One has
\[ \sum_{p \leq x} \frac{\log p}{p - a \pmod{q}} - \sum_{p \leq x} \frac{\log p}{p^k - a \pmod{q}} \]
where
\[ \left| \sum_{p \leq x/\sqrt{2}} \frac{\log p}{p - a \pmod{q}} \right| \leq \frac{1}{\sqrt{\log x}} \cdot \log x = (\log x) \pi(x^{1/2}) \ll x^{1/2}. \]
Thus
\[ \theta(x; q, a) = \psi(x; q, a) + O(x^{1/2}). \]

By applying the Riemann-Siegel integration, we find that
\[ \Pi(x; q, a) = \int_{2-}^{x} \frac{1}{\log y} \vartheta(y; q, a) = \left[ \frac{\theta(y; q, a)}{\log y} \right]_{2-}^{x} + \int_{2-}^{x} \frac{\theta(y; q, a)}{y (\log y)^2} \, dy \]
\[ = \frac{\theta(x; q, a)}{\log x} + O \left( \int_{2-}^{x} \frac{dy}{(\log y)^2} \right), \quad \text{since} \quad \theta(y; q, a) \leq \theta(y) \log y. \]
\[ = \frac{\theta(x; q, a)}{\log x} + O \left( \frac{x}{(\log x)^2} \right) \quad \text{(apply a dyadic division, say)}. \]

We are told to recall that, for a suitable \( a(q) > 0 \), giving an upper bound for the \( \varphi(q) \) term, one has
\[ \sum_{p \leq x} \frac{\log p}{p - a \pmod{q}} \leq a(q). \]
Thus, when \( C = C(q) \) is large, we deduce that
\[ \sum_{x/C < p \leq x} \frac{\log p}{p - a \pmod{q}} \geq \left( \frac{1}{\varphi(q)} \log x - a(q) \right) \left( \frac{1}{\varphi(q)} \log (x/C) + a(q) \right) \]
\[ = \frac{\log C}{\varphi(q)} - 2a(q). \]
If we take \( C(q) = \exp(3a(q)\varphi(q)) \), therefore, then it follows that
\[ \sum_{x/C < p \leq x} \frac{\log p}{p - a \pmod{q}} \geq \frac{3a(q)\varphi(q)}{\varphi(q)} - 2a(q) = a(q), \]
and hence the desired conclusion holds with \( C(q) = a(q) \).

(i) One has
\[ \sum_{x/C < p \leq x} \frac{\log p}{p - a \pmod{q}} \leq \sum_{x/C < p \leq x} \frac{\log p}{\log x} \leq \frac{C \log x}{x} \sum_{p \leq x} \frac{1}{p - a \pmod{q}} \]

Hence
\[ \Pi(x; q, a) \geq C(q) \cdot \frac{x}{C \log x} = \frac{C(q)}{C(q)} \cdot \frac{x}{\log x} \iff \frac{x}{\log x}. \]
(i) One has $\chi(n) = 0$ when $x \not\equiv (q, n)_2 = 1$. Thus, by orthogonality,
\[
\frac{1}{\varphi(q)} \sum_{\chi \in \chi(q)} \left| \sum_{n=1}^{q} a_n \chi(n) \right|^2 = \frac{1}{\varphi(q)} \sum_{\chi \in \chi(q)} \sum_{n=1}^{q} a_n \chi(n) \sum_{m=1}^{q} \overline{a_m} \chi(m) \\
= \frac{1}{\varphi(q)} \sum_{n=1}^{q} \sum_{m=1}^{q} a_n \overline{a_m} \sum_{\chi \in \chi(q)} \chi(n \overline{m}) = 0, \quad \text{when } n \overline{m} \not\equiv 1 \pmod{q} \\
= \frac{1}{\varphi(q)} \sum_{n=1}^{q} (a_n \overline{a_n}) = \phi(q), \quad \text{since the only summands which contribute are those with } n \equiv m \pmod{q} \\
= \frac{1}{\varphi(q)} \sum_{n=1}^{q} |a_n|^2.
\]

(ii) Similarly, one has
\[
\frac{1}{\varphi(q)} \left| \sum_{\chi \in \chi(q)} \sum_{x=1}^{q} \chi_1(x) \chi_2(x) \right|^2 = \frac{1}{\varphi(q)} \sum_{\chi_1, \chi_2 \in \chi(q)} \sum_{n=1}^{q} \chi_1(n) \chi_2(n) \\
= \frac{1}{\varphi(q)} \sum_{\chi_1, \chi_2 \in \chi(q)} \sum_{n=1}^{q} \chi_1(n) \chi_2(n) = 0, \quad \text{when } \chi_1 \chi_2 \not\equiv \chi_0 \pmod{q} \\
= \frac{1}{\varphi(q)} \sum_{\chi \in \chi(q)} |a_{\chi}|^2 = \phi(q), \quad \text{since the only summands which contribute are those with } \chi_1 = \chi_2 \\
= \sum_{\chi \in \chi(q)} |a_{\chi}|^2.
\]

