Solutions to problem sheet 4.

Q1 (i) When $\sigma > 1$, one has
$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \Lambda(n) n^{-s},$$
whence by Perron's formula, when $x \notin \mathbb{Z}$,
$$\psi(x) = \sum_{1 \le n \le x} \Lambda(n) = -\lim_{T \to \infty} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\zeta'}{\zeta}(s) \frac{x^s}{s}\, ds \qquad (\sigma_0 > 1).$$

(ii) When $\sigma > 1$, one has
$$\frac{1}{\zeta(s)} = \prod_p (1 - p^{-s}) = \sum_{n=1}^{\infty} \mu(n) n^{-s},$$
whence by Perron's formula, when $x \notin \mathbb{Z}$,
$$\sum_{1 \le n \le x} \mu(n) = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{1}{\zeta(s)} \frac{x^s}{s}\, ds \qquad (\sigma_0 > 1).$$

(iii) When $\sigma > 1$, one has
$$\sum_{1 \le n \le x} \mu(n)^2 n^{-s} = \prod_p (1 + p^{-s}) = \prod_p \left(\frac{1 - p^{-2s}}{1 - p^{-s}} \right) = \frac{\zeta(s)}{\zeta(2s)},$$
whence by Perron's formula, when $x \notin \mathbb{Z}$,
$$\sum_{1 \le n \le x} \mu(n)^2 = \lim_{T \to \infty} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\zeta(s)}{\zeta(2s)} \frac{x^s}{s}\, ds \qquad (\sigma_0 > 1).$$

Q2 One has, by examining the Taylor series expansion of $h^{(m)}(z)$,
$$\left| \frac{h^{(m)}(z)}{m!} \right| \leqslant \sum_{k=0}^{\infty} \left| \frac{h^{(k+m)}(0)}{m!\,k!} \right| r^k \le \frac{2M}{R^m} \sum_{k=0}^{\infty} \frac{(k+m)!}{m!\,k!} \left(\frac{r}{R} \right)^k = \frac{2MR}{(R-r)^{m+1}} \qquad (m \ge 1),$$
using $\frac{d^m}{dz^m} \frac{1}{1-z} = \frac{m!}{(1-z)^{m+1}}$

Q3 (i) By applying Lemma 10.3, we obtain
$$\frac{\frac{d}{dz}\big((z - z_0) f(z) \big)}{(z - z_0) f(z)} = \sum_{k=1}^{n} \frac{1}{z - z_k} + O\left(\log\left(\frac{M}{|z_0 f(0)|} \right) \right)$$
$$\Rightarrow \quad \frac{f'(z)}{f(z)} + \frac{1}{z - z_0} = \sum_{k=1}^{n} \frac{1}{z - z_k} + O\left(\log\left(\frac{M}{|z_0 f(0)|} \right) \right),$$
and the desired conclusion follows. □

(ii) We apply part (i), noting that $(s-1)\zeta(s) = 1 + c_0(s-1) + \cdots$ is the power series expansion around $s = 1$. Thus $|(s-1)\zeta(s)| \ll 1$ for $|s-1| \le 1$, ② and, moreover, one has $(s-1)\zeta(s) \ll \tau^2$ for $\delta \le \sigma \le 2$ and all t.

By applying part (i) to the function $f(z) = \zeta(z + (\frac{3}{2}+it))$ with $z_0 = -\frac{1}{2}-it$, $R = 5/6$ and $r = 2/3$, just as in the proof of Lemma 10.5, we obtain

$$-\frac{f'}{f}(z) = \frac{1}{z-z_0} - \sum_{\rho} \frac{1}{(z+\frac{3}{2}+it)-\rho} + O\left(\log\left(\frac{\tau^2}{\tau/|f(0)|}\right)\right)$$

$$\Downarrow$$

$$-\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} - \sum_{\rho} \frac{1}{s-\rho} + O(\log(|t|+4)),$$

all for $\frac{5}{6} \le \sigma \le 2$ and any $t \in \mathbb{R}$. □ A couple of words of explanation are in order. First, in order for part (i) to be applicable, we need $|z_0| < 1$, which restricts $|t| \le \sqrt{3}/2$. When $|t| > \sqrt{3}/2$, the argument of the proof of Lemma 10.5 may be applied, since in such circumstances

