Let \( \pi(x; q, a) \) denote the number of prime numbers \( p \) with \( p \equiv a \pmod{q} \) and \( p \leq x \).

Also, let \( \theta(x; q, a) = \sum_{p \leq x \atop p \equiv a \pmod{q}} \log p \) and \( \psi(x; q, a) = \sum_{n \leq x \atop n \equiv a \pmod{q}} \Lambda(n) \).

A1. By considering the contribution of the prime powers \( p^k \) with \( k \geq 2 \), show that
\[
\theta(x; q, a) = \psi(x; q, a) + O(x^{1/2}).
\]

A2. By applying Riemann-Stieltjes integration, or otherwise, show that
\[
\pi(x; q, a) = \frac{\theta(x; q, a)}{\log x} + O\left(\frac{x}{(\log x)^2}\right).
\]

B3. Recall that when \((a, q) = 1\), one has
\[
\sum_{p \leq x \atop p \equiv a \pmod{q}} \log p = \frac{1}{\phi(q)} \log x + O_q(1).
\]

(i) Show that, when \( C = C(q) \) is sufficiently large in terms of \( q \), and \( x \) is sufficiently large in terms of \( C \), then there is a positive number \( c = c(q) \) for which
\[
\sum_{x/C < p \leq x \atop p \equiv a \pmod{q}} \log p > c.
\]

(ii) Deduce that for all large enough \( x \) in terms of \( q \), one has \( \pi(x; q, a) \gg q x / \log x \).

B4. Let \( X(q) \) denote the set of Dirichlet characters modulo \( q \).

(i) By applying orthogonality, prove that whenever \( a_i \in \mathbb{C} \) \((1 \leq i \leq q)\), one has
\[
\frac{1}{\phi(q)} \sum_{\chi \in X(q)} \left| \sum_{n=1}^{q} a_n \chi(n) \right|^2 = \sum_{n=1 \atop (n, q)=1}^{q} |a_n|^2.
\]

(ii) By applying orthogonality, prove that whenever \( a_\chi \in \mathbb{C} \) \((\chi \in X(q))\), one has
\[
\frac{1}{\phi(q)} \sum_{n=1}^{q} \left| \sum_{\chi \in X(q)} a_\chi \chi(n) \right|^2 = \sum_{\chi \in X(q)} |a_\chi|^2.
\]
C5. When $q$ and $n$ are natural numbers, define the function

$$c_q(n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e(an/q).$$

(i) Prove that

$$\sum_{d|q} c_d(n) = \delta_q(n),$$

where

$$\delta_q(n) = \begin{cases} q, & \text{when } q|n, \\ 0, & \text{when } q \nmid n. \end{cases}$$

(ii) Show that the function $\delta_q(n)$ is a multiplicative function of $q$.

(iii) By applying the Möbius inversion formula, prove that

$$c_q(n) = \sum_{d|(q,n)} d\mu(q/d).$$

(iv) Deduce that $c_q(n)$ is a multiplicative function of $q$ which, as a function of $n$, is periodic with period $q$.

(v) By considering the value of $c_q(n)$ when $q$ is a prime power, or otherwise, prove that

$$c_q(n) = \frac{\mu(q/(q,n))}{\phi(q/(q,n))}\phi(q),$$

whence $|c_q(n)| \leq (q,n)$.

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