(a) Let \( A(x) = \sum_{n \leq x} (\log n)^{2020} = x(\log x)^{2020} + O(x(\log x)^{2019}) \). Then since \( A(x) \) is not bounded, it follows from Theorem 2.4 that
\[
\sigma_c = \limsup_{x \to \infty} \frac{\log (x(\log x)^{2020} + O(x(\log x)^{2019}))}{\log x} = \lim_{x \to \infty} \frac{\log x + O(\log \log x)}{\log x} = 1.
\]

(b) Let \( A(x) = \sum_{n \leq x} (n^2 + 1)^{3/2} = \sum_{n \leq x} (n^2 + O(1)) = \frac{1}{2} x^2 + O(x^3) \).

Then since \( A(x) \) is unbounded,
\[
\sigma_c = \limsup_{x \to \infty} \frac{\log \left(\frac{1}{2} x^2 + O(x^3)\right)}{\log x} = \lim_{x \to \infty} \frac{4 \log x + O(1)}{\log x} = 4.0.
\]

(c) Let \( A(x) = \sum_{n \leq x} 2^{(\log n)^{3/2}} \). Then \( 2^{(\log(x-1))^{3/2}} \leq A(x) \leq 2^{(\log x)^{3/2}} \).

Since \( A(x) \) is unbounded, we have
\[
\sigma_c \geq \limsup_{x \to \infty} \frac{\log \left(2^{(\log(x-1)^{3/2}}\right)}{\log x} = \lim_{x \to \infty} \frac{\log (x-1)^{3/2}}{\log x} = -\infty.
\]

(a) Let \( A(x) = \sum_{n \leq x} (-1)^{\lfloor x \rfloor} = \begin{cases} 1, & \text{when } LxJ \text{ odd;} \\ 0, & \text{when } LxJ \text{ even.} \end{cases} \)

Since \( A(x) \) is bounded, one has \( \sigma_c \leq 0 \). Also, since
\[ A^+(x) = \sum_{n \leq x} |(-1)^{\lfloor x \rfloor}| = LxJ, \]

is unbounded, one sees that
\[
\sigma_c = \limsup_{x \to \infty} \frac{\log |A^+(x)|}{\log x} = \lim_{x \to \infty} \frac{\log LxJ}{\log x} = 1.
\]

Then since \( \sigma_c \leq \sigma_c + 1 \), it follows that \( \sigma_c = 0.0 \).

(c) Let \( A(x) = \sum_{1 \leq n \leq x} \sin \left(\frac{\pi x}{n} + \frac{2\pi}{n}\right) = \begin{cases} 1 + O(1/n), & \text{when } n \equiv 1 \pmod{4}, \\ 0(1/n), & \text{when } 2/n, \\ -1 + O(1/n), & \text{when } n \equiv 3 \pmod{4}. \end{cases} \)
Then, if \( A(x) \) were unbounded, one obtains
\[
\sigma_c = \limsup_{x \to \infty} \frac{\log |A(x)|}{\log x} = \lim_{x \to \infty} \frac{\log (C \log x)}{\log x} = 0.
\]

If \( A(x) \) were bounded, meanwhile, then \( \sigma_c = 0 \). Meanwhile, since
\[
A^+(x) = \sum_{1 \leq n \leq x} \log \left( \frac{n^c + 2^x}{n} \right) = \frac{x^c}{\log 2} + O \left( \sum_{n \leq x} \frac{1}{n} \right),
\]
we see that
\[
\sigma_c = \limsup_{x \to \infty} \frac{\log A^+(x)}{\log x} = \lim_{x \to \infty} \frac{\log (\frac{x^c}{\log 2} + O (\log x))}{\log x} = 1.
\]

Thus, since \( \sigma_a \leq \sigma_c + 1 \), it follows that \( \sigma_c = 0, \sigma_a \).

Q.31 (i) One has
\[
\left| \sum_{n > x} a_n n^{-s} \right| \leq \sum_{n > x} |a_n| n^{-r},
\]
so if \( \text{rhs} \to 0 \) as \( n \to \infty \) (which happens for \( s > \sigma_a \)), then \( \text{lhs} \to 0 \) as \( x \to \infty \), whence \( \sum_{n} a_n n^{-s} \) converges. Thus \( \sigma_c \leq \sigma_a \). \( \square \)

(ii) Suppose that \( s > \sigma_c \); say \( s = \sigma_c + 2\varepsilon \). Then
\[
\sum_{n} a_n n^{-\sigma_c - \varepsilon} \to 0 \quad \text{as} \quad n \to \infty.
\]
This implies that
\[
\sum_{n > x} |a_n| n^{-s} \leq \sum_{n > x} n^{-(s - \sigma_c - \varepsilon)} \to 0 \quad \text{as} \quad x \to \infty.
\]
Thus for any \( \varepsilon > 0 \), we have \( \sigma_a \leq \sigma_c + 1 + 2\varepsilon \), whence \( \sigma_a \geq \sigma_c + 1 \). \( \square \)

Q.41 (i) One has
\[
\sum_{n \leq x} a_n n^k = \int_{\infty}^{2x} u^k \, d \left( \sum_{n \leq u} a_n \right).
\]
But
\[
\sum_{n \leq u} a_n = \sum_{n=1}^{2x} a_n - \sum_{n \geq u} a_n \quad \text{for} \quad u \leq 2x,
\]
so
\[
\sum_{n \leq x} a_n n^k = -\int_{x}^{2x} u^k \, d \left( A(u) \right). \quad \square
\]

(ii) Apply integration by parts and the estimate \( |A(u)| \leq u^\theta \).
\[ \sum a_n n^k = -u^k A(u) \left\lfloor \frac{2^{x+1}}{x} \right\rfloor + k \int_1^{2x} A(u) u^{k-1} du \]
\[ \ll x^{k+\theta} + k \int_x^{2x} u^{k+\theta-1} du. \]

This is \( \ll x^{k+\theta} \) when \( k+\theta \neq 0 \), and when \( k+\theta = 0 \) we get \( \sum a_n n^k \ll \int_1^{2x} A(u) u^{k-1} du \). [The factor \( \log x \) was for safety!]

