L} (s, \chi_0) - 4 \frac{L'}{L} (s + it, \chi) - \frac{L'}{L} (s + 2it, \chi^2) \right) > 0,
\]

and apply the argument underlying the proof of Theorem 10.7. Thus, we consider a small positive number \( \delta \) and note that when \( s = \sigma + it \) and \( \sigma = 1 + \delta \), then for all zeros \( \rho \) of \( L(s, \chi) \) we have \( \Re (s - \rho) > 0 \) and \( \Re (\frac{1}{s - \rho}) > 0 \). Hence, by Lemma 14.2, for a suitable positive number \( c_1 \), we have

\[
\begin{align*}
- \Re \left( \frac{L'}{L} (1 + \delta, \chi_0) \right) &= \Re \left( \frac{1}{\delta} - \sum_{\rho} \frac{1}{1 + \delta - \rho} \right) + O \left( \log q \right) \\
&\leq \frac{1}{\delta} + c_1 \log q,
\end{align*}
\]

\[
\begin{align*}
- \Re \left( \frac{L'}{L} (1 + \delta + i\gamma_0, \chi) \right) &= \Re \left( - \sum_{\rho} \frac{1}{1 + \delta + i\gamma_0 - \rho} \right) + O \left( \log(q(1 + \gamma_0 + 4)) \right) \\
&\leq - \frac{1}{1 + \delta - \beta_0} + c_1 \log q(1 + \gamma_0 + 4),
\end{align*}
\]

and

\[
\begin{align*}
- \Re \left( \frac{L'}{L} (1 + \delta + 2i\gamma_0, \chi^2) \right) &\leq \Re \left( - \sum_{\rho'} \frac{1}{1 + \delta + 2i\gamma_0 - \rho'} \right) + O \left( \log(q(2 + \gamma_0 + 4)) \right) \\
&\leq (14.2)
\end{align*}
\]
In the last expression, we omit any poles \( \rho \neq \frac{1}{2} \). Notice that \( \chi^2 \neq \chi_0 \), so our application of Lemma 14.2 avoids a pole from the term \( \frac{E_\rho(\chi^2)}{s-1} \).

By taking the evident linear combination of these terms, we see that from Lemma 14.3,

\[
0 \leq \text{Re} \left( -3 \frac{L'}{L}(\sigma, \chi_0) - 4 \frac{L'}{L}(\sigma + it, \chi) - \frac{L'}{L}(\sigma + 2it, \chi^2) \right) \leq \frac{3}{8} - \frac{4}{1+5-s_\rho} + 8c_1 \log(1+O(1)).
\]

It is apparent that \( s_\rho < 1 \), for otherwise, if one were to have \( s_\rho = 1 \), we obtain

\[
-\frac{1}{8} + 8c_1 \log(1+O(1)) \geq 0,
\]

which yields a contradiction on taking \( s \) sufficiently small. We may therefore take \( s = 6(1-\beta_0) \), where

\[
\frac{3}{6(1-\beta_0)} - \frac{4}{7(1-\beta_0)} + 8c_1 \log(1+O(1)) \geq 0.
\]

Equivalently,

\[
1-\beta_0 \geq \frac{1}{14 \cdot 8c_1 \log(1+O(1))}.
\]

Then \( L(s, \chi) \) has no zeros in the region \( \sigma > 1 - \frac{c}{\log(q \gamma_0)} \), provided that \( c < \frac{1}{112c_1} \).\( \square \)

The situation in which \( \chi \) is real and non-principal, which is to say, quadratic, is more complicated, and requires a further subdivision into cases.

(ii) Suppose that \( \chi \) is quadratic and \( 1/\gamma_0 \geq 6(1-\beta_0) \).
In this situation with \( l \) large, a very similar argument applies. Note that we have proved earlier that \( L(1, X) \neq 0 \) (in our proof of Dirichlet's theorem). If one were to have \( \gamma_0 = 0 \), then our initial hypothesis implies that \( 1 - \beta_0 \leq \frac{1}{2} \gamma_0 = 0 \), whence \( \beta_0 \geq 1 \), and this yields \( \beta_0 = 1 \). Then we have \( \gamma_0 \neq 0 \).

We now have \( \chi^2 = \chi_0 \), so (14.2) must be replaced by the bound

\[
- \text{Re} \left( \frac{L'}{L} (1 + \delta + 2i\gamma_0, \chi^2) \right) \leq \text{Re} \left( \frac{1}{\delta + 2i\gamma_0} \right) + c_1 \log \left( q (1\gamma_0 + 4) \right)
\]

\[
\leq \frac{\delta}{\delta^2 + 4\gamma_0^2} + c_1 \log \left( q (1\gamma_0 + 4) \right).
\]

By Lemma 14.3, we now obtain

\[
0 \leq \frac{3}{8} - \frac{4}{1 + 8 - \beta_0} + \frac{\delta}{\delta^2 + 4\gamma_0^2} + 8c_1 \log \left( q (1\gamma_0 + 4) \right).
\]

Once again, if one were to have \( \beta_0 = 1 \), then we obtain a contradiction on taking \( \delta \to 0^+ \). Thus we may suppose that \( \beta_0 \geq 1 \). Since \( 1\gamma_0 \geq 6 (1 \beta_0) \), it follows that on taking \( \delta = 6(1 - \beta_0) \), we obtain the bound

\[
0 \leq \frac{1}{1 - \beta_0} \left( \frac{3}{6} - \frac{4}{7} + \frac{6}{\delta^2 + 4\gamma_0^2} \right) + 8c_1 \log \left( q (1\gamma_0 + 4) \right),
\]

whence

\[
\frac{1}{1 - \beta_0} \left( \frac{1}{2} + \frac{4}{7} - \frac{1}{\delta^2} \right) \leq 8c_1 \log \left( q (1\gamma_0 + 4) \right),
\]

which is to say

\[
1 - \beta_0 \geq \frac{1}{\frac{105}{4} \cdot 8c_1 \log \left( q (1\gamma_0 + 4) \right)} = \frac{1}{210c_1 \log \left( q (1\gamma_0 + 4) \right)}.
\]

Then \( L(\delta, X) \) has no zeros in the region \( \delta > 1 - \frac{1}{210c_1 \log (q)} \), provided that \( c < \frac{1}{210c_1} \), subject to the condition \( 1\gamma_0 \geq 6(1 - \beta_0) \).
Suppose that \( \lambda \) is quadratic and \( 0 < 1 - \gamma_0 \leq 6 (1 - \beta_0) \).

We now consider the situation in which there is a non-real zero \( \rho_0 = \rho_0 + i \gamma_0 \) in the primitive zero-free region, with \( \gamma_0 \) "small".

