MA598AANT ANALYTIC NUMBER THEORY. PROBLEMS 4

TO BE HANDED IN BY FRIDAY 23RD OCTOBER 2020

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

A1. Show that when $\sigma_0 > 1$ and $x > 0$ is not an integer, then

$$\psi(x) = -\lim_{T\to\infty} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\zeta'}{\zeta}(s) \frac{x^s}{s} \, \mathrm{d}s,$$

$$\sum_{1\leqslant n\leqslant x} \mu(n) = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{1}{\zeta(s)} \frac{x^s}{s} \, \mathrm{d}s,$$

$$\sum_{1\leqslant n\leqslant x} \mu(n)^2 = \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\zeta(s)}{\zeta(2s)} \frac{x^s}{s} \, \mathrm{d}s.$$

A2. Suppose that $h(z)$ is analytic in a domain containing the disc $|z| \leqslant R$. Suppose also that $h(0) = 0$, and that $\Re(h(z)) \leqslant M$ for $|z| \leqslant R$. By applying the upper bound

$$\left| \frac{h^{(k)}(0)}{k!} \right| \leqslant \frac{2M}{R^k} \quad (k \geqslant 1),$$

obtained in the course of the proof of the Borel-Carathéodory Lemma, prove that whenever $|z| \leqslant r < R$, one has

$$\left| \frac{h^{(m)}(z)}{m!} \right| \leqslant \frac{2MR}{(R-r)^{m+1}} \quad (m \geqslant 1).$$

B3.(i) Suppose that $f(z)$ is analytic in a domain containing the disc $|z| \leqslant 1$, except for a simple pole at $z = z_0$, where $0 < |z_0| < 1$. Suppose also that $|(z - z_0)f(z)| \leqslant M$ in this disc, and that $f(0) \neq 0$. Let r and R be fixed real numbers with $0 < r < R < 1$. By applying Lemma 10.3 to the function $(z - z_0)f(z)$, or otherwise, show that when $|z| \leqslant r$ and $z \neq z_0$, one has

$$-\frac{f'}{f}(z) = \frac{1}{z - z_0} - \sum_{k=1}^{n} \frac{1}{z - z_k} + O\left(\log\left(\frac{M}{|z_0 f(0)|} \right) \right),$$

where the summation is taken over all zeros $z_1, \ldots, z_n$ of f for which $|z_k| \leqslant R$.
(ii) Show that when $5/6 \leqslant \sigma \leqslant 2$ and $s \neq 1$, then

$$-\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} - \sum_{\rho} \frac{1}{s - \rho} + O(\log(|t| + 4)),$$

where the sum is taken over all zeros ρ of $\zeta(s)$ for which $|\rho - (3/2 + it)| \leqslant 5/6$.

B4. Suppose that $x \geqslant 2$ and $T \geqslant 2$.
(i) Show that when $1 < \sigma \leqslant 2$, one has

$$-\frac{\zeta'}{\zeta}(\sigma) \ll \frac{1}{\sigma - 1},$$

and hence deduce that

$$\frac{4^\sigma + x^\sigma}{T} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\sigma} \ll \frac{(4x)^\sigma}{T(\sigma - 1)}.$$

(ii) Prove that

$$\sum_{x/2<n<2x} \Lambda(n) \min\left\{1, \frac{x}{T|x-n|}\right\} \ll (\log x)\left(1 + \frac{x}{T} \sum_{1\leqslant k\leqslant x} \frac{1}{k}\right).$$

(iii) Use the simplified version of the quantitative form of Perron's formula to show that when $2 \leqslant T \leqslant x$ and $\sigma_0 = 1 + 1/\log x$, one has

$$\psi(x) = -\frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\zeta'}{\zeta}(s) \frac{x^s}{s} \, \mathrm{d}s + O\left(\frac{x}{T}(\log x)^2\right).$$

C5.(i) Let $c(n) = \sum_{d|n} \Lambda(d)\Lambda(n/d)$. Show that when $\sigma > 1$, one has

$$\sum_{n=1}^{\infty} c(n) n^{-s} = \left(\frac{\zeta'}{\zeta}(s)\right)^2.$$

(ii) Prove that when $2 \leqslant T \leqslant x$ and $\sigma_0 = 1 + 1/\log x$, one has

$$\sum_{1\leqslant n\leqslant x} c(n) = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \left(\frac{\zeta'}{\zeta}(s)\right)^2 \frac{x^s}{s} \, \mathrm{d}s + O\left(\frac{x}{T}(\log x)^3\right).$$

(iii) Hence deduce that there is a positive number c for which

$$\sum_{1\leqslant n\leqslant x} \sum_{d|n} \Lambda(d)\Lambda(n/d) = x\log x - (2C_0 + 1)x + O\left(x \exp(-c\sqrt{\log x})\right).$$