(iii) When \( \varepsilon > 0 \), one has \( \sum_{n=1}^{\infty} a_n n^{-e-\varepsilon} \) converges, whence \( \sum_{n>x} a_n n^{-e-\varepsilon} \ll 1 \).

Then we deduce that \( \sum_{n>x} a_n \ll x^{e+\varepsilon} \) (whence \( \sigma_c < 0 \) \( \varepsilon < \varepsilon(1) \)).

So that by summing over dyadic intervals, we deduce that \( \sum a_n \ll x^{e+\varepsilon} \).

It follows that
\[ \sigma^* = \limsup_{x \to \infty} \frac{\log |A(x)|}{\log x} \leq \lim_{x \to \infty} \frac{(e+\varepsilon) \log x}{\log x} = e+\varepsilon. \]

Since this holds for all small \( \varepsilon > 0 \), we find that \( \sigma^* \leq e \).

On the other hand, defining \( \sigma^* \) in this way, we have \( |A(x)| \ll x^{e+\varepsilon} \) for all small \( \varepsilon > 0 \). Take \( \varepsilon = 10^{-1} \). Then the argument of (iii) shows that
\[ \sum_{1 \leq n \leq x} a_n n^k \ll x^{k+e+\varepsilon} \log x, \]

whence
\[ \alpha(s-k) = \sum_{n=1}^{\infty} a_n n^{k-s} \]
has abscissa of convergence \( \sigma_c' = \limsup_{x \to \infty} \frac{\log |\sum_{1 \leq n \leq x} a_n n^k|}{\log x} \leq k+e+\varepsilon. \]

It follows that \( \alpha(s) \) has abscissa of convergence \( \sigma_c \leq \sigma^* + \varepsilon. \) Since this holds for all \( \varepsilon > 0 \), we see that \( \sigma_c \leq \sigma^* \).

We conclude that \( \sigma_c \leq \sigma^* \leq \sigma_c \), whence \( \sigma_c = \sigma^* \). That is,
\[ \sigma_c = \limsup_{x \to \infty} \frac{\log |A(x)|}{\log x}. \]

(i) Whenever \( \gcd(n,m) = 1 \), one has
\[ \tau(nm) = \sum_{d|nm} 1 = \sum_{d|n} \sum_{d|m} 1, \]
where \( d_1 = (d,n) \) \( d_2 = (d,m) \), since \( \gcd(n,m) = 1 \).
Thus \( \tau(nm) = \left( \sum_{d \mid n} 1 \right) \left( \sum_{d \mid m} 1 \right) = \tau(n)\tau(m) \), so \( \tau \) is multiplicative. Similarly, when \( (n,m) = 1 \), the prime divisors of \( n \) and \( m \) are disjoint; so

\[
\Omega_0(nm) = \prod_{p \mid nm} p = \left( \prod_{p \mid n} p \right) \left( \prod_{p \mid m} p \right) = \Omega_0(n)\Omega_0(m),
\]

whence \( \Omega_0 \) is multiplicative.

(ii) Using the multiplicative property of \( \tau \) and \( \Omega_0 \), we see that when \( \sigma > 1 \) one has

\[
\sum_{n=1}^{\infty} \frac{1}{n^\sigma \Omega_0(n)^\sigma} = \sum_{n=1}^{\infty} \prod_{p \mid n} \left( 1 + \sum_{h=1}^{\infty} \frac{(h+1)^{-\sigma} - 1}{p^h} \right)
= \prod_{p} \left( 1 + \sum_{h=1}^{\infty} \frac{(h+1)^{-\sigma} - 1}{p^h} \right) = \prod_{p} \left( 1 + \frac{\zeta(\sigma) - 1}{p^\sigma} \right).
\]

The absolute convergence of the product on the right hand side is assured, since for \( \sigma > 1 \) one has \( \zeta(\sigma) > 1 \), and

\[
\sum_{p} \frac{\zeta(\sigma) - 1}{p^\sigma} \ll \sum_{p} \frac{1}{p^\sigma} < \infty.
\]

The absolute convergence of the product shows the sum on the left hand side to be absolutely convergent, and since

\[
\left| \sum_{n=1}^{\infty} \frac{1}{n^\sigma \Omega_0(n)^\sigma} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^\sigma \Omega_0(n)^\sigma},
\]

we find that \( \chi(s) \) is absolutely convergent for \( \sigma > 1 \). It follows that

\[
\chi(s) = \prod_{p} \left( 1 + \frac{\zeta(\sigma) - 1}{p^\sigma} \right) \text{ is absolutely convergent for } \Re(s) > 1,
\]

whence \( \chi(s) \) is analytic for \( \sigma > 1 \). \( \Box \)