Solutions to problem sheet 5.

Q1 (i) By Dickman's theorem, one has $\psi(x,x^\theta) = \rho(\frac{1}{\theta})x + O(\frac{x}{\log x})$, where $\rho(\frac{1}{\theta}) = 1 - \log(\frac{1}{\theta}) > 1 - \log(\sqrt{e}) = \frac{1}{2}$ (since $\theta > 1/\sqrt{e}$ and $\rho(u)$ decreasing). Thus, when x is large enough, one has $\psi(x, x^\theta) > \frac{1}{2}x$. □

(ii) $\operatorname{card}\{n \in [1,N] : N-n \in S(N,N^\theta)\} = \operatorname{card}\{m \in [0,N-1] : m \in S(N,N^\theta)\}$

$> N/2$ for N large. □

(iii) More than $N/2$ of the integers n with $1 \le n \le N-1$ are N^θ-smooth. By the Pigeon-hole principle, any set having $N/2$ integers lying in $1 \le n \le N-1$ must therefore contain an N^θ-smooth integer. But $\{N-n : n \in S(N,N^\theta)\}$ has $> N/2$ elements, so contains an N^θ-smooth integer. Then N is the sum of two N^θ-smooth integers. □

Q2 By applying Riemann-Stieltjes integration and the Prime Number Theorem in the form $\theta(x) = x + O(x\exp(-2c\sqrt{\log x}))$ (suitable $c>0$), we obtain

$$\sum_{y<p\le x} \frac{1}{p} = \int_x^y \frac{d\,\theta(u)}{u\log u} = \int_y^x \frac{du}{u\log u} + \int_y^x \frac{d(\theta(u)-u)}{u\log u}$$

$$= \Big[\log\log u\Big]_y^x + \Big[\frac{\theta(u)-u}{u\log u}\Big]_y^x + \int_y^x \frac{(\theta(u)-u)(1+\log u)}{(u\log u)^2}\,du$$

$$= \log\log x - \log\log y + O(\exp(-2c\sqrt{\log y})) + O\Big(\int_y^x \frac{du}{u(\log u)^2}\cdot\exp(-c\sqrt{\log y})\Big)$$

$$= \log\Big(\frac{\log x}{\log y}\Big) + O\big(\exp(-c\sqrt{\log y})\big). \;\square$$

Q3 (i) We aim to show that $\pi(n(\log n + \log\log n)) > n$, whence $p_n < n(\log n + \log\log n)$. To verify this claim, note that when x is large, there is a $c>0$ such that

$$\pi(x) = li(x) + O(x\exp(-c\sqrt{\log x}))$$
$$> \frac{x}{\log x} + \frac{x}{(\log x)^2}.$$

Thus, when n is large,

$$\pi(n(\log n + \log\log n)) > \frac{n(\log n + \log\log n)}{\log n + \log\log n + \log(1+\frac{\log\log n}{\log n})} + \frac{n(\log n + \log\log n)}{\left(\log n + \log\log n + \log(1+\frac{\log\log n}{\log n})\right)^2}$$

But $\log\left(1 + \frac{\log\log n}{\log n}\right) < \frac{\log\log n}{\log n}$, whence

$$\pi(n(\log n + \log\log n)) > n\left(1 + \frac{\log\log n}{(\log n)^2}\right)^{-1} + \frac{n}{2\log n}$$

$$> n\left(1 - \frac{2\log\log n}{(\log n)^2}\right) + \frac{n}{2\log n} > n. \quad \square$$

(ii) Similarly, to show that $\pi(n(\log n + \log\log n - 1)) < n$, whence $p_n > n(\log n + \log\log n - 1)$, we apply the bound

$$\pi(x) < \frac{x}{\log x} + \frac{x}{(\log x)^2} + \frac{3x}{(\log x)^3} \quad (x \text{ large}).$$

Thus, when n is large,

$$\pi(n(\log n + \log\log n - 1)) < \frac{n(\log n + \log\log n - 1)}{\log n + \log\log n + \log\left(1 + \frac{\log\log n - 1}{\log n}\right)}$$
$$+ \frac{n(\log n + \log\log n - 1)}{\left(\log n + \log\log n + \log\left(1 + \frac{\log\log n - 1}{\log n}\right)\right)^2} + \frac{6n\log n}{(\log n)^3}.$$

But $\log\left(1 + \frac{\log\log n - 1}{\log n}\right) > 0$ for large n, so

$$\pi(n(\log n + \log\log n - 1)) < n\left(1 + \frac{\log\left(1 + \frac{\log\log n - 1}{\log n}\right)}{\log n + \log\log n}\right)^{-1} - \frac{n}{\log n + \log\log n + \log\left(1 + \frac{\log\log n - 1}{\log n}\right)}$$
$$+ \frac{n}{\log n + \log\log n + \log\left(1 + \frac{\log\log n - 1}{\log n}\right)} + \frac{6n}{(\log n)^2}$$
$$< n\left(1 + \frac{1}{2}\frac{\log\log n}{(\log n)^2}\right)^{-1} + \frac{6n}{(\log n)^2}$$
$$< n\left(1 - \frac{1}{4}\frac{\log\log n}{(\log n)^2}\right) < n. \quad \square$$

Q4 (i) We have

$$\pi_2(x) = \sum_{\substack{p_1 < p_2 \le x \\ p_1 p_2 \le x}} 1 = \sum_{\substack{p_1 < p_2 \le x/p_1 \\ p_1 \le \sqrt{x}}} 1 = \sum_{p_1 \le \sqrt{x}} \left(\pi(x/p_1) - \pi(p_1)\right)$$
$$= \sum_{p \le \sqrt{x}} \pi(x/p) + O\left(\sum_{p \le \sqrt{x}} \frac{\sqrt{x}}{\log x}\right)$$
$$= \sum_{p \le \sqrt{x}} \pi(x/p) + O\left(\frac{x}{(\log x)^2}\right). \quad \square$$

