Solutions to problem sheet 2.

Q1 (i) When $\sigma > 1$, one has

$$\zeta(s)\zeta(2s) = \left(\sum_{n=1}^{\infty} n^{-s}\right)\left(\sum_{m=1}^{\infty} (m^2)^{-s}\right) = \sum_{k=1}^{\infty} \sum_{k=nm^2} k^{-s} = \sum_{k=1}^{\infty}\left(\sum_{d^2|k} 1\right) k^{-s} = \sum_{k=1}^{\infty} \tau^*(k) k^{-s}. \quad \square$$

(ii) One has

$$\sum_{1\le n\le x} \tau^*(n) = \sum_{1\le n\le x} \sum_{d^2|n} 1 = \sum_{1\le d\le x^{1/2}} \sum_{1\le m\le x/d^2} 1 = \sum_{1\le d\le x^{1/2}} \left(\frac{x}{d^2} + O(1)\right)$$

$$= x\sum_{d=1}^{\infty} \frac{1}{d^2} - x\underbrace{\sum_{d>x} \frac{1}{d^2}}_{=O(x^{-1/2})} + O(x^{1/2}) = x\,\zeta(2) + O(x^{1/2}). \quad \square$$

Q2 (i) Recall ~~that~~ we have $\sum_{p\le x} \frac{\log p}{p} = \log x + O(1)$. Then by Riemann–Stieltjes integration, it follows that when $\delta>0$ one has

$$\sum_p \frac{1}{p(\log p)^\delta} = \int_{2^-}^{\infty} (\log u)^{-1-\delta}\, d A(u), \qquad \text{where } A(u) = \sum_{p\le u} \frac{\log p}{p} = \log u + O(1).$$

$$= \left[A(u)(\log u)^{-1-\delta}\right]_{2^-}^{\infty} + \int_{2^-}^{\infty} (1+\delta)(\log u)^{-2-\delta}\frac{A(u)}{u}\, du$$

$$\ll 0 + (1+\delta)\int_{2^-}^{\infty} (\log u)^{-1-\delta}\frac{du}{u} < \infty. \quad \square$$

(ii) Similarly, using $\sum_{p\le x} \frac{1}{p} = \log\log x + O(1)$, we obtain

$$\sum_{p\le x} \frac{1}{p\log\log p} = \frac{1}{2\log\log 2} + \int_{3^-}^{x} (\log\log u)^{-1}\, d B(u), \quad \text{where } B(u) = \sum_{p\le u}\frac{1}{p} = \log\log u + O(1).$$

By integrating by parts, the integral here is equal to

$$\left[B(u)(\log\log u)^{-1}\right]_{3^-}^{x} + \int_{3^-}^{x} \frac{B(u)}{u(\log u)(\log\log u)^2}\, du$$

$$= \int_{3^-}^{x} \frac{du}{u\log u\,\log\log u} + O\left(\int_{3^-}^{x} \frac{du}{u(\log u)(\log\log u)^2}\right) + O(1)$$

$$= \log\log\log x + O(1).$$

Thus $$\sum_{p\le x} \frac{1}{p\log\log p} = \log\log\log x + O(1). \quad \square$$

②

Q3 (i) When $(m,n)=1$, one has that either m or n or both are odd, whence

$$mn-1 = (m-1)(n-1)+(m-1)+(n-1) \equiv (m-1)+(n-1) \pmod 2.$$

Thus $(-1)^{mn-1} = (-1)^{m-1}\cdot(-1)^{n-1}$, or equivalently, $f(mn)=f(m)f(n)$. Also, one has $f(1)=1$, so f is multiplicative. □

When $\sigma>1$, the series $\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}$ is absolutely convergent, so we may employ the Euler product to deduce that

$$\sum_{n\geq 1}^{\infty}(-1)^{n-1}n^{-s} = \left(1-\sum_{h=1}^{\infty}2^{-hs}\right)\prod_{p>2}\left(1+\sum_{h=1}^{\infty}p^{-hs}\right)$$

$$= \frac{(1-2^{-s}/(1-2^{-s}))}{(1-2^{-s})^{-1}}\prod_{p}(1-p^{-s})^{-1} = (1-2^{1-s})\zeta(s).$$