Q5 (i) One has
\[
\sum_{d|q} c_d(n) = \sum_{d|q} \sum_{a=1}^{d} e\left(\frac{a(q/d)}{q} n\right) = \sum_{b=1}^{q} e\left(\frac{b}{q} n\right) = \left\{\begin{array}{ll}
q, & \text{if } q|n \\
0, & \text{if } q|n\end{array}\right.
\]

Hence
\[
\sum_{d|q} c_d(n) = \delta_q(n).
\]

(ii) When $(q_1, q_2) = 1$, one has
\[
q_1 q_2 | n \Leftrightarrow q_1 | n \quad \& \quad q_2 | n.
\]
Whence \( \delta_{q,iq} (n) = qiq = \delta_{q,1}(n) \delta_{q,iq}(n) \) when \( qiq \parallel n \),

\( \delta_{qiq} (n) = 0 = \delta_{q,1}(n) \delta_{q,iq}(n) \) when \( qiq \nmid n \).

Thus \( \delta_{q,iq} (n) = \delta_{q,1}(n) \delta_{q,iq}(n) \) in all circumstances. Moreover, one has \( \delta_{q,1}(n) = 1 \), so \( \delta_{q,1}(n) \) is a multiplicative function of \( q \).

(iii) By Möbius inversion, one has

\[
\sum_{d \mid \varphi} c_d(n) = \delta_{q,1}(n) \Rightarrow c_q(n) = \sum_{d \mid \varphi} \delta_{q,1}(n) \varphi(\varphi/d) = \sum_{d \mid \varphi} d \varphi(\varphi/d),
\]

where \( c_q(n) = \sum_{d \mid \varphi} d \varphi(\varphi/d). \)

(iv) Since \( \delta_{q,1}(n) \) is a multiplicative function of \( q \) and \( \varphi(\varphi/d) \) is a multiplicative function of \( q \), we see that \( \sum_{d \mid \varphi} \delta_{q,1}(n) \varphi(\varphi/d) = \delta_{q,1}(n) \varphi(q) \) is a multiplicative function of \( q \). Thus \( c_q(n) \) is a multiplicative function of \( q \).

Moreover, since

\[
c_q(n+q) = \sum_{a=1}^{q} e\left(\frac{a(n+q)}{q}\right) = \sum_{a=1}^{q} e\left(\frac{aq}{q}\right) = c_q(n),
\]

we find that \( c_q(n) \) is a periodic function of \( n \) with period dividing \( q \).

To see that the period is at least \( q \), observe that when \( r \) is not divisible by \( q \), one has

\[
|c_q(r)| = \left| \sum_{a=1}^{q} e\left(\frac{ar}{q}\right) \right| \leq |\varphi(q) - 1 + e\left(\frac{q}{q}\right)| < \varphi(q),
\]

where \( |c_q(r)| < |c_q(q)| = \varphi(q) \). So it is impossible that \( c_q(n) \) is a periodic function of \( n \) with period \( r \).

(v) Use the multiplicativity of \( c_q(n) \) as a function of \( q \). When \( q = p^h \), one has

\[
c_{p^h}(n) = \sum_{d \mid (p^h,n)} d \varphi(p^h/d) = \begin{cases} 0, & \text{when } (p^h,n) = p^t \text{ with } t \leq h-2, \\
p^{h-1}, & \text{when } (p^h,n) = p^{h-1}, \\
p^h - p^{h-1}, & \text{when } (p^h,n) = p^h,
\end{cases}
\]

But

\[
\frac{\varphi(p^h)}{\phi(p^h)} = \begin{cases} 0, & \text{when } (p^h,n) = p^t \text{ with } t \leq h-2, \\
-\frac{\varphi(p^h)}{\phi(p)} = -p^{h-1}, & \text{when } (p^h,n) = p^{h-1}, \\
\phi(p^h) = p^h - p^{h-1}, & \text{when } (p^h,n) = p^h.
\end{cases}
\]
Thus, for every prime power $p^k$, we have

$$C_{p^k}(n) = \frac{\mu\left(p^k/(p^k,n)\right)}{\phi(p^k)} \phi(p^k),$$

Whence

$$C_q(n) = \prod_{p^k \vert q} C_{p^k}(n) = \prod_{p^k \vert q} \frac{\mu\left(p^k/(p^k,n)\right)}{\phi(p^k)} \phi(p^k)$$

$$= \frac{\mu\left(q/(q,n)\right)}{\phi(q/(q,n))} \phi(q), \quad \text{using multiplicative property of the \phi functions here.}$$

In particular, one sees that

$$|C_q(n)| \leq \frac{\phi(q)}{\phi(q/(q,n))} = \prod_{p^k \vert q} \frac{\phi(p^k)}{\phi(p^k/(p^k,n))} \leq \prod_{p^k \vert q} (p^k,n) = (q,n). \quad \square$$