In this situation we are able to derive a second such zero by conjugation, for by the Schwartz reflection principle (noting that \( L(s, \lambda) \) is real when \( s \) is real), we have

\[
L(\rho_0 + i \gamma_0, \lambda) = \overline{L(\rho_0 - i \gamma_0, \lambda)} = 0.
\]

Thus, both \( \rho_0 + i \gamma_0 \) and \( \rho_0 - i \gamma_0 \) are zeros of \( L(s, \lambda) \). We therefore deduce from Lemma 14.2 that when \( 1 < \sigma \leq 2 \), one has

\[
\Re \left( -\frac{L'(\sigma, \lambda)}{L(\sigma, \lambda)} \right) \leq -\Re \left( \frac{1}{\sigma - \rho_0} + \frac{1}{\sigma - \overline{\rho_0}} \right) + c_1 \log (4q),
\]

for a suitable positive constant \( c_1 > 0 \). Hence

\[
\Re \left( -\frac{L'(\sigma, \lambda)}{L(\sigma, \lambda)} \right) \leq -\frac{2 (\sigma - \rho_0)}{(\sigma - \rho_0)^2 + \gamma_0^2} + c_1 \log (4q). \quad (14.3)
\]

In this case we can take a simpler path to showing that this primitive zero lies outside \( \Re(s) > 1 - \frac{c}{\log (4q)} \) by allowing the close pair of zeros to conspire with each other. Note that when \( \sigma > 1 \), one has

\[
-\frac{L'(\sigma, \lambda)}{L(\sigma, \lambda)} - \frac{L'(\frac{5}{2}, \lambda)}{L(\frac{5}{2}, \lambda)} = \sum_{\substack{n=1 \\
(n, q) = 1 \atop \Lambda(n) \neq 0}} \frac{\Lambda(n)}{n^\sigma} (1 + \chi(n)) \geq 0.
\]

Put \( \sigma = 1 + 13 (1 - \beta_0) \) and recall that

\[
\Re \left( \frac{L'(\sigma, \lambda)}{L(\sigma, \lambda)} (1 + \delta, \lambda_0) \right) \leq \frac{1}{\delta} + c_1 \log q.
\]

Then we see that

\[
0 \leq \frac{1}{\delta} - \frac{2 (\sigma - \beta_0)}{(\sigma - \beta_0)^2 + \gamma_0^2} + 2 c_1 \log (4q),
\]
Whence
\[
\frac{1}{1 - \beta_0} \left( \frac{1}{13} - \frac{2(13 + 1)}{(13 + 1)^2 + 6^2} \right) + 2c_1 \log (4q) \geq 0.
\]

Thus
\[
1 - \beta_0 \geq \frac{1}{\frac{1}{33} \cdot \frac{254}{2} \cdot 2c_1 \log (4q)}.
\]

Then \(L(s, \chi)\) has no zeros in the region \(\sigma > 1 - c/\log(4q)\), provided that \(c < 1/46c_1\), subject to the condition \(0 < \gamma_0 \leq 6(1 - \beta_0).\)

(iv) Suppose that \(\chi\) is quadratic and \(\gamma_0 = 0\); so \(\beta_0\) is real.

If \(\beta_0\) is a real zero of \(L(s, \chi)\), then \(\beta_0 < 1\) as a consequence of our proof that \(L(1, \chi) \neq 0\). Suppose that \(\beta_0\) and \(\beta_1\) are two such zeros, say with \(\beta_0 < \beta_1 < 1\). Then it follows from Lemma 14.2 that when \(1 < \sigma < 2\), one has
\[
\text{Re}\left( -\frac{L'(\sigma, \chi)}{L(\sigma, \chi)} \right) \leq -\frac{1}{\sigma - \beta_0} - \frac{1}{\sigma - \beta_1} + c_1 \log (4q)
\]
\[
\leq -\frac{2}{\sigma - \beta_0} + c_1 \log (4q).
\]

Thus, proceeding as in case (iii), we find that
\[
-\frac{L'(\sigma, \chi_0)}{L(\sigma, \chi_0)} = -\frac{L'(\sigma, \chi)}{L(\sigma, \chi)} \geq 0
\]

whence with \(\sigma = 1 + \delta\),
\[
0 \leq \frac{1}{\delta} - \frac{2}{\sigma - \beta_0} + 2c_1 \log (4q).
\]

Put \(\delta = 2(1 - \beta_0)\). Then we obtain
\[
\sigma \leq \frac{1}{1 - \beta_0} \left( \frac{1}{2} - \frac{2}{3} \right) + 2c_1 \log (4q),
\]
and thus
\[
1 - \beta_0 \geq \frac{1}{12c_1 \log (4q)}.
\]
Then \(L(s, \chi)\) has at most one zero \(\beta\), in the region 
\[\sigma > 1 - \frac{c}{\log(4q)}\] provided that \(c < \frac{1}{12c_1}\).

This completes the proof of the theorem.

We now turn to an analogue of Theorem 10.8 for \(L(s, \chi)\).

**Theorem 14.5.** Let \(\chi\) be a non-principal character modulo \(q\). Let \(c > 0\) be an absolute constant with the property that \(L(s, \chi) \neq 0\) for 
\[\sigma > 1 - \frac{c}{\log(q^r)}\] where, in the case that \(\chi\) is quadratic, we permit a possible exceptional zero \(\beta \in \mathbb{R}\) with \(\beta < 1\). Then whenever 
\[\sigma > 1 - \frac{c}{(2 \log(q^r))}\], one has the following:

(i) When \(L(s, \chi)\) has no exceptional zero, and when \(\beta_i\) is an exceptional zero of \(L(s, \chi)\) satisfying \(|s - \beta_i| \geq \frac{1}{\log q}\), one has 
\[
\frac{L'(s, \chi)}{L(s, \chi)} \ll \log(q^r) \quad \frac{1}{|L(s, \chi)|} \ll \log(q^r) \quad \text{and} \quad |\log L(s, \chi)| \ll \log(q^r) + O(1);
\]

(ii) When \(\beta_i\) is an exceptional zero of \(L(s, \chi)\) and \(|s - \beta_i| \leq \frac{1}{\log q}\), then 
\[
\frac{L'(s, \chi)}{L(s, \chi)} = \frac{1}{s - \beta_i} + O\left(\frac{\log q}{s - \beta_i}\right) \quad (s \neq \beta_i),
\]

\[
|\arg L(s, \chi)| \leq \frac{1}{2} \log \log q + O(1) \quad (s \neq \beta_i),
\]

and 
\[
|s - \beta_i| \ll |L(s, \chi)| \ll (s - \beta_i)(\log q)^2.
\]

**Proof.** We begin by establishing the bound \(\frac{L'(s, \chi)}{L(s, \chi)} \ll \log(q^r)\) in (i). This is simple when \(\sigma > 1 + \frac{1}{\log(q^r)}\), since then we have 
\[
\left|\frac{L'(s, \chi)}{L(s, \chi)}\right| \ll \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma}} = -\frac{\zeta'(\sigma)}{\zeta}\ll \frac{1}{\sigma - 1} \ll \log(q^r).
\]

In order to handle general \(s = \sigma + it\) with \(\sigma \geq 1 - \frac{c}{\log(q^r)}\), we seek to shift from \(s_1 = 1 + \frac{1}{\log(q^r)} + it\) to \(s\). We have
\[
\frac{L'}{L}(s, \chi) \ll \log(qT),
\]

where Lemma 14.2 yields
\[
\sum_P \frac{1}{s_1 - \rho} + O(\log(qT)) = \frac{L'}{L}(s, \chi) \ll \log(qT),
\]

where the summation is over zeros \(\rho \neq \frac{1}{2} + it\) of \(L(s, \chi)\) with \(|\rho - \frac{1}{2} + it| \leq \frac{5}{6}\). Hence
\[
\sum_P \frac{1}{s_1 - \rho} \ll \log(qT).
\]

This implies, in turn, that
\[
\sum_P \frac{1}{s_1 - \rho} = \sum_P \left( \frac{1}{s_1 - \rho} - \frac{1}{s_1 - \rho} \right) + O(\log(qT)),
\]

where for each \(\rho\) one has
\[
\frac{1}{s_1 - \rho} - \frac{1}{s_1 - \rho} = \frac{1}{s_1 - \rho} - \frac{1}{s_1 - \rho} = \frac{1 + 1/\log(qT) - \sigma}{(s_1 - \rho)(s_1 - \rho)} \ll \frac{1}{\log(qT)} \ll \text{Re}\left(\frac{1}{s_1 - \rho}\right).
\]