The function on the right hand side is analytic for $\mathrm{Re}(s)>0$, since $1-2^{1-s}$ is analytic in this half-plane with a zero at $s=1$, and $\zeta(s)$ is analytic in $\sigma>0$ except for a simple pole at $s=1$ (this being cancelled by the zero of $1-2^{1-s}$ at $s=1$). Thus, since $\sum_{n=1}^{\infty}(-1)^{n-1}n^{-\sigma}$ is convergent for $\sigma>0$, and hence has abscissa of convergence $\sigma_c=0$, we see that

$$\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s} = (1-2^{1-s})\zeta(s) \quad \text{for } \sigma>0. \ \square$$

(ii) By considering Laurent series, we see that

$$(1-2^{1-s})\zeta(s) = (1-e^{(1-s)\log 2})\left(\frac{1}{s-1}+c_0+c_1(s-1)+\dots\right)$$

$$= \left((s-1)\log 2-\frac{(s-1)^2}{2}(\log 2)^2+\dots\right)\left(\frac{1}{s-1}+c_0+c_1(s-1)+\dots\right)$$

$$= \log 2 + \left(c_0\log 2-\tfrac{1}{2}(\log 2)^2\right)(s-1)+\dots\ .$$

Then $$\sum_{n=1}^{\infty}(-1)^{n-1}/n = \lim_{s\to 1}(1-2^{1-s})\zeta(s) = \log 2$$

and $$\sum_{n=1}^{\infty}(-1)^{n}\frac{\log n}{n} = \lim_{s\to 1}\frac{d}{ds}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^s} = \lim_{s\to 1}\frac{d}{ds}\left((1-2^{1-s})\zeta(s)\right)$$

$$= c_0\log 2-\tfrac{1}{2}(\log 2)^2. \ \square$$

Q4 (i) The function $\tau_k(n)$ is multiplicative by induction on k (use the fact that f & g multiplicative $\Rightarrow \sum_{d|n}f(d)g(n/d)$ multiplicative). Thus

$$\sum_{1\leq n\leq x}\tau_k(n)^r/n \leq \sum_{1\leq n\leq x}\prod_{p^h\|n}\frac{\tau_k(p^h)^r}{p^h} \leq \sum_{p\leq x}\sum_{h=0}^{\infty}\frac{\tau_k(p^h)^r}{p^h} = F(x). \ \square$$

③

(ii) One has $\sum_{h=0}^{\infty} \tau_k(p^h)^r p^{-h} = 1 + \frac{k^r}{p} + \sum_{h=2}^{\infty} \frac{a(h)}{p^h}$, for suitable integers $a(h) = a(k,r,h) > 0$ with $\log(a(h)) \ll \log h$. Thus

$$(1-p^{-1})^{k^r} \sum_{h=0}^{\infty} \tau_k(p^h)^r p^{-h} = \left(1 - \frac{k^r}{p} + \frac{\binom{k^r}{2}}{p^2} + \cdots\right)\left(1 + \frac{k^r}{p} + \frac{a(2)}{p^2} + \cdots\right)$$

$$= 1 - \frac{\left(k^{2r} - \binom{k^r}{2} - a(2)\right)}{p^2} + \cdots$$

$$= 1 + \sum_{h=2}^{\infty} \frac{b(h)}{p^h}, \quad \text{for suitable integers } b(h) = b(k,r,h)$$

with $\log(b(h)) \ll \log h$.

