Solutions to problem sheet 1.

Q1 (i) Put $A(x) = \sum_{n \le x} (\log n)^{2020} = x(\log x)^{2020} + O(x(\log x)^{2019})$. Then since $A(x)$ is not bounded, it follows from Theorem 2.4 ~~that~~

$$\sigma_c = \limsup_{x\to\infty} \frac{\log\left(x\log(x)^{2020} + O(x(\log x)^{2019})\right)}{\log x} = \lim_{x\to\infty} \frac{\log x + O(\log\log x)}{\log x} = 1 \qquad \square$$

(ii) Put $A(x) = \sum_{n\le x}(n^2+1)^{3/2} = \sum_{n\le x}(n^3 + O(n)) = \frac{1}{4}x^4 + O(x^3)$.

Then since $A(x)$ is unbounded,

$$\sigma_c = \limsup_{x\to\infty} \frac{\log\left(\frac{1}{4}x^4 + O(x^3)\right)}{\log x} = \lim_{x\to\infty}\frac{4\log x + O(1)}{\log x} = 4. \ \square$$

(iii) Put $A(x) = \sum_{n\le x} 2^{(\log n)^{3/2}}$. Then $2^{(\log(x-1))^{3/2}} \le A(x) \le x\, 2^{(\log x)^{3/2}}$.

Since $A(x)$ is unbounded, we have

$$\sigma_c \ge \limsup_{x\to\infty}\frac{\log\left(2^{(\log(x-1))^{3/2}}\right)}{\log x} = \limsup_{x\to\infty}\frac{(\log(x-1))^{3/2}\log 2}{\log x}$$

$$= (\log 2)\limsup_{x\to\infty}(\log x)^{1/2} = +\infty. \quad \square.$$

Q2 (i) Put $A(x) = \sum_{n\le x}(-1)^{n-1} = \begin{cases} 1, & \text{when } \lfloor x\rfloor \text{ odd},\\ 0, & \text{when } \lfloor x\rfloor \text{ even}.\end{cases}$

Since $A(x)$ is bounded, one has $\sigma_c \le 0$. Also, since

$$A^+(x) := \sum_{n\le x}|(-1)^{n-1}| = \lfloor x\rfloor,$$

is unbounded, one sees ~~that~~

$$\sigma_a = \limsup_{x\to\infty}\frac{\log|A^+(x)|}{\log x} = \lim_{x\to\infty}\frac{\log\lfloor x\rfloor}{\log x} = 1.$$

Then since $\sigma_a \le \sigma_c + 1$, it follows ~~that~~ $\sigma_c = 0$. $\square$.

(ii) Put $A(x) = \sum_{1\le n\le x} \underbrace{\sin\left(\frac{n\pi}{2} + \frac{2\pi}{n}\right)} = \left\{\begin{matrix}1\\0\end{matrix}\right. + O\left(\sum_{n\le x}\frac{1}{n}\right)$.

$$\| $$

$$\begin{cases} 1 + O(\frac{1}{n}), & \text{when } n \equiv 1 \pmod 4,\\ O(\frac{1}{n}), & \text{when } 2\mid n,\\ -1 + O(\frac{1}{n}), & \text{when } n\equiv 3 \pmod 4.\end{cases}$$

 Then, if $A(x)$ were unbounded, one obtains

$$\sigma_c = \limsup_{x\to\infty} \frac{\log |A(x)|}{\log x} \leq \lim_{x\to\infty} \frac{\log(C\log x)}{\log x} = 0.$$

If $A(x)$ were bounded, meanwhile, then $\sigma_c \leq 0$. Meanwhile, since

$$A^+(x) := \sum_{1\leq n\leq x} \left|\sin\left(\frac{n\pi}{2} + \frac{2\pi}{n}\right)\right| = \left\lfloor \frac{x}{2} \right\rfloor + O\left(\sum_{n\leq x} \frac{1}{n}\right),$$

we see that

$$\sigma_a = \limsup_{x\to\infty} \frac{\log A^+(x)}{\log x} = \lim_{x\to\infty} \frac{\log\left(\frac{x}{2} + O(\log x)\right)}{\log x} = 1.$$

Thus, since $\sigma_a \leq \sigma_c + 1$, it follows that $\sigma_c = 0$. □.

Q3 (i) One has

$$\left|\sum_{n>x} a_n n^{-s}\right| \leq \sum_{n>x} |a_n| n^{-\sigma},$$

so if rhs $\to 0$ as $n\to\infty$ (as happens for $\sigma > \sigma_a$), then lhs $\to 0$ as $x\to\infty$, whence $\sum_n a_n n^{-s}$ converges. Thus $\sigma_c \leq \sigma_a$. □

(ii) Suppose that $\sigma > \sigma_c$, say $\sigma = \sigma_c + 2\varepsilon$. Then

$$\sum_n a_n n^{-\sigma_c - \varepsilon} < \infty \Rightarrow a_n n^{-\sigma_c - \varepsilon} \to 0 \text{ as } n\to\infty.$$

This implies that $\displaystyle\sum_{n>x} |a_n| n^{-1-\sigma} < \sum_{n>x} n^{-1-(\sigma-\sigma_c-\varepsilon)}$ for x sufficiently large

$$= \sum_{n>x} n^{-1-\varepsilon} < \infty.$$

Then for any $\varepsilon > 0$, we have $\sigma_a \leq \sigma_c + 1 + 2\varepsilon$, whence $\sigma_a \leq \sigma_c + 1$. □

Q4 (i) One has

$$\sum_{x\leq n\leq 2x} a_n n^k = \int_{x^-}^{2x^+} u^k \, d\left(\sum_{n\leq u} a_n\right).$$

