MA598AANT ANALYTIC NUMBER THEORY. PROBLEMS 2

TO BE HANDED IN BY FRIDAY 25TH SEPTEMBER 2020

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

A1. Let τ^* be the arithmetic function defined by putting $\tau^*(n) = \sum_{d^2|n} 1$.
(i) Show that when $\sigma > 1$, one has

$$\sum_{n=1}^{\infty} \tau^*(n) n^{-s} = \zeta(s)\zeta(2s).$$

(ii) Prove that

$$\sum_{1 \leqslant n \leqslant x} \tau^*(n) = x\zeta(2) + O(\sqrt{x}).$$

A2. (i) By applying Riemann-Stieltjes integration, show that when $\delta > 0$,

$$\sum_{p} \frac{1}{p(\log p)^{\delta}} < \infty.$$

(ii) By applying Riemann-Stieltjes integration, show that when $x \geqslant 3$, one has

$$\sum_{p \leqslant x} \frac{1}{p \log\log p} = \log\log\log x + O(1).$$

B3. (i) Prove that the arithmetic function f defined by taking $f(n) = (-1)^{n-1}$ $(n \in \mathbb{N})$ is multiplicative. Hence, by considering Euler products, deduce that for $\sigma > 0$, one has

$$\sum_{n=1}^{\infty} (-1)^{n-1} n^{-s} = (1 - 2^{1-s})\zeta(s).$$

(ii) By considering the Laurent series expansion of $\zeta(s)$ around $s = 1$, prove that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log 2$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n} \log n}{n} = C_0 \log 2 - \tfrac{1}{2}(\log 2)^2.$$

B4. (i) Let $\tau_k(n)$ denote the multiplicative function defined by

$$\tau_k(n) = \sum_{d_1 d_2 \dots d_k = n} 1,$$

and write

$$F(u) = \prod_{p \leqslant u} \left(\sum_{h=0}^{\infty} \tau_k(p^h)^r p^{-h} \right).$$

(i) Show that when $k \geqslant 2$ and $r \geqslant 1$, one has

$$\sum_{1 \leqslant n \leqslant x} \tau_k(n)^r/n \leqslant F(x).$$

(ii) Prove that there is a real number $c = c(k,r) > 0$ for which one has

$$\prod_{p \leqslant x} \left(1 - p^{-1}\right)^{k^r} \left(\sum_{h=0}^{\infty} \tau_k(p^h)^r p^{-h} \right) = c + O(1/x).$$

(iii) Deduce that when x is large, one has

$$F(x) = c'(\log x)^{k^r} \left(1 + O(1/\log x)\right),$$

for a suitable constant $c' > 0$, whence

$$\sum_{1 \leqslant n \leqslant x} \tau_k(n)^r/n \ll (\log x)^{k^r}.$$

C5. (i) Prove that

$$\sum_{1 \leqslant n \leqslant x} \tau_4(n) \ll x(\log x)^3.$$

(ii) By interpreting

$$\sum_{1 \leqslant n \leqslant x} \tau_2(n)^2$$

in terms of the number of solutions of the equation $x_1x_2 = y_1y_2$, prove that

$$\sum_{1 \leqslant n \leqslant x} \tau_2(n)^2 \leqslant \sum_{1 \leqslant n \leqslant x} \tau_4(n) \ll x(\log x)^3.$$

(iii) Let $k_1, \dots k_r$ be integers with $k_i \geqslant 2$ $(1 \leqslant i \leqslant r)$. Prove that

$$\sum_{1 \leqslant n \leqslant x} \tau_{k_1}(n) \dots \tau_{k_r}(n) \ll x(\log x)^{k_1 \dots k_r - 1}.$$