MA598AANT ANALYTIC NUMBER THEORY. PROBLEMS 5

TO BE HANDED IN BY FRIDAY 6TH NOVEMBER 2020

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

A1. Suppose that $\theta > 1/\sqrt{e}$.
(i) Show that when x is large, one has $\psi(x, x^\theta) > x/2$.
(ii) Let N be a large natural number. Deduce that

$$\text{card}\{n \in [1, N] : N - n \in S(N, N^\theta)\} > N/2.$$

(iii) Conclude that for some $n \in S(N, N^\theta)$, one has $N - n \in S(N, N^\theta)$, whence N is the sum of two N^θ-smooth numbers.

A2. Apply the Prime Number Theorem (with error term) to show that there is a positive number c with the property that, whenever $y < x$ and y is large,

$$\sum_{y<p\leqslant x} \frac{1}{p} = \log\left(\frac{\log x}{\log y}\right) + O(\exp(-c\sqrt{\log y})).$$

B3. Let $p_1 < p_2 < \dots$ be the prime numbers in order.
(i) Apply the Prime Number Theorem (with error term) to show that when n is large,

$$p_n < n(\log n + \log\log n).$$

(ii) Apply the Prime Number Theorem (with error term) to show that when n is large,

$$p_n > n(\log n + \log\log n - 1).$$

B4. Let $\pi_2(x)$ denote the number of integers not exceeding x that are the product of precisely two distinct prime numbers.
(i) Show that

$$\pi_2(x) = \sum_{p\leqslant\sqrt{x}} \pi(x/p) + O\left(\frac{x}{(\log x)^2}\right).$$

(ii) Deduce that

$$\pi_2(x) = \sum_{p\leqslant\sqrt{x}} \frac{x}{p\log(x/p)} + O\left(\frac{x\log\log x}{(\log x)^2}\right).$$

(iii) Apply Riemann-Stieltjes integration to establish that

$$\pi_2(x) = \frac{x\log\log x}{\log x} + O\left(\frac{x}{\log x}\right).$$

C5. In this question, you may assume any results from the course.

(i) Show that there is a positive number c with the property that

$$\sum_{1\leqslant n\leqslant x} \mu(n) \ll x\exp(-c\sqrt{\log x}).$$

(ii) Show that there is a positive number c with the property that

$$\sum_{1\leqslant n\leqslant x} \mu(n)n^{-1} \ll \exp(-c\sqrt{\log x}).$$

(iii) Show that for any fixed real number t with $t \neq 0$, one has

$$\sum_{n=1}^{\infty} \mu(n)n^{-1-it} = \frac{1}{\zeta(1+it)}.$$