MA598AANT ANALYTIC NUMBER THEORY. PROBLEMS 3

TO BE HANDED IN BY FRIDAY 9TH OCTOBER 2020

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

Let $\pi(x;q,a)$ denote the number of prime numbers p with $p \equiv a \pmod{q}$ and $p \leqslant x$. Also, let

$$\theta(x;q,a) = \sum_{\substack{p \leqslant x \\ p \equiv a \ (\mathrm{mod}\ q)}} \log p \quad \text{and} \quad \psi(x;q,a) = \sum_{\substack{n \leqslant x \\ n \equiv a \ (\mathrm{mod}\ q)}} \Lambda(n).$$

A1. By considering the contribution of the prime powers p^k with $k \geqslant 2$, show that

$$\theta(x;q,a) = \psi(x;q,a) + O(x^{1/2}).$$

A2. By applying Riemann-Stieltjes integration, or otherwise, show that

$$\pi(x;q,a) = \frac{\theta(x;q,a)}{\log x} + O\left(\frac{x}{(\log x)^2}\right).$$

B3. Recall that when $(a,q) = 1$, one has

$$\sum_{\substack{p \leqslant x \\ p \equiv a \ (\mathrm{mod}\ q)}} \frac{\log p}{p} = \frac{1}{\phi(q)} \log x + O_q(1).$$

(i) Show that, when $C = C(q)$ is sufficiently large in terms of q, and x is sufficiently large in terms of C, then there is a positive number $c = c(q)$ for which

$$\sum_{\substack{x/C < p \leqslant x \\ p \equiv a \ (\mathrm{mod}\ q)}} \frac{\log p}{p} > c.$$

(ii) Deduce that for all large enough x in terms of q, one has $\pi(x;q,a) \gg_q x/\log x$.

B4. Let $\mathrm{X}(q)$ denote the set of Dirichlet characters modulo q.
(i) By applying orthogonality, prove that whenever $a_i \in \mathbb{C}$ $(1 \leqslant i \leqslant q)$, one has

$$\frac{1}{\phi(q)} \sum_{\chi \in \mathrm{X}(q)} \left| \sum_{n=1}^{q} a_n \chi(n) \right|^2 = \sum_{\substack{n=1 \\ (n,q)=1}}^{q} |a_n|^2.$$

(ii) By applying orthogonality, prove that whenever $a_\chi \in \mathbb{C}$ $(\chi \in \mathrm{X}(q))$, one has

$$\frac{1}{\phi(q)} \sum_{n=1}^{q} \left| \sum_{\chi \in \mathrm{X}(q)} a_\chi \chi(n) \right|^2 = \sum_{\chi \in \mathrm{X}(q)} |a_\chi|^2.$$

C5. When q and n are natural numbers, define the function

$$c_q(n)=\sum_{\substack{a=1\\(a,q)=1}}^{q} e(an/q).$$

(i) Prove that

$$\sum_{d|q} c_d(n)=\delta_q(n),$$

where

$$\delta_q(n)=\begin{cases} q, & \text{when } q|n,\\ 0, & \text{when } q\nmid n.\end{cases}$$

(ii) Show that the function $\delta_q(n)$ is a multiplicative function of q.
(iii) By applying the Möbius inversion formula, prove that

$$c_q(n)=\sum_{d|(q,n)} d\mu(q/d).$$

(iv) Deduce that $c_q(n)$ is a multiplicative function of q which, as a function of n, is periodic with period q.
(v) By considering the value of $c_q(n)$ when q is a prime power, or otherwise, prove that

$$c_q(n)=\frac{\mu(q/(q,n))}{\phi(q/(q,n))}\phi(q),$$

whence $|c_q(n)|\leqslant (q,n)$.