MA598AANT ANALYTIC NUMBER THEORY. PROBLEMS 1
TO BE HANDED IN BY FRIDAY 13TH SEPTEMBER 2020

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

A1. Determine the abscissa of convergence of the following Dirichlet series:
\[ \sum_{n=1}^{\infty} (\log n)^{2020} n^{-s}, \sum_{n=1}^{\infty} (n^2 + 1)^{3/2} n^{-s}, \sum_{n=1}^{\infty} 2^{(\log n)^{3/2}} n^{-s}. \]

A2. Determine the abscissa of convergence, and the abscissa of absolute convergence, of the following Dirichlet series:
\[ \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s}, \sum_{n=1}^{\infty} \sin \left( \frac{n\pi}{2} + 2\pi \frac{u}{n} \right) n^{-s}. \]

B3. (i) Let \( \alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s} \) be a Dirichlet series. Let \( \sigma_c \) be the abscissa of convergence of \( \alpha(s) \), and \( \sigma_a \) the corresponding abscissa of absolute convergence.
   (i) Prove that \( \sigma_c \leq \sigma_a \);
   (ii) Observe that whenever \( \sigma > \sigma_c \), one has \( a_n n^{-\sigma} \to 0 \) as \( n \to \infty \). Hence deduce that \( \sigma_a \leq \sigma_c + 1 \).

B4. Let \((a_n)_{n=1}^{\infty}\) be a complex sequence satisfying the property that, for some number \( \theta \leq 0 \), one has
\[ A(x) := \sum_{n>x} a_n \ll x^\theta. \]
   (i) Apply Riemann-Stieltjes integration to show that for each real number \( k \), and for all \( x \geq 1 \), one has
\[ \sum_{x \leq n \leq 2x} a_n n^k = -\int_{x-}^{2x+} u^k dA(u). \]
   (ii) Conclude that
\[ \sum_{x \leq n \leq 2x} a_n n^k \ll x^{k+\theta} \log x. \]
   (iii) Let \( \alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s} \) have abscissa of convergence \( \sigma_c \in (-\infty, 0) \). Prove that
\[ \sigma_c = \limsup_{x \to \infty} \frac{\log |A(x)|}{\log x}. \]

C5. (i) Let \( \tau(n) \) denote the number of positive divisors of \( n \), and let \( \Box_0(n) \) denote the squarefree kernel of \( n \). Thus
\[ \tau(n) = \sum_{1 \leq l, m \leq n \atop l|m=n} 1 \quad \text{and} \quad \Box_0(n) = \prod_{p|n} p. \]
Prove that both \( \tau(n) \) and \( \Box_0(n) \) are multiplicative functions of \( n \).
(ii) Prove that for $\sigma > 2$ the series

$$\Upsilon(s) = \sum_{n=1}^{\infty} \frac{1}{\tau(n)^s \mu_0(n)^s}$$

converges absolutely, and further that

$$\Upsilon(s) = \prod_p \left( 1 + \frac{\zeta(s) - 1}{p^s} \right).$$

Deduce that $\Upsilon(s)$ is analytic for $\sigma > 2$. [It is possible, but more challenging, to establish the same conclusion for $\sigma > 1$.]