[Note: \(\text{Re}\left(\frac{1}{s_1 - \rho}\right) = \text{Re}\left(\frac{1 + 1/\log(qT) + it}{1 - c/2\log(qT)}\right) \gg 1/\log(qT)\) for \(\rho \neq \frac{1}{2}\), and \(\text{Re}\left(\frac{1}{s_1 - \rho}\right) \gg 1/\log(qT)\). Thus we deduce that
\[
\sum_P \frac{1}{s_1 - \rho} \ll \text{Re}\left(\sum_P \frac{1}{s_1 - \rho}\right) + O(\log(qT)),
\]

so from Lemma 14.2, we have
\[
\frac{L'}{L}(s, \chi) = \sum_P \frac{1}{s_1 - \rho} + O(\log(qT)) \ll \text{Re}\left(\sum_P \frac{1}{s_1 - \rho}\right) + O(\log(qT))
\]

\[= \text{Re}\left(\frac{L'}{L}(s, \chi)\right) + O(\log(qT)) \ll \log(qT). \quad \Box
\]

Next we consider the bound \(|L(s, \chi)| \leq \log(qT) + O(1) \neq 0\).

Again, when \(\sigma > 1 + 1/\log(qT)\), we have
\[
|\log L(s, \chi)| \leq \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-\sigma} = \log G(\sigma) \leq \log\left(\frac{\sigma}{\sigma - 1}\right) \leq \log(\log(qT)).
\]

We again put \(s_1 = 1 + 1/\log(qT) + it\), and note that when
\[1 + 1/\log(qT) > \sigma > 1 - c/2\log(qT),\]

we have
\[
\log L(s, \chi) - \log L(s, \chi) \leq \int_{s_1}^s \frac{L'}{L}(s, \chi) ds \ll \log(qT) < 1.\]
Hence \( |\log L(s, X)| \leq \log \log q + o(1) \),

and this in turn implies that

\[
\log \left( \frac{1}{L(s, X)} \right) = -\text{Re} \left( \log L(s, X) \right) \leq \log \log (q^2) + o(1)
\]

\[
\Rightarrow \frac{1}{L(s, X)} \leq \log(q^2).
\]

It remains to handle part (ii) in which we may assume that

\( L(s, X) \) has an exceptional zero \( \beta_1 \) with \( |s - \beta_1| \leq 1/\log q \). We may then suppose that

\[
1 - \frac{\sqrt{2}\log(q)}{\log(q)} \leq \sigma \leq 1 + \frac{1}{\log q},
\]

so by Lemma 14.2,

\[
\frac{L'(s, X)}{L(s, X)} = \frac{1}{s - \beta_1} + \sum_{\rho} \frac{1}{s - \rho} + O \left( \log q \right),
\]

in which the summation over \( \rho \) is over \( \rho - (1/2 + it) \) with \( \sigma \leq \frac{1}{2} \), excluding \( \beta_1 \). We have, just as before,

\[
\sum_{\rho} \frac{1}{s - \rho} \ll \text{Re} \left( \sum_{\rho} \frac{1}{s - \rho} \right) + O \left( \log(q^2) \right),
\]

where

\[
\frac{L'(s, X)}{L(s, X)} = \sum_{\rho} \frac{1}{s - \rho} + O \left( \log(q^2) \right) = \frac{1}{s - \beta_1} + \text{Re} \left( \frac{L'(s, X)}{L(s, X)} \right) + O \left( \log(q^2) \right)
\]

\[
= \frac{1}{s - \beta_1} + O \left( \log(q) \right).
\]

In a similar fashion to our treatment in (i), we find that

\[
\log L(s, X) - \log L(s_1, X) = \int_{s_1}^{s} \frac{1}{z - \beta_1} \, dz + O \left( |s_1 - s| \log(q^2) \right)
\]

\[
= \log \left( \frac{s - \beta_1}{s_1 - \beta_1} \right) + O(1),
\]

where

\[
\left| \log L(s, X) - \log \left( \frac{s - \beta_1}{s_1 - \beta_1} \right) \right| \leq \left| \log L(s_1, X) \right| + o(1) \leq \log \log q + o(1).
\]

Thus, since \( \arg (s - \beta_1) = 1 \), \( \arg (s_1 - \beta_1) = 1 \) and \( \log |s_1 - \beta_1| \geq -\log \log q + o(1) \),

we deduce that \( \left| \arg L(s, X) \right| \leq \log \log q + o(1) \) and
\[
\log \left| \frac{L(s, \chi)}{s - \beta_1} \right| \leq \left( \log |s - \beta_1| ^{1} + \log \log q + O(1) \right) = 2 \log \log q + o(1)
\]

and
\[
\log \left| \frac{L(s, \chi)}{s - \beta_1} \right| \geq -\log |s - \beta_1| - \log \log q + o(1) = O(1),
\]

whence
\[
1 \ll \left| \frac{L(s, \chi)}{s - \beta_1} \right| \ll (\log q)^2.
\]

§15. The Prime Number Theorem in arithmetic progressions, I.

In this section we explore the consequences of the weak information so far available to us concerning exceptional zeros. We write

\[
\pi(x; q, a) = \sum_{p \leq x \atop p \equiv a \pmod{q}} 1, \quad \theta(x; q, a) = \sum_{p \leq x \atop p \equiv a \pmod{q}} \log p, \quad \psi(x; q, a) = \sum_{n \leq x \atop n \equiv a \pmod{q}} \Lambda(n),
\]

\[
\pi(x, \chi) = \sum_{p \leq x} \chi(p), \quad \theta(x, \chi) = \sum_{p \leq x} \chi(p) \log p, \quad \psi(x, \chi) = \sum_{n \leq x} \chi(n) \Lambda(n).
\]

We are able to relate the above quantities to the latter via orthogonality. Thus, for example,

\[
\psi(x; q, a) = \frac{1}{\phi(q)} \sum_{\chi \in \chi_{(q)}} \overline{\chi(a)} \psi(x, \chi).
\]

We are able to provide an asymptotic formula for \( \psi(x, \chi) \) (and hence also for \( \psi(x; q, a) \)) by imitating the argument of the proof of Theorem 11.1 using the \( \omega \)-free region made available in §14.

Theorem 15.1. There is a constant \( c > 0 \) having the following property.

Suppose that \( q \leq \exp \left( 2 c \sqrt{\log x} \right) \). Then, when \( L(s, \chi) \) has no
\[
\psi(x, X) = E_0(X) x + O(x \exp(-c \sqrt{\log x})).
\]

Meanwhile, when \( L(s, X) \) has an exceptional zero \( \beta_1 \), one instead has
\[
\psi(x, X) = -\frac{x^{\beta_1}}{\beta_1} + O(x \exp(-c \sqrt{\log x})).
\]

**Proof.** By our explicit version of Perron's formula,
\[
\psi(x, X) = -\frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{L'(s, X)}{L(s, X)} \frac{x^s}{s} \, ds + R(T),
\]
where \( \sigma_0 > 1 \) and
\[
R(T) \ll \sum_{x/2 < \alpha \leq x \leq 2x} \Lambda(n) \min \left\{ 1, \frac{x}{T|\alpha - n|} \right\} + \frac{(4x)^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma_0}}.
\]

As in the proof of Theorem 1.1, we suppose that \( 2sT \leq x \) and put \( \sigma_0 = 1 + \frac{1}{5 \log x} \), giving \( R(T) \ll \frac{x}{T} (\log x)^2 \). The definition of the appropriate contour \( \mathcal{C} \) used to complete the path of integration must now take account of the possible existence of the exceptional zero \( \beta_1 \).