$$\frac{1}{s-1} \ll \frac{1}{|t|} \ll \log \tau.$$

<u>Q4</u> (i) When $\sigma > 1$, one has

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{s+1}}\,du = \frac{1}{s-1} + O(1) \quad \text{as } s \to 1,$$

whence
$$\frac{\zeta'}{\zeta}(s) = \frac{-\frac{1}{(s-1)^2} - \int_1^\infty \frac{\{u\}}{u^{s+1}}\,du + s(s+1)\int_1^\infty \frac{\{u\}}{u^{s+2}}\,ds}{\frac{1}{s-1} + O(1)}$$

$$\Rightarrow \quad -\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} + O(1).$$

Then $$-\frac{\zeta'}{\zeta}(\sigma) = \frac{1}{\sigma-1} + O(1) \ll \frac{1}{\sigma-1} \quad \text{for } 1 < \sigma \le 2.$$ □

(ii) We have

$$\sum_{\frac{x}{2}<n<2x} \underbrace{\Lambda(n)}_{\ll \log x} \min\left\{1, \frac{x}{T|x-n|}\right\} \ll (\log x)\left(1 + \frac{x}{T}\sum_{1\le k\le x}\frac{1}{k}\right)$$
$$\ll \log x + \frac{x}{T}(\log x)^2 \ll \frac{x}{T}(\log x)^2.$$ □

when $2 \le T \le x$.

(iii) By the simplified version of the quantitative form of Perron's formula, one has

$$\psi(x) = \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds + R(T),$$

③ where

$$R(T) \ll \sum_{\frac{x}{2}<n<2x} \Lambda(n) \min\left\{1, \frac{x}{T|x-n|}\right\} + \frac{(4x)^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\sigma_0}}$$

$$\ll \frac{x}{T}(\log x)^2 + \frac{(4x)^{1+1/\log x}}{T}\left(-\frac{\zeta'}{\zeta}(1+1/\log x)\right)$$

$$\ll \frac{x}{T}(\log x)^2 + \frac{x}{T}\log x \qquad \text{(using (i) and (ii)).}$$

$$\ll \frac{x}{T}(\log x)^2. \qquad \square$$

Q5 (i) When $\sigma > 1$, the Dirichlet series

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n=1}^{\infty} \Lambda(n) n^{-s}$$

converges absolutely, whence

$$\left(\frac{\zeta'}{\zeta}(s)\right)^2 = \sum_{n=1}^{\infty} \underbrace{\left(\sum_{d|n} \Lambda(d)\Lambda(n/d)\right)}_{c(n)} n^{-s}. \qquad \square$$

(ii) By the simplified version of the ~~quantitative~~ version of Perron's formula, one has

$$\sum_{1\le n\le x} c(n) = \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} \left(\frac{\zeta'}{\zeta}(s)\right)^2 \frac{x^s}{s}\,ds + R(T), \qquad (x \notin \mathbb{Z}) \text{ or } x \in \mathbb{Z}.$$

where

$$R(T) \ll \underset{\substack{\uparrow \\ \text{when } x\in\mathbb{Z}, \\ 0 \text{ otherwise}}}{|c(x)|} + \sum_{\frac{x}{2}<n\le 2x} \underset{\substack{\uparrow \\ \ll (\log x)^2}}{c(n)} \min\left\{1, \frac{x}{T|x-n|}\right\} + \frac{(4x)^{\sigma_0}}{T}\sum_{n=1}^{\infty}\frac{c(n)}{n^{\sigma_0}}.$$

[Note: $\sum_{d|n} \Lambda(d)\Lambda(n/d) \le (\log n)\sum_{d|n}\Lambda(d) \le (\log n)^2$.]