Hence $\prod_p (1-p^{-1})^{k^r} \left(\sum_{h=0}^{\infty} \tau_k(p^h)^r p^{-h}\right) = \prod_p \left(1 + \sum_{h=2}^{\infty} \frac{b(h)}{p^h}\right)$ is absolutely convergent, since the same is true for $\sum_p \log\left(1 + \sum_{h=2}^{\infty} \frac{b(h)}{p^h}\right) \ll \sum_p \left(\frac{b(2)}{p^2} + \cdots\right) < \infty$. Put

$$c = \prod_p \left(1 + \sum_{h=2}^{\infty} \frac{b(h)}{p^h}\right).$$

Then $\log\left(\prod_{p>x} (1-p^{-1})^{k^r} \left(\sum_{h=0}^{\infty} \tau_k(p^h)^r / p^h\right)\right) = \sum_{p>x} \log\left(1 + \sum_{h=2}^{\infty} \frac{b(h)}{p^h}\right) \ll \sum_{n>x} n^{-2} = O\left(\frac{1}{x}\right)$,

Whence

$$\prod_{p>x} (1-p^{-1})^{k^r} \left(\sum_{h=0}^{\infty} \tau_k(p^h)^r p^{-h}\right) = 1 + O(1/x).$$

Consequently,

$$\prod_{p \le x} (1-p^{-1})^{k^r} \left(\sum_{h=0}^{\infty} \tau_k(p^h)^r p^{-h}\right) = c/(1 + O(\tfrac{1}{x})) = c + O\left(\frac{1}{x}\right). \quad \square$$

(iii) We conclude that

$$F(x) = \prod_{p \le x} \sum_{h=0}^{\infty} \tau_k(p^h)^r p^{-h} = \left(c + O\left(\tfrac{1}{x}\right)\right)\left(\prod_{p \le x} (1-p^{-1})^{-1}\right)^{k^r}$$

$$= c' (\log x)^{k^r} (1 + O(1/x)), \quad \square$$

by apply Mertens' theorem $\prod_{p \le x} (1-p^{-1})^{-1} = e^{c_0} (\log x) + O(1)$. ~~But then,~~

$$\sum_{1 \le n \le x} \tau_k(n)^r / n \le \prod_{p \le x} \left(\sum_{h=0}^{\infty} \frac{\tau_k(p^h)^r}{p^h}\right) = F(x) \ll (\log x)^{k^r}. \quad \square$$

Q5 (i) We have $\sum_{1 \le n \le x} \tau_4(n) = \sum_{1 \le n \le x} \sum_{d|n} \tau_3(d) = \sum_{1 \le d \le x} \tau_3(d) \sum_{1 \le m \le x/d} 1$

$$\le \sum_{1 \le d \le x} \tau_3(d) \frac{x}{d} = x \sum_{1 \le d \le x} \tau_3(d)/d \ll x (\log x)^3 \text{ by Q4.} \quad \square$$

(ii) Observe that $\sum_{1 \le n \le x} \tau_2(n)^2 = \# \{ x_1 x_2 = y_1 y_2 = n : 1 \le n \le x, \quad x_i, y_i \in \mathbb{N} \}$

$$= \# \{ x_1 x_2 = y_1 y_2 \le x \}.$$

Put $a_1 = (x_1, y_1)$, $a_2 = x_1/a_1$, $b_1 = (x_2, y_1/a_1)$, $b_2 = x_2/b_1$. Then we have $x_1 = a_1 a_2$, $x_2 = b_1 b_2$, $y_1 = a_1 b_1$, $y_2 = a_2 b_2$, and we see that

$$\sum_{1 \le n \le x} \tau_2(n)^2 \le \# \{ a_1 a_2 b_1 b_2 \le x \} = \sum_{1 \le n \le x} \tau_4(n) \ll x (\log x)^3 \text{ by (i).}$$

$\square$

(iii) Similarly, we can interpret

$$\sum_{1 \le n \le x} \tau_{k_1}(n) \cdots \tau_{k_r}(n) = \# \{ x_{11} \cdots x_{1k_r} = x_{21} \cdots x_{2k_2} = \cdots = x_{r1} \cdots x_{rk_r} \le x \}$$

$$\le \# \Big\{ \prod_{1 \le i_1 \le k_1} \cdots \prod_{1 \le i_r \le k_r} a_{i_1 \dots i_r} \le x \Big\}$$

$$= \sum_{1 \le n \le x} \tau_{k_1 \cdots k_r}(n) \ll x \sum_{1 \le d \le x} \tau_{k_1 \cdots k_r}(d)/d$$

$$\ll x (\log x)^{k_1 \cdots k_r - 1}. \quad \square$$