(i) **Suppose that there is no exceptional zero.**

We take \( \sigma_1 = 1 - \frac{c_0}{5 \log(qT)} \), where \( c_0 \) is any constant having the property that \( L(s, X) \) has no zeros in the region \( \sigma > 1 - \frac{c_1}{\log(qT)} \) (as we are permitted to assume in the case in Theorem 4.4). When \( X \neq X_0 \), it follows that \( \frac{L'(s, X)}{L(s, X)} \) is analytic inside and on \( \mathcal{C} \),
and when $x = x_0$ then the same is true except for a simple pole at $s = 1$ with residue $x$. We recall in this context that

$$\frac{-1'}{L}(s, x_0) = -\frac{5'}{5}(s) - \sum \frac{\log p}{\log p^3 - 1} \uparrow \uparrow \frac{1}{s - 1} + o(1) \text{ poles at } s = \frac{\text{zink}_{\log p}}{\log p}.$$ 

Hence

$$\frac{-1}{2\pi i} \int \frac{L'(s, x)}{L} \frac{x^s}{s} \, ds = E_0(x) \cdot x.$$ 

We follow the argument of the proof of Theorem 11.1 to handle the left vertical and horizontal segments of $C$ using Theorem 14.50. Thus

$$\psi(x, X) - E_0(x) \cdot x \ll x \frac{\log x^2}{2} \left( \frac{1}{T} + \exp \left( - c \frac{\log x}{5 \log (qT)} \right) \right).$$ 

By taking $T = \exp(2c \sqrt{\log x})$ and assuming that $q \leq \exp(2c \sqrt{\log x})$, with $c = \sqrt{c_1 / 40}$, we find that

$$\psi(x, X) - E_0(x) \cdot x \ll x (\log x^2) \left( \exp \left( - 2c \sqrt{\log x} \right) + \exp \left( - c_1 \frac{\log x}{5 \cdot 4c \sqrt{\log x}} \right) \right) \ll x \exp \left( - c \sqrt{\log x} \right).$$

(ii) Suppose that there is an exceptional zero $\beta_1$ and $\beta_1 > 1 - c_1 / 4 \log(qT)$.

Here, we emphasize that $c_1$ has the same meaning as in case (i). In this case we may take $c_1 = 1 - c_1 / 3 \log(qT)$. In this case there is a pole of $-\frac{L'(s, x)}{L} \frac{x^s}{s}$ at $s = \beta_1$ inside $C$ having residue $-\frac{x^{\beta_1}}{\beta_1}$. [Recall that $-\frac{L'(s, x)}{L} = \frac{-1}{s - \beta_1} + o(1)$ when $s \to \beta_1$].

The remaining parts of the estimation from case (i) remain applicable, and so
\[ \psi(x, \lambda) = -\frac{x^{\beta_1}}{\beta_1} + O \left( x \left( \log x \right)^2 \left( \frac{1}{t} + \exp \left( -c_1 \frac{\log x}{5 \log (qT)} \right) \right) \right) \]

\[ = -\frac{x^{\beta_1}}{\beta_1} + O \left( x \exp \left( -c \sqrt{\log x} \right) \right). \quad \text{II} \]

(iii) Suppose that there is an exceptional zero \( \beta_1 \), and \( \beta_1 < 1 - \frac{c_1}{4 \log (qT)} \).

In this case we again take \( \sigma_1 = 1 - \frac{c_1}{5 \log (qT)} \) and proceed as in case (i). In this case the exceptional zero does not lie within or on \( \mathfrak{C} \), and we again obtain

\[ \psi(x, \lambda) - \xi(x) x \ll x \left( \log x \right)^2 \left( \frac{1}{t} + \exp \left( -c_1 \frac{\log x}{5 \log (qT)} \right) \right). \]

But in this case we have

\[ -\frac{x^{\beta_1}}{\beta_1} \ll \exp \left( \beta_1 \log x \right) < x \exp \left( -\frac{c_1 \log x}{4 \log (qT)} \right) \ll x \exp \left(-c \sqrt{\log x} \right), \]

whence

\[ \psi(x, \lambda) = -\frac{x^{\beta_1}}{\beta_1} + O \left( x \exp \left( -c \sqrt{\log x} \right) \right). \quad \text{II} \]

We can obtain an asymptotic formula for \( \psi(x; \lambda) \) from here by gluing together the corresponding estimates for \( \psi(x, \lambda) \) for \( x \in \mathfrak{X}(q) \). But how many exceptional zeros might there be as we consider the \( \phi(q) \) \( L \)-functions \( L(s, \chi) \left( x \in \mathfrak{X}(q) \right) ? \) This is a matter that we now address.

We again make use of a positivity lemma.
Lemma 15.2. (Landau) Suppose that \( \chi_1 \) and \( \chi_2 \) are quadratic characters. Then whenever \( \sigma > 1 \), one has

\[
-\frac{\zeta'(\sigma)}{\zeta(\sigma)} - \frac{L'(\sigma, \chi_1)}{L(\sigma, \chi_1)} - \frac{L'(\sigma, \chi_2)}{L(\sigma, \chi_2)} \geq 0.
\]

Proof. The left hand side here may be written as the Dirichlet series

\[
\sum_{n=1}^{\infty} a_n \Lambda(n) n^{-\sigma},
\]

where

\[
a_n = 1 + \chi_1(n) + \chi_2(n) + \chi_1(n)\chi_2(n).
\]

\[
= (1 + \chi_1(n))(1 + \chi_2(n)) > 0.
\]

The desired conclusion therefore follows via absolute convergence. \( \Box \)

Now we use putative zeros of \( L(5, \chi_1) \) and \( L(5, \chi_2) \) against each other.

Theorem 15.3. (Landau) There is a constant \( c > 0 \) with the following property. Suppose that \( \chi_i \) is a quadratic character modulo \( q_i \) for \( i = 1 \) and \( 2 \), and further suppose that \( \chi_1 \chi_2 \) is non-principal. Then \( L(5, \chi_1) L(5, \chi_2) \) has at most one real zero \( \beta \) with \( 1 - \frac{c}{\log(q_1 q_2)} < \beta < 1 \).

Proof. It follows from Theorem 14.4 that, should any zero \( \beta \) of the type described exist, then there can be at most one associated with \( L(5, \chi_1) \), and one associated with \( L(5, \chi_2) \).

