MA598AANT ANALYTIC NUMBER THEORY. PROBLEMS 1

TO BE HANDED IN BY FRIDAY 13TH SEPTEMBER 2020

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

A1. Determine the abscissa of convergence of the following Dirichlet series:

$$\sum_{n=1}^{\infty}(\log n)^{2020}n^{-s}, \quad \sum_{n=1}^{\infty}(n^2+1)^{3/2}n^{-s}, \quad \sum_{n=1}^{\infty}2^{(\log n)^{3/2}}n^{-s}.$$

A2. Determine the abscissa of convergence, and the abscissa of absolute convergence, of the following Dirichlet series:

$$\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}, \quad \sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{2}+\frac{2\pi}{n}\right)n^{-s}.$$

B3. (i) Let $\alpha(s)=\sum_{n=1}^{\infty}a_n n^{-s}$ be a Dirichlet series. Let σ_c be the abscissa of convergence of $\alpha(s)$, and σ_a the corresponding abscissa of absolute convergence.
(i) Prove that $\sigma_c \leqslant \sigma_a$;
(ii) Observe that whenever $\sigma > \sigma_c$, one has $a_n n^{-\sigma} \to 0$ as $n \to \infty$. Hence deduce that $\sigma_a \leqslant \sigma_c + 1$.

B4. Let $(a_n)_{n=1}^{\infty}$ be a complex sequence satisfying the property that, for some number $\theta \leqslant 0$, one has

$$A(x) := \sum_{n>x} a_n \ll x^{\theta}.$$

(i) Apply Riemann-Stieltjes integration to show that for each real number k, and for all $x \geqslant 1$, one has

$$\sum_{x\leqslant n\leqslant 2x} a_n n^k = -\int_{x-}^{2x+} u^k \, \mathrm{d}A(u).$$

(ii) Conclude that

$$\sum_{x\leqslant n\leqslant 2x} a_n n^k \ll x^{k+\theta}\log x.$$

(iii) Let $\alpha(s)=\sum_{n=1}^{\infty}a_n n^{-s}$ have abscissa of convergence $\sigma_c \in (-\infty, 0)$. Prove that

$$\sigma_c = \limsup_{x\to\infty}\frac{\log|A(x)|}{\log x}.$$

C5. (i) Let $\tau(n)$ denote the number of positive divisors of n, and let $\square_0(n)$ denote the squarefree kernel of n. Thus

$$\tau(n) = \sum_{\substack{1\leqslant l,m\leqslant n\\ lm=n}} 1 \quad \text{and} \quad \square_0(n) = \prod_{p\mid n} p.$$

Prove that both $\tau(n)$ and $\square_0(n)$ are multiplicative functions of n.

(ii) Prove that for $\sigma > 2$ the series

$$\Upsilon(s) = \sum_{n=1}^{\infty} \frac{1}{\tau(n)^s \square_0(n)^s}$$

converges absolutely, and further that

$$\Upsilon(s) = \prod_p \left(1 + \frac{\zeta(s) - 1}{p^s} \right).$$

Deduce that $\Upsilon(s)$ is analytic for $\sigma > 2$. [It is possible, but more challenging, to establish the same conclusion for $\sigma > 1$.]