Solutions to problem sheet 3.

Q1 One has
$$\theta(x;q,a) = \sum_{\substack{p\le x\\ p\equiv a \,(\mathrm{mod}\, q)}} \log p = \psi(x;q,a) - \sum_{\substack{p^k\le x\\ p^k\equiv a\,(\mathrm{mod}\, q)\\ k\ge 2}} \log p,$$
whence
$$|\theta(x;q,a) - \psi(x;q,a)| \le \sum_{p\le x^{1/2}} \left(\frac{\log x}{\log p}\right)\cdot \log p = (\log x)\,\pi(x^{1/2}) \ll x^{1/2}.$$
Thus $\theta(x;q,a) = \psi(x;q,a) + O(x^{1/2})$. //

Q2 By applying Riemann-Stieltjes integration, we find that
$$\pi(x;q,a) = \int_{2^-}^{x} \frac{1}{\log y}\, d\,\theta(y;q,a) = \left[\frac{\theta(y;q,a)}{\log y}\right]_{2^-}^{x} + \int_{2^-}^{x} \frac{\theta(y;q,a)}{y(\log y)^2}\,dy$$
$$= \frac{\theta(x;q,a)}{\log x} + O\left(\int_{2^-}^{x}\frac{dy}{(\log y)^2}\right), \text{ since } \theta(y;q,a)\le\theta(y)\asymp y.$$
$$= \frac{\theta(x;q,a)}{\log x} + O\left(\frac{x}{(\log x)^2}\right) \quad \text{(apply a dyadic dissection, say).} //$$

Q3 (i) We are told to recall that, for a suitable $a(q)>0$ giving an upper bound for the $O_q(1)$ term, one has
$$\left|\sum_{\substack{p\le x\\ p\equiv a\,(\mathrm{mod}\, q)}} \frac{\log p}{p} - \frac{1}{\phi(q)}\log x\right| \le a(q).$$
Thus, when $C=C(q)$ is large, we deduce ~~that~~
$$\sum_{\substack{x/C<p\le x\\ p\equiv a\,(\mathrm{mod}\, q)}} \frac{\log p}{p} \ge \left(\frac{1}{\phi(q)}\log x - a(q)\right) - \left(\frac{1}{\phi(q)}\log(x/C) + a(q)\right)$$
$$= \frac{\log C}{\phi(q)} - 2a(q).$$
If we take $C(q) > \exp(3a(q)\phi(q))$, ~~therefore,~~ then it follows ~~that~~
$$\sum_{\substack{x/C<p\le x\\ p\equiv a\,(\mathrm{mod}\, q)}} \frac{\log p}{p} > \frac{3a(q)\phi(q)}{\phi(q)} - 2a(q) = a(q),$$
and hence the desired conclusion holds with $c(q)=a(q)$. □

(ii) One has
$$\sum_{\substack{x/C<p\le x\\ p\equiv a\,(\mathrm{mod}\, q)}} \frac{\log p}{p} \le \sum_{\substack{x/C<p\le x\\ p\equiv a\,(\mathrm{mod}\, q)}} \frac{\log x}{x/C} \le \frac{C\log x}{x}\sum_{\substack{p\le x\\ p\equiv a\,(\mathrm{mod}\, q)}} 1,$$
Whence
$$\pi(x;q,a) \ge c(q)\cdot\frac{x}{C\log x} = \frac{c(q)}{C(q)}\,\frac{x}{\log x} \gg_q \frac{x}{\log x}. \quad □ //$$

②

Q4 (i) One has $\chi(n)=0$ when $\chi\in X(q)$ & $(q,n)>1$. Thus, by orthogonality,

$$\frac{1}{\phi(q)}\sum_{\chi\in X(q)}\Big|\sum_{n=1}^{q}a_n\chi(n)\Big|^2=\frac{1}{\phi(q)}\sum_{\chi\in X(q)}\sum_{\substack{n=1\\(n,q)=1}}^{q}a_n\chi(n)\sum_{\substack{m=1\\(m,q)=1}}^{q}\bar{a}_m\bar{\chi}(m)$$

$$=\frac{1}{\phi(q)}\sum_{\substack{n=1\\(n,q)=1}}^{q}\sum_{\substack{m=1\\(m,q)=1}}^{q}a_n\bar{a}_m\underbrace{\sum_{\chi\in X(q)}\chi(nm^{-1})}_{\begin{cases}=0, & \text{when } nm^{-1}\not\equiv 1 \pmod q\\ =\phi(q), & \text{when } nm^{-1}\equiv 1\pmod q\end{cases}}$$

$$=\frac{1}{\phi(q)}\sum_{\substack{n=1\\(n,q)=1}}^{q}|a_n|^2\phi(q),$$ since the only summands which contribute are those with $n\equiv m \pmod q$

$$=\sum_{\substack{n=1\\(n,q)=1}}^{q}|a_n|^2. \qquad \square$$

(ii) Similarly, one has

$$\frac{1}{\phi(q)}\sum_{n=1}^{q}\Big|\sum_{\chi\in X(q)}a_\chi\chi(n)\Big|^2=\frac{1}{\phi(q)}\sum_{\substack{n=1\\(n,q)=1}}^{q}\sum_{\chi_1\in X(q)}a_{\chi_1}\chi_1(n)\sum_{\chi_2\in X(q)}\bar{a}_{\chi_2}\bar{\chi}_2(n)$$