[Note that \( 1 - \frac{c}{\log(q_1 q_2)} \approx \max \{ 1 - \frac{c}{\log q_1}, 1 - \frac{c}{\log q_2} \} \).]
Suppose that both \( L(s, \chi_1) \) and \( L(s, \chi_2) \) have such a zero, say \( \beta_1 \) and \( \beta_2 \) respectively. There is no loss of generality in supposing that \( 5/6 \leq \beta_i \leq 1 \) \((i=1, 2)\). We may also suppose that \( \chi_1 \chi_2 \) is non-principal modulo \( q_1q_2 \). Then we deduce from Lemma \( 14.2 \) that when \( 0 < s \leq 1 \), and for a suitable constant \( c > 0 \),

\[
- \frac{\delta'}{3} (1 + \delta) \leq \frac{1}{\delta} + c_1 \log 41,
\]

\[
- \frac{1}{L'} (1 + \delta, \chi_i) \leq \frac{-1}{1 + \delta - \beta_i} + c_1 \log q_i \quad (i=1, 2),
\]

\[
- \frac{1}{L'} (1 + \delta, \chi \chi_k) \leq c_1 \log (q_1q_2).
\]

Then by substitution into the conclusion of Lemma \( 15.2 \), we infer that

\[
0 \leq \frac{1}{\delta} - \frac{1}{1 + \delta - \beta_1} - \frac{1}{1 + \delta - \beta_2} + 3c_1 \log (q_1q_2),
\]

If we suppose that \( \beta_1 \leq \beta_2 \) and put \( \delta = 2(1 - \beta_1) \), it follows that

\[
0 \leq \frac{1}{\delta} - \frac{2}{1 + \delta - \beta_1} + 3c_1 \log (q_1q_2)
\]

\[
= \frac{1}{1 - \beta_1} \left( \frac{1}{2} - \frac{2}{3} \right) + 3c_1 \log (q_1q_2),
\]

whence

\[
1 - \beta_1 \geq \frac{1}{18c_1 \log (q_1q_2)}.
\]

We therefore conclude that whenever \( c < \frac{1}{18c_1} \), then

\( L(s, \chi_1) L(s, \chi_2) \) has at most one real zero \( \beta \) with \( 1 - \beta > \frac{c}{\log (q_1q_2)} \).
Corollary 15.4 (Landau) There is a constant \( c > 0 \) with the property
that \( \frac{\pi}{\log q} \) has at most one zero in the region \( x \in \chi(q) \)
\( \sigma > 1 - \frac{c}{\log(q)} \). If such a zero exists, then it is necessarily
real and the associated character is quadratic.

Proof. Suppose, by way of contradiction, that there are two zeros
or more in the region \( \sigma > 1 - \frac{c}{\log(q)} \). As a consequence
of Theorem 14.4, when \( c \) is sufficiently small, this is impossible.
Unless both zeros are real and the associated characters \( \chi_1 \) and
\( \chi_2 \) are quadratic. But then Theorem 15.3 applies to give a
contradiction.

Notice that there may be several quadratic characters modulo
\( q \), so the fact that there is at most one exceptional zero is
significant. By making use of this conclusion in combination with
Theorem 15.1, we derive our final version of the Prime Number
Theorem in arithmetic progressions.

Theorem 15.5 (Page). There is a constant \( c > 0 \) having the
following property. Suppose that \( q \leq \exp \left( 2c \sqrt{\log x} \right) \) and
\( a \in \mathbb{Z} \) satisfy \( (a, q) = 1 \).

(i) When there is no exceptional character modulo \( q \), one
has
\[
\psi(x; q, a) = \frac{x}{\phi(q)} + O \left( x \exp \left( -c \sqrt{\log x} \right) \right).
\]
(ii) When there is an exceptional character $\chi_1$ modulo $q$, and $\beta_1$ is the associated exceptional zero of $L(s, \chi_1)$, one has instead

$$\psi(x; q, \alpha) = \frac{x}{\phi(q)} - \frac{\chi_1(a) x^{\beta_1}}{\phi(q) \beta_1} + O\left( x \exp\left( -c \sqrt{\log x} \right) \right).$$

Proof. We have

$$\psi(x; q, \alpha) = \frac{1}{\phi(q)} \sum_{\chi \in \chi(q)} \overline{\chi}(a) \psi(x, \chi),$$

and so the desired estimates are immediate from the relations derived in Theorem 15.1, namely

$$\psi(x, \chi) = E_0(x) x + O\left( x \exp\left( -c \sqrt{\log x} \right) \right) \quad (x \neq \chi_1)$$

and

$$\psi(x, \chi) = -\frac{x^{\beta_1}}{\beta_1} + O\left( x \exp\left( -c \sqrt{\log x} \right) \right) \quad (x = \chi_1).$$

Notice that the claimed asymptotic formulae would be worse than trivial when $q > \exp\left( 2c \sqrt{\log x} \right)$, for then

$$\psi(x; q, \alpha) \leq \left( \frac{x}{q} + 1 \right) \log x \ll x \exp\left( -2c \sqrt{\log x} \right) \log x,$$

which is an estimate smaller than the respective error terms in the statement of the theorem.

In order to say more concerning the exceptional zeros $\beta_1$, we must say more concerning characters $\chi$ modulo $q$, and in particular we focus on so-called primitive characters.
In the final part of the course, we aim to prove an important theorem of Siegel that provides an upper bound on \( \beta \), showing that it cannot lie too close to 1.

**Theorem 15.6.** (Siegel). For each \( \varepsilon > 0 \), there is a positive constant \( c_1(\varepsilon) \) such that, when \( \chi \) is a quadratic character modulo \( q \), then

\[
L(1, \chi) > c_1(\varepsilon) q^{-\varepsilon}.
\]

**Corollary 15.7.** For any \( \varepsilon > 0 \), there is a positive constant \( c_2(\varepsilon) \) such that, when \( \chi \) is a quadratic character modulo \( q \) and \( \beta \) is a real zero of \( L(s, \chi) \), then

\[
\beta < 1 - c_2(\varepsilon) q^{-\varepsilon}.
\]

**Proof.** We may suppose that there is a constant \( c > 0 \) having the property that there is at most one character \( \chi \) modulo \( q \) (quadratic) having a real zero \( \beta \) satisfying

\[
1 - \frac{c}{\log q} < \beta < 1.
\]

If such a zero exists, then it follows from Theorem 14.5 (ii) that

\[
L(1, \chi) \ll (1 - \beta)^2 (\log q)^2.
\]

Then since \( L(1, \chi) > c_1(\varepsilon) q^{-\varepsilon/2} \), it follows that there is a constant \( c_3 \) with the property that

\[
(1 - \beta)^2 (\log q)^2 > c_3^{-1} \cdot c_1(\varepsilon/2) q^{-\varepsilon/2}
\]

\[
\land
\]

\[
c_3 (1 - \beta)^{\varepsilon/2} \Rightarrow 1 - \beta > c_2(\varepsilon) q^{-\varepsilon},
\]

with \( c_2(\varepsilon) = c_1(\varepsilon/2) / c_3^2 \).

This yields an alternative to Theorem 15.5 in which the role of the exceptional zero is suppressed.
Theorem 15.8. There is a constant $c > 0$ having the following property. Let $A > 0$ be fixed. Then whenever $x \geq x_0(A)$ and $q \leq (\log x)^A$, one has
\[
\psi(x; \chi) = E_0(\chi) x + O(x \exp(- c\sqrt{\log x})).
\]

Proof. This is a consequence of Theorem 15.1. All that is required is to handle the possible exceptional zero $\beta_1$ of $L(s, \chi)$ for a quadratic character $\chi$. In such a situation, we have
\[
\psi(x; \chi) = -\frac{x^{\beta_1}}{\beta_1} + O(x \exp(- c_1 \sqrt{\log x})), \quad (c_1 > 0)
\]
where (from Corollary 15.7), we may suppose that for a suitable $c_2(\varepsilon) > 0$, we have $\beta_1 < 1 - c_2(\varepsilon) q^{-\varepsilon}$.