Since $|c(x)| \ll (\log x)^2$ we deduce ~~that~~

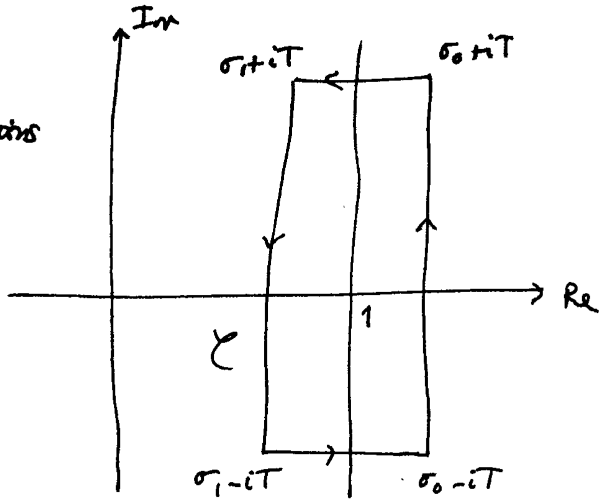
$$R(T) \ll (\log x)^2 + (\log x)^2 \sum_{\frac{x}{2}<n<2x} \min\left\{1, \frac{x}{T|x-n|}\right\} + \frac{x}{T}\left(\frac{\zeta'}{\zeta}(\sigma_0)\right)^2$$

$$\ll (\log x)^3 \frac{x}{T} \quad \text{as in Q4, using } \frac{\zeta'}{\zeta}(1+1/\log x) \ll \log x. \qquad \square$$

(iii) Apply the method of the proof of Theorem 11.1. Take

$$\sigma_1 = 1 - b/\log T,$$

where b is any positive constant for which $\zeta(s)$ has no ~~zeros~~ s with $\sigma > 1 - 2b/\log t$. We have $-\frac{\zeta'}{\zeta}(s) \ll \log T$ on both



horizontal segments and on left hand vertical segment of $\mathcal{C}$. Thus the contributions from these segments to

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \left(-\frac{\zeta'}{\zeta}(s)\right)^2 \frac{x^s}{s}\,ds$$

are (as in proof of Theorem 11.1),

$$\ll (\log T)^2 \cdot \left(\frac{x^{\sigma_0}}{T}(\sigma_0-\sigma_1)\right) + (\log T)^2 \left(x^{\sigma_1}\log T + x^{\sigma_1}/(1-\sigma_1)\right)$$

$$\ll x^{\sigma_1}(\log T)^3 + \frac{x}{T}\log T.$$

Hence, in view of part (ii), we have

$$\sum_{1\le n\le x} c(n) = \frac{1}{2\pi i}\int_{\mathcal{C}} \left(-\frac{\zeta'}{\zeta}(s)\right)^2 \frac{x^s}{s}\,ds + (\log T)^3\left(\frac{x}{T} + x^{1-b/\log T}\right).$$

$$= \frac{1}{2\pi i}\int_{\mathcal{C}} \left(-\frac{\zeta'}{\zeta}(s)\right)^2 \frac{x^s}{s}\,ds + O\left(x\exp\left(-c\sqrt{\log x}\right)\right),$$

some $c>0$, on taking $T = \exp(\sqrt{\log x})$.

The argument of the integral $\left(-\frac{\zeta'}{\zeta}(s)\right)^2 \frac{x^s}{s}$ is analytic inside and on $\mathcal{C}$, except for a double pole at $s=1$. This has residue

$$\lim_{s\to 1} \frac{d}{ds}\left(\underbrace{(s-1)^2 \frac{\zeta'}{\zeta}(s)^2}\ \frac{x^s}{s}\right).$$

But

$$(s-1)\frac{\zeta'}{\zeta}(s) = (s-1)\left(\log\left(\frac{1}{s-1} + c_0 + \cdots\right)\right)' = (s-1)\left(-\log(s-1) + \log(1+c_0(s-1)+\cdots)\right)'$$

$$= (s-1)\left(-\frac{1}{s-1} + c_0 + \cdots\right) = -1 + c_0(s-1) + \cdots,$$

Whence

$$\left((s-1)\frac{\zeta'}{\zeta}(s)\right)^2 = 1 - 2c_0(s-1) + \cdots,$$

and

$$\lim_{s\to 1}\frac{d}{ds}\left((s-1)^2\frac{\zeta'}{\zeta}(s)^2 \frac{e^{s\log x}}{s}\right) = -2c_0\frac{x^s}{s}\Big|_{s=1} + \left((\log x)\frac{x^s}{s} - \frac{x^s}{s^2}\right)\Big|_{s=1}$$

$$= x\log x - (2c_0+1)x.$$

Thus

$$\sum_{1\le n\le x}\sum_{d\mid n}\Lambda(d)\Lambda(n/d) = x\log x - (2c_0+1)x + O\left(x\exp\left(-c\sqrt{\log x}\right)\right).$$

$\square$