$$=\frac{1}{\phi(q)}\sum_{\chi_1\in X(q)}\sum_{\chi_2\in X(q)}a_{\chi_1}\bar{a}_{\chi_2}\underbrace{\sum_{\substack{n=1\\(n,q)=1}}^{q}\chi_1\bar{\chi}_2(n)}_{\begin{cases}=0, & \text{when } \chi_1\bar{\chi}_2\neq\chi_0\\ =\phi(q), & \text{when } \chi_1\bar{\chi}_2=\chi_0\end{cases}}$$

$$=\frac{1}{\phi(q)}\sum_{\chi\in X(q)}|a_\chi|^2\phi(q),$$ since the only summands which contribute are those with $\chi_1=\chi_2$

$$=\sum_{\chi\in X(q)}|a_\chi|^2. \qquad \square \quad //$$

Q5 (i) One has $$\sum_{d|q}c_d(n)=\sum_{d|q}\sum_{\substack{a=1\\(a,d)=1}}^{d}e\left(\frac{a(q/d)}{q}n\right)=\sum_{b=1}^{q}e\left(\frac{b}{q}n\right)=\begin{cases}q, & \text{if } q|n\\ 0, & \text{if } q\nmid n\end{cases}.$$

Hence $$\sum_{d|q}c_d(n)=\delta_q(n). \quad \square$$

(ii) When $(q_1,q_2)=1$, one has $q_1q_2|n \Leftrightarrow q_1|n$ & $q_2|n$, Whence $\quad \delta_{q_1q_2}(n) = q_1q_2 = \delta_{q_1}(n)\,\delta_{q_2}(n) \quad$ when $q_1q_2 \mid n$,

$\quad \delta_{q_1q_2}(n) = 0 = \delta_{q_1}(n)\,\delta_{q_2}(n) \quad$ when $q_1q_2 \nmid n$.

Thus $\delta_{q_1q_2}(n) = \delta_{q_1}(n)\,\delta_{q_2}(n)$ in all circumstances. Moreover, one has $\delta_1(n) = 1$, so $\delta_q(n)$ is a multiplicative function of q. □

(iii) By Möbius inversion, one has

$$\sum_{d\mid q} c_d(n) = \delta_q(n) \Rightarrow c_q(n) = \sum_{d\mid q} \delta_d(n)\,\mu(q/d) = \sum_{\substack{d\mid q\\ d\mid n}} d\,\mu(q/d),$$

whence $\quad c_q(n) = \displaystyle\sum_{d\mid (q,n)} d\,\mu(q/d)$. □

(iv) Since $\delta_q(n)$ is a multiplicative function of q and $\mu(q)$ is a multiplicative function of q, we see that $\displaystyle\sum_{d\mid n} \delta_d(n)\,\mu(q/d) = \delta_{\cdot}(n) * \mu(\cdot)$ is a multiplicative function of q. Thus $c_q(n)$ is a multiplicative function of q. Moreover, since

$$c_q(n+q) = \sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(\frac{a(n+q)}{q}\right) = \sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(\frac{an}{q}\right) = c_q(n),$$

we find that $c_q(n)$ is a periodic function of n with period dividing q. To see that the period is at least q, observe that when r is not divisible by q, one has

$$|c_q(r)| = \left|\sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(\frac{ar}{q}\right)\right| \leqslant \left|\phi(q) - 1 + e\left(\frac{r}{q}\right)\right| < \phi(q),$$

whence $\quad \underset{\displaystyle |c_q(q+r)|}{\overset{\|}{|c_q(r)|}} < |c_q(q)| = \phi(q)$. So it is impossible that $c_q(n)$ is a periodic function of n with period r. □

(v) Use the ~~periodic~~ multiplicativity of $c_q(n)$ as a function of q. When $q = p^h$, one has

$$c_{p^h}(n) = \sum_{d\mid (p^h,n)} d\,\mu(p^h/d) = \begin{cases} 0, & \text{when } (p^h,n) = p^t \text{ with } t \leqslant h-2,\\ -p^{h-1}, & \text{when } (p^h,n) = p^{h-1},\\ p^h - p^{h-1}, & \text{when } (p^h,n) = p^h. \end{cases}$$

But

$$\frac{\mu(p^h/(p^h,n))}{\phi(p^h/(p^h,n))}\,\phi(p^h) = \begin{cases} 0, & \text{when } (p^h,n) = p^t, \text{ with } t \leqslant h-2,\\ -\dfrac{\phi(p^h)}{\phi(p)} = -p^{h-1}, & \text{when } (p^h,n) = p^{h-1},\\ \phi(p^h) = p^h - p^{h-1}, & \text{when } (p^h,n) = p^h. \end{cases}$$

④

Thus, for every prime power p^h, we have

$$c_{p^h}(n) = \frac{\mu(p^h/(p^h,n))}{\phi(p^h/(p^h,n))}\phi(p^h),$$

whence

$$c_q(n) = \prod_{p^h \| q} c_{p^h}(n) = \prod_{p^h \| q} \frac{\mu(p^h/(p^h,n))}{\phi(p^h/(p^h,n))}\phi(p^h)$$

$$= \frac{\mu(q/(q,n))}{\phi(q/(q,n))}\phi(q),$$ using multiplicativity of ~~the three~~ functions here.

In particular, one sees ~~that~~

$$|c_q(n)| \leq \frac{\phi(q)}{\phi(q/(q,n))} = \prod_{p^h \| q} \frac{\phi(p^h)}{\phi(p^h/(p^h,n))} \leq \prod_{p^h \| q} (p^h,n) = (q,n).$$

□