But
\[
x^{\beta_1} = x \exp\left(-\frac{1}{\alpha} \log x\right) \leq x \exp\left(- c_2(\varepsilon) q^{-\varepsilon} \log x\right)
\leq x \exp\left(- c_2(\varepsilon) (\log x)^{1-A\varepsilon}\right).
\]

If we take $\varepsilon = 1/(3A)$, then we deduce that
\[
x^{\beta_1} \leq 2x \exp\left(- c_2(\varepsilon) (\log x)^{2/3}\right) < x \exp\left(- c_1 \sqrt{\log x}\right).
\]

Thus we conclude that
\[
\psi(x; \chi) = O\left(x \exp(- c_1 \sqrt{\log x})\right)
\]
even in the case that an exceptional zero exists. \(\Box\)

Corollary 15.9 (Siegel - Walfisz Theorem). There is a constant $c > 0$ with the following property. Let $A > 0$ be fixed. Then whenever $x \geq x_0(A)$ and $q \leq (\log x)^A$, and $(a, q) = 1$, one has
\[
\psi(x; q, a) = \frac{x}{\phi(q)} + O_A\left(x \exp(- c\sqrt{\log x})\right).
\]
§ 16. Primitive Dirichlet characters and Gauss sums.

Definition 16.1. Suppose that \( d \mid q \), that \( \chi^* \) is a Dirichlet character modulo \( d \), and that \( \chi \) is a Dirichlet character modulo \( q \).
Then we say that \( \chi^* \) induces \( \chi \) when
\[
\chi(n) = \begin{cases} 
\chi^*(n), & \text{when } (n, q) = 1, \\
0, & \text{when } (n, q) > 1.
\end{cases}
\]

Thus, if \( \chi_0 \) is the principal character modulo \( q \), we have \( \chi = \chi^* \chi_0 \). Notice that \( \chi(n) \neq \chi^*(n) \) whenever \( (n, q) > 1 \) and \( (n, d) = 1 \). Moreover, it is evident that
\[
L(s, \chi) = L(s, \chi^*) \prod_{p | q} \left(1 - \frac{\chi^*(p)}{p^s}\right).
\]

How do we identify this phenomenon — a character \( \chi(\text{mod } q) \) induced by (and normally equal to) a character \( \chi^*(\text{mod } d) \) with \( d \mid q \) and \( d < q \)?

Definition 16.2. Let \( \chi \) be a Dirichlet character modulo \( q \). We say that \( d \) is a quasi-period of \( \chi \) if \( \chi(m) = \chi(n) \) whenever \( m \equiv n \pmod{d} \) and \( (mn, q) = 1 \).

Note that the least quasi-period of \( \chi \) is a divisor of \( q \). For if \( d \) is a quasi-period of \( \chi \), one can show that \( e = (d, q) \mid q \) is also a quasi-period. Indeed, whenever \( m \equiv n \pmod{e} \) and \( (mn, q) = 1 \), one has for some \( l \in \mathbb{Z} \) that
\[
m - n = le = l(d, q) = l(dx + qy), \quad \text{some } x, y \in \mathbb{Z}.
\]

Hence \( \chi(m) = \chi(m - ly \cdot q) = \chi(n + lx \cdot d) = \chi(n) \), so that \( e \) is indeed a quasi-period of \( \chi \).

A similar argument shows that when \( d_1, d_2 \) are both quasi-periods of \( \chi \), then so too is \( (d_1, d_2) \), and hence the
Definition 16.3. (i) The least quasi-period \( d \) of a Dirichlet character \( \chi \) is called the **conductor** of \( \chi \).

(ii) A character \( \chi \) modulo \( q \) is called **primitive** when \( q \) is the least quasi-period of \( \chi \).

**Theorem 9.4.** Let \( \chi \) be a Dirichlet character modulo \( q \), and let \( d \) be the conductor of \( \chi \). Then \( d \mid q \), and there is a unique character \( \chi^* \) modulo \( d \) that induces \( \chi \).

**Proof.** We have seen already that the least quasi-period of \( \chi \), namely \( d \), is a divisor of \( q \). We begin by showing that \( \chi \) is induced by a character \( \chi^* \) modulo \( d \). Suppose that \( n \in \mathbb{Z} \) satisfies \( (n, d) = 1 \). Let \( d_0 : (d^\infty, q) = \lim_{m \to \infty} (d^m, q) \), and put \( r = q / d_0 \), so that \( (r, d) = 1 \). Then there exists \( k \in \mathbb{Z} \) with \( (n + kd, r^k) = 1 \), whence \( 1 = (n + kd, rd_0) = (n + kd, q) \).

For any such integer \( k \), we define

\[
\chi^*(n) := \chi(n + kd).
\]

Notice that since \( d \) is a quasi-period of \( \chi \), then the choice of \( k \) does not impact the definition of \( \chi^*(n) \). This defines \( \chi^*(n) \) when \( (n, d) = 1 \), and when \( (n, d) > 1 \) we put \( \chi^*(n) = 0 \).

It is evident that \( \chi^* \) has period \( d \). Also, when \( (n, m) = 1 \) one has for suitable integers \( u \) and \( v \) that

\[
\chi^*(n) \chi^*(m) = \chi(n + ud) \chi(m + vd) = \chi(nm + (um + vnt + uvd) d) = \chi^*(nm). \quad \text{(Notice issues on coprimality with } q \text{ here).}
\]
Thus $\chi^*$ is multiplicative with period $d$, and satisfies

$$\chi^*(n) = 0 \quad \text{for} \quad (n,d) > 1,$$

and thus is a Dirichlet character modulo $d$.

Indeed, if $\chi_0$ is the principal character modulo $q$, one has

$$\chi(n) = \chi_0(n) \chi^*(n),$$

and $\chi^*$ induces $\chi$.

If $\chi^*$ were to have a quasi-period smaller than $d$, then so too would $\chi$, contradicting minimality. Then $\chi^*$ is primitive. Finally, to see that $\chi^*$ is unique we observe that if some other character $\chi^+$ modulo $d$ also induces $\chi$, then for any $n \in \mathbb{Z}$ with $(n,d) = 1$, we could find $k \in \mathbb{Z}$ with

$$\chi^*(n) = \chi^*(n + kd) = \chi(n + kd) = \chi^+(n + kd) = \chi^+(n),$$

so that $\chi^* = \chi^+$. //

We can decompose primitive characters according to the prime power decomposition of the conductor $q$.

**Lemma 16.5.** Suppose that $q_1, q_2 \in \mathbb{N}$ and $(q_1, q_2) = 1$. Suppose also that $\chi_1$ and $\chi_2$ are characters modulo $q_1$ and $q_2$, respectively, and put $\chi(n) = \chi_1(n) \chi_2(n)$. Then the character $\chi$ is primitive modulo $q_1 q_2$ if and only if both $\chi_1$ and $\chi_2$ are primitive.

**Proof.** $(\Rightarrow)$ Put $q = q_1 q_2$, and suppose that $\chi$ is primitive modulo $q$. Also, with $d_i = \text{cond}(\chi_i)$ ($i = 1, 2$), we seek to show that $q = d_1 d_2$, whence $d_1 = q_1$ and $d_2 = q_2$, so that both $\chi_1$ and $\chi_2$ are primitive.

To see this, observe that whenever $m \equiv n \pmod{d_1, d_2}$ and $(m, q_1 q_2) = 1$, one has $\chi_i (m) = \chi_i (n)$ ($i = 1, 2$), whence $\chi(m) = \chi(n)$. Thus $d_1 d_2$ is a quasi-period of $\chi$, so by primality of $\chi$, it follows that $q = d_1 d_2$. //
Suppose \( \chi_i \) is primitive modulo \( q_i \) \((i=1,2)\), and let \( d = \gcd(q_1, q_2) \). We aim to show that \( d = q \), whence \( \chi \) is primitive.