But

$$\sum_{n\leq u} a_n = \sum_{n=1}^{2x^+} a_n - \underbrace{\sum_{n>u} a_n}_{A(u)} \quad \text{for } u \leq 2x,$$

so

$$\sum_{x\leq n\leq 2x} a_n n^k = -\int_{x^-}^{2x^+} u^k \, d\,A(u). \quad \square.$$

(ii) Apply integration by parts and the estimate $A(u) \ll u^\theta$:

③

$$\sum_{x \le n \le 2x} a_n n^k = -u^k A(u) \Big|_{x^-}^{2x+} + k \int_x^{2x} A(u) u^{k-1} du$$

$$\ll x^{k+\theta} + k \int_x^{2x} u^{k+\theta-1} du.$$

This is $\ll x^{k+\theta}$ when $k+\theta \ne 0$, and when $k+\theta = 0$ we get

$$\sum_{x \le n \le 2x} a_n n^k \ll x^{k+\theta} + k \log u \Big|_x^{2x} \ll 1.$$ [The factor $\log x$ was for safety!] □

(iii) When $\varepsilon > 0$, one has $\sum_{n=1}^{\infty} a_n n^{-\sigma_c - \varepsilon}$ converges, whence $\sum_{n>x} a_n n^{-\sigma_c-\varepsilon} \ll 1$

Then we deduce ~~that~~ $\sum_{x \le n \le 2x} a_n \ll x^{\sigma_c + \varepsilon}$ (when $\sigma_c < 0$ & $\varepsilon < \frac{1}{2}|\sigma_c|$)

so ~~that~~ by summing over dyadic intervals, we deduce that $\sum_{n>x} a_n \ll X^{\sigma_c+\varepsilon}$.

It follows that

$$\sigma^* := \limsup_{x\to\infty} \frac{\log |A(x)|}{\log x} \le \lim_{x\to\infty} \frac{(\sigma_c+\varepsilon)\log x}{\log x} = \sigma_c + \varepsilon.$$

Since this holds for all small $\varepsilon > 0$, we find that $\sigma^* \le \sigma_c$.

On the other hand, defining σ^* in this way, we have $|A(x)| \ll X^{\sigma^*+\varepsilon}$ for all small $\varepsilon > 0$. Take $k > |\sigma^*|$. Then the argument of (ii) shows that

$$\sum_{1 \le n \le x} a_n n^k \ll X^{k+\sigma^*+\varepsilon} \log x,$$

Whence

$$\alpha(s-k) = \sum_{n=1}^{\infty} a_n n^{k-s}$$

has abscissa of convergence $\sigma_c' = \limsup_{x\to\infty} \dfrac{\log \left| \sum_{1\le n \le x} a_n n^k \right|}{\log x} \le k + \sigma^* + \varepsilon$.

It follows that $\alpha(s)$ has abscissa of convergence $\sigma_c \le \sigma^* + \varepsilon$. Since this holds for all $\varepsilon > 0$, we see that $\sigma_c \le \sigma^*$.

We conclude that $\sigma_c \le \sigma^* \le \sigma_c$, whence $\sigma_c = \sigma^*$. That is,

$$\sigma_c = \limsup_{x\to\infty} \frac{\log |A(x)|}{\log x}.$$ □

Q5 (i) Whenever $(n,m) = 1$, one has

$$\tau(nm) = \sum_{d|nm} 1 = \sum_{d_1|n} \sum_{d_2|m} 1,$$ where $d_1 = (d,n)$ & $d_2 = (d,m)$, since $(n,m) = 1$.

④ Thus $\tau(nm) = \left(\sum_{d_1|n} 1\right)\left(\sum_{d_2|m} 1\right) = \tau(n)\tau(m)$, so τ is multiplicative. Similarly, when $(n,m)=1$, the prime divisors of n and m are disjoint, so

$$\square_0(nm) = \prod_{p|nm} p = \left(\prod_{p|n} p\right)\left(\prod_{p'|m} p'\right) = \square_0(n)\square_0(m),$$

whence $\square_0$ is multiplicative. $\square$.

(ii) Using the multiplicative property of τ and $\square_0$, we see that when $\sigma > 1$ one has

$$\sum_{n=1}^{\infty} \frac{1}{\tau(n)^\sigma \square_0(m)^\sigma} = \sum_{n=1}^{\infty} \prod_{p^h \| n} (h+1)^{-\sigma} p^{-\sigma} = \prod_p \left(1 + \sum_{h=1}^{\infty} (h+1)^{-\sigma} p^{-\sigma}\right)$$

$$= \prod_p \left(1 + p^{-\sigma}\left(\sum_{h=1}^{\infty} h^{-\sigma} - 1\right)\right) = \prod_p \left(1 + \frac{\zeta(\sigma)-1}{p^\sigma}\right).$$

The absolute convergence of the product on the right hand side is assured, since for $\sigma > 1$ one has $\zeta(\sigma) > 1$, and

$$\sum_p \frac{\zeta(\sigma)-1}{p^\sigma} \ll \sum_p p^{-\sigma} < \infty.$$

The absolute convergence of the product shows the sum on the left hand side to be absolutely convergent, and since

$$\left|\sum_{n=1}^{\infty} \frac{1}{\tau(n)^s \square_0(m)^s}\right| < \sum_{n=1}^{\infty} \frac{1}{\tau(n)^\sigma \square_0(m)^\sigma},$$

we find that $\Upsilon(s)$ is absolutely convergent for $\sigma > 1$. It follows that

$$\Upsilon(s) = \prod_p \left(1 + \frac{\zeta(s)-1}{p^s}\right) \text{ is absolutely convergent for } \mathrm{Re}(s) > 1,$$

whence $\Upsilon(s)$ is analytic for $\sigma > 1$. $\square$