Put \( d_i = (d, q_i) \) \((i=1,2)\). Then whenever \( m \equiv n \pmod{d_i} \) and \((m, q_1) = 1\), we may choose integers \( m', n' \) with \( m' \equiv m \pmod{q_1} \) and \( n' \equiv n \pmod{q_1} \) with \( m' \equiv n' \equiv 1 \pmod{q_2} \). We then have \( m' \equiv n' \pmod{d_1d_2} \), that is \( m \equiv n \pmod{d} \), and also \((m', q_2) = 1\). Hence \( \chi(m') = \chi(n') \).

\[
\begin{align*}
\chi_1(m') &= \chi_2(n') \\
\chi_1(m) &= \chi_2(1)
\end{align*}
\]

where \( \chi_1(m) = \chi_1(n) \), so that \( d_1 \) is a quasi-period of \( \chi_1 \). But \( \chi_1 \) is primitive, so \( d_1 = q_1 \). A similar argument shows that \( d_2 = q_2 \), and consequently \( d = d_1d_2 = d \), which gives \( d = q \). So \( \chi \) is primitive.

It follows from Lemma 16.5 that primitive characters modulo \( q \) are determined by primitive characters modulo \( p^h \) for \( p^h \mid q \).

**p odd:**
\[
\chi(n) = e\left(\frac{k \text{ Ind}_q n}{\phi(p^h)}\right)
\]
for \( q \) primitive modulo \( p^h \).

- **h=1:** \( \chi \) primitive if and only if \( \chi \neq \chi_0 \), so \((p-1)+k\).
- **h>1:** \( \chi \) primitive if and only if \( p \nmid k \).

**p even:**
\((p=2)\).

- **h=1:** \( \chi = \chi_0 \) not primitive
- **h=2:** \( \chi \) either principal (not primitive) or not, in which case \( \chi(4k+1) = 1, \chi(4k-1) = -1 \), and this is primitive.
- **h>2:** \( \chi(n) = e\left(\frac{kn}{2} + i\frac{k}{2^{h-2}}\right) \) with \( n \equiv (-1)^u 2^v \pmod{2^h} \). \( \chi \) primitive if and only if \( k \) odd.
Observation: (i) If $\chi$ is primitive, then whenever $d|q$ and $d < q$, there exists $c \equiv 1 \pmod{d}$ with $(c, q) = 1$ such that $\chi(c) \neq 1$.

Proof. There exist $m \equiv n \pmod{d}$ with $\chi(m) \neq \chi(n)$ and $\chi(mn) \neq 0$.
Then with any $c \in \mathbb{Z}$ s.t. $(c, q) = 1$ and $cm \equiv n \pmod{q}$, we have
$$\chi(cm) = \chi(n) \Rightarrow \chi(c) = \chi(n) \chi(m) \neq 1.$$ Further
$$\chi(c) \chi(m)$$

(ii) If $d|q$ and $d < q$, then for all $a \in \mathbb{Z}$ we have
$$\sum_{\substack{n=1 \atop n \equiv a \pmod{d}}}^{q} \chi(n) = 0.$$

Proof. We have an integer $c$ satisfying (i), and then
$$\sum_{n=1 \atop n \equiv a \pmod{d}}^{q} \chi(n) = \sum_{k=1}^{q/d} \chi(a + kd) = \sum_{k=1}^{q/d} \chi(c(a + kd))$$

permutes this arithmetic progression $a \pmod{d}$ working modulo $q$

$$= \chi(c) \sum_{n=1 \atop n \equiv a \pmod{d}}^{q} \chi(n)$$

$$= \chi(c) \sum_{n=1 \atop n \equiv a \pmod{d}}^{q} \chi(n) = 0 \quad \text{since} \quad \chi(c) \neq 1.$$ $\square$


We have seen a proof of the fact that $L(1, \chi) \neq 0$ using Landau's lemma in our proof of Dirichlet's theorem. We now work harder so as to obtain more quantitative information. For this purpose, we require a lemma of Estermann.
Lemma 4.1 (Estermann). Suppose that $f(s)$ is analytic for $|s - 2| < 3/2$, and that $|f(s)| \leq M$ for $s$ within this disc. Suppose further that $F(s)$ can be written as a Dirichlet series for $s > 1$.

Put $F(s) = \sum_{n=1}^{\infty} r(n) n^{-s}$ ($s > 1$).

Suppose in addition that $r(1) = 1$ and $r(n) \geq 0$ ($n \in \mathbb{N}$).

Then whenever there exists $\sigma \in [1/20, 1)$ such that $f(\sigma) > 0$, one has

$$f(1) \geq \frac{1}{2} (1 - \sigma) M^{-3(1-\sigma)}.$$

Proof. We expand $F(s)$ as a power series in $(2-s)$. Thus, we have

$$F(s) = \sum_{k=0}^{\infty} b_k (2-s)^k,$$

for $|s-2| < 1$, say. By Cauchy's formula, we see that

$$b_k = \frac{(-1)^k}{k!} \frac{F^{(k)}(2)}{k} = \frac{1}{k!} \sum_{n=1}^{\infty} r(n) n^{-2} (\log n)^{k-1}.$$

Then our hypotheses on $r(n)$ ensure that $b_k \geq 0$ for all $k$, and moreover

$$b_0 = \sum_{n=1}^{\infty} r(n) n^{-2} \geq 1.$$

On noting that

$$\frac{1}{s-1} = \frac{1}{1/(2-s)} = \sum_{k=0}^{\infty} (2-s)^k \quad (|s-2| < 1),$$

we deduce that

$$F(s) - \frac{f(1)}{s-1} = \sum_{k=0}^{\infty} \left( b_k - f(1) \right) (2-s)^k \quad (|s-2| < 1).$$

The l.h.s. of this last relation is analytic for $|s-2| < 3/2$, so the series on the r.h.s. converges in this disc (which goes beyond the disc $|s-2| < 1$!). Our goal now is to estimate
the terms appearing here so as to obtain a lower bound on \( f(1) \).

We begin by bounding the coefficients \( b_k - f(1) \) in modulus by bounding the lhs of (16.1). Observe that when \( 15 - 21 = \frac{3}{2} \), we have \( |s - 1| > \frac{1}{2} \), \( |s| \leq \frac{\sqrt{2}}{2} \) and \( |t| > \frac{1}{2} \). Also,

\[
|\lambda(s)| = |1 + \frac{1}{s - 1} + 5 \int_1^\infty \frac{\log u - \frac{u}{\log u}}{u - 1} \, du| \\
\leq 1 + \frac{1}{|s - 1|} + \frac{|s|}{|t|} \leq 1 + 2 + \frac{3/2}{1/2} = 10,
\]

and \( |f(1)| \leq M \), so that

\[
\left| \frac{f(1)}{s - 1} \right| \leq \frac{M}{1/2} = 2M.
\]

We therefore deduce that

\[
\left| F(s) - \frac{f(1)}{s - 1} \right| \leq 15|s| f(s) + \left| \frac{f(1)}{s - 1} \right| \leq 10M + 2M = 12M,
\]

hence from the Cauchy coefficient inequalities, we see that

\[
|b_k - f(1)| \leq \left( \sup_{15 - 21 = \frac{3}{2}} \left| F(s) - \frac{f(1)}{s - 1} \right| \right) / \left( \frac{3}{2} \right)^k \\
\leq 12M \left( \frac{2}{3} \right)^k.
\]

We now have a means of bounding the rhs of (16.1). It transpires that it is most convenient to truncate the infinite series at a cut-off \( K \) to be chosen later. Thus, whenever \( \frac{1}{2} < \sigma < \frac{3}{2} \), we deduce that

\[
5(\sigma) f(\sigma) - \frac{f(1)}{\sigma - 1} \geq \sum_{k=0}^{K} (b_k - f(1)) (2 - \sigma)^k - 12M \sum_{k>K} \left( \frac{2}{3} (2 - \sigma) \right)^k.
\]

Since \( b_0 \geq 1 \), \( b_k \geq 0 \) \( (k \geq 1) \) and \( \frac{2}{3} (2 - \sigma) \leq \frac{9}{10} \) for \( \frac{19}{20} \leq \sigma < 1 \),

we see that the rhs here is
\[ 1 - f(1) \frac{1 - (2 - \sigma)^{K+1}}{1 - (2 - \sigma)} - \left( \frac{\sigma}{10} \right)^{K+1} \frac{12M}{1 - \frac{\sigma}{10}}, \]

Whence from (16.2),

\[ 1 \leq - f(1) \frac{1 - (2 - \sigma)^{K+1}}{1 - \sigma} + 5(\sigma) f(\sigma) - \frac{f(1)}{\sigma - 1} + 40M \left( \frac{\sigma}{10} \right)^{K+1}. \]

\[ = f(1) \frac{(2 - \sigma)^{K+1}}{1 - \sigma} + 5(\sigma) f(\sigma) + 40M \left( \frac{\sigma}{10} \right)^{K+1}. \]

At this point we choose \( K \) in such a way that the final term is not troublesome. If we take

\[ K = \left[ \frac{\log (80M)}{\log \left( \frac{10}{7} \right)} \right], \]

then the final term is at most \( 1/2 \). Also, since \( \frac{19}{20} \leq \sigma < 1 \), we have \( 5(\sigma) > 0 \), and further \( f(\sigma) > 0 \) by hypothesis.

Thus

\[ f(1) \frac{(2 - \sigma)^{K+1}}{1 - \sigma} \geq 1 - \frac{1}{2}, \]

Whence

\[ f(1) \geq \frac{1}{2} (1 - \sigma) (2 - \sigma)^{-K-1} \]

\[ \geq \frac{1}{2} (1 - \sigma) (2 - \sigma) \left( \left[ \frac{\log (80M)}{\log \left( \frac{10}{7} \right)} \right] + 1 \right) \]

\[ \geq \frac{1}{2} (1 - \sigma) (2 - \sigma) \left( \frac{10}{21} (1 - \sigma) (80M) \right) \]

\[ \geq \frac{1}{2} (80) \left( \frac{\log (10/7)}{\log (10/7)} \right) \frac{1}{\sigma} - \log (2 - \sigma) / \log (10/7) \]

\[ \geq \frac{1}{4} M - \log (2 - \sigma) / \log (10/7). \]

But \( \log (10/7) \geq 1/3 \) and \( \log (2 - \sigma) \leq 1 - \sigma \), so

\[ f(1) \geq \frac{1}{4} M - 3(1 - \sigma), \] as required. \( \Box \)
This lemma of Estermann paves the way for Siegel's theorem.

**Theorem 15.6 (Siegel).** For each $\varepsilon > 0$, there is a positive constant $C_1(\varepsilon)$ such that, when $\chi$ is a quadratic character modulo $q$, then

$$L(1, \chi) > C_1(\varepsilon) q^{-\varepsilon}.$$

**Proof.** Consider any positive number $\varepsilon$ with $\varepsilon \leq \frac{1}{5}$. We observe first that there is no loss of generality in restricting to primitive characters $\chi$. For if $\chi$ is not primitive modulo $q$, then $\chi$ is induced by some primitive character $\chi^*$ modulo $d | q$, say $q = dq$. But then, assuming the conclusion of the theorem for $\chi^*$, we see that

$$L(1, \chi) = L(1, \chi^*) \prod_{p} \left(1 - \frac{\chi^*(p)}{p}\right) \geq L(1, \chi^*) \frac{\varphi(r)}{r} \geq C_1(\varepsilon) d^{-\varepsilon} r^{-\varepsilon} = C_1(\varepsilon) q^{-\varepsilon}.$$

Suppose then that $\chi$ is primitive. We divide into two cases.

(i) Suppose that there exists no quadratic character $\chi_1$ such that $L(s, \chi_1)$ has a real zero $\beta_1 \in \left[1 - \frac{\varepsilon}{4}, 1\right]$.

In this case we take $f(s) = L(s, \chi)$ and $\sigma = 1 - \frac{\varepsilon}{4}$.

By Lemma 14.1, we have

$$\sup_{1 - 2 \leq s \leq 3/2} \left| L(s, \chi) \right| \leq \sup_{1/2 \leq \sigma \leq 3/2} \left(1 + (q^2)^{1-\sigma}\right) \min \left\{ \frac{1}{1-\sigma}, \log(q^2) \right\}$$

so we may apply Lemma 17.1 with $M \ll q^{1/2}$. Thus we deduce that

$$\| L(1, \chi) \| \geq \frac{1}{4} \left(\frac{\varepsilon}{4}\right) M^{-3\varepsilon/4} \geq \varepsilon q^{-3\varepsilon/8}.$$
The conclusion \( L(1, \chi) > C_1(\varepsilon) q^{-E} \) then follows for a suitable \( C_1(\varepsilon) > 0 \).

(i) Suppose that there exists a quadratic character \( \chi_1 \) such that \( L(\frac{1}{2}, \chi_1) \) has a real zero \( \beta_1 > 1 - \frac{5}{4} \).

In this case, since \( L(1, \chi_1) > 0 \), there is a constant \( C_2(\varepsilon) > 0 \) such that \( L(1, \chi_1) \geq C_2(\varepsilon) q^{-E} \) (just take \( C_2(\varepsilon) = q^E L(1, \chi_1) \)).

We now investigate the implications of this assumption for other primitive Dirichlet characters \( \chi \) modulo \( q \) with \( \chi \neq \chi_1 \). We apply Lemma 17.1 with

\[
\frac{f(s)}{L(s, \chi)} = L(s, \chi_1) L(s, \chi). 
\]

We must check in this instance that \( F(s) f(s) \) has a Dirichlet series with non-negative coefficients. To verify this property, note that

\[
\log \left( \frac{F(s) f(s)}{L(s, \chi)} \right) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\log n} \left( 1 + \chi(n) + \chi_1(n) \right) n^{-s} 
\]

has non-negative coefficients. Thus, on exponentiation we find that

\[
F(s) f(s) = \exp \left( \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\log n} \left( 1 + \chi(n) + \chi_1(n) \right) n^{-s} \right) 
\]

has non-negative coefficients. Observe next that

\[
\sup_{|s-1/2| \leq 3/2} |F(s)| \ll q^{1/2} q_1^{1/2} (q q_1)^{1/2} \ll q q_1. 
\]

Then we may apply Lemma 17.1 with \( M = C_3 q q_1 \), \( \delta \), and \( \sigma = \beta_1 \).
Thus \( f(1) \geq \frac{1}{4} \left( C_{3/9} q \right)^{-3(1-\varepsilon)} \geq \frac{1}{4} \left( C_{3/9} q \right)^{-3\varepsilon/4} \).

Moreover, also from Lemma 14.1,

\[
f(1) = L(1, X) L(1, X_1) L(1, X_2) \ll L(1, X) (\log q_1)(\log(q_1 q)) \]

whence

\[
L(1, X)(\log(q_1 q))^2 \gg (q_1 q)^{-3\varepsilon/4},
\]

and thus

\[
L(1, X) \gg \frac{q^{-3\varepsilon/4}}{\log q} \gg q^{-\varepsilon}.
\]

We consequently conclude also, in this case that \( L(1, X) \gg C_1(\varepsilon) q^{-\varepsilon} \), for a suitable \( C_1(\varepsilon) > 0 \). \( \square \)