(ii) Apply the Prime Number Theorem in the shape $\pi(y) = \frac{y}{\log y} + O\left(\frac{y}{(\log y)^2}\right)$ to obtain

$$\sum_{p \le \sqrt{x}} \pi(x/p) = \sum_{p \le \sqrt{x}} \left(\frac{x/p}{\log(x/p)} + O\left(\frac{x/p}{(\log(x/p))^2}\right)\right) = \sum_{p \le \sqrt{x}} \frac{x/p}{\log(x/p)} + O\left(\frac{x}{(\log x)^2} \sum_{p \le \sqrt{x}} \frac{1}{p}\right).$$

Since (Mertens' theorem) $\sum_{p \le \sqrt{x}} \frac{1}{p} \ll \log\log \sqrt{x}$, we conclude that

$$\sum_{p \le \sqrt{x}} \pi(x/p) = \sum_{p \le \sqrt{x}} \frac{x/p}{\log(x/p)} + O\left(\frac{x \log\log x}{(\log x)^2}\right), \text{ whence}$$

$$\pi_2(x) = \sum_{p \le \sqrt{x}} \frac{x}{p \log(x/p)} + O\left(\frac{x \log\log x}{(\log x)^2}\right). \quad \square$$

(iii) Apply the Prime Number Theorem in the shape $\theta(u) = u + O(u \exp(-c\sqrt{\log u}))$, and Riemann–Stieltjes integration. We obtain for a suitable $c > 0$,

$$\sum_{p \le \sqrt{x}} \frac{1}{p \log(x/p)} = \int_{2^-}^{\sqrt{x}} \frac{d\,\theta(u)}{u \log u \log(x/u)} = \int_{2^-}^{\sqrt{x}} \frac{1}{\log x}\left(\frac{1}{u \log u} + \frac{1}{u \log(x/u)}\right) du$$

$$+ \int_{2^-}^{\sqrt{x}} \frac{d\,(\theta(u) - u)}{u \log u \log(x/u)}$$

$$= \frac{1}{\log x}\Big[\log\log u - \log\log(x/u)\Big]_2^{\sqrt{x}} + \left[\frac{\theta(u) - u}{u \log u \log(x/u)}\right]_2^{\sqrt{x}}$$

$$+ O\left(\int_{2^-}^{\sqrt{x}} \frac{u \exp(-c\sqrt{\log u})}{\log x \cdot u}\, du\right)$$

$$= \frac{\log\log x}{\log x} + O\left(\frac{1}{\log x}\right).$$

Thus $\pi_2(x) = x \frac{\log\log x}{\log x} + O\left(\frac{x}{\log x}\right)$ (from part (ii)). $\square$

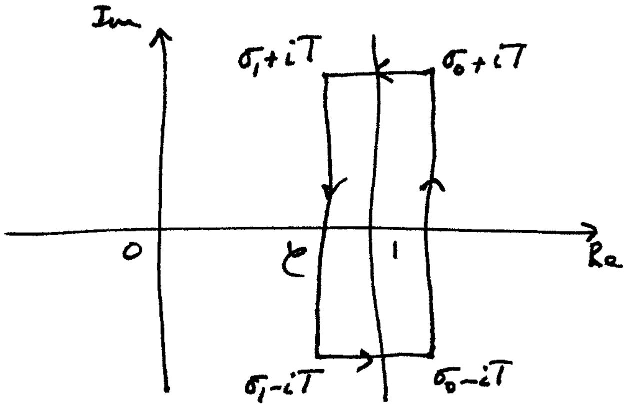
Q5 (i) Apply the method of the proof of Theorem 11.1, noting that with

$$\sigma_1 = 1 - b/\log T,$$

where b is any positive constant for which $\zeta(s)$ has no ~~zeros~~ s with $\sigma > 1 - 2b/\log t$, then we have $\frac{1}{\zeta(s)} \ll \log T$ on both horizontal segments and on left hand vertical segment of $\mathcal{C}$ (overleaf). Thus the contributions from these segments to

$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{1}{\zeta(s)}\frac{x^s}{s}\,ds$ are, as in the proof of Theorem 11·1,

$$\ll x^{\sigma_1}(\log T)^2 + \frac{x}{T}\log T.$$



$\sigma_0 = 1 + 1/\log x$

$\sigma_1 = 1 - b/\log T.$

Hence

$$\sum_{1\le n\le x}\mu(n) = \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{1}{\zeta(s)}\frac{x^s}{s}\,ds + O\left((\log T)^2\left(\frac{x}{T} + x^{1-b/\log T}\right)\right)$$

$$= \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{1}{\zeta(s)}\frac{x^s}{s}\,ds + O\left(x\exp\left(-c\sqrt{\log x}\right)\right),$$

some $c>0$, on taking $T = \exp(\sqrt{\log x})$.

The argument of the integral $\frac{1}{\zeta(s)}\frac{x^s}{s}$ is analytic inside and on $\mathcal{C}$ (no poles!). Hence $\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{1}{\zeta(s)}\frac{x^s}{s}\,ds = 0$, whence

$$\sum_{1\le n\le x}\mu(n) \ll x\exp\left(-c\sqrt{\log x}\right). \qquad \square$$

(ii) and (iii). The above arguments applies, with some modification, in case we wish to ~~estimate~~

$$\sum_{1\le n\le x}\mu(n)n^{-1-it} \qquad (t\in\mathbb{R}, \text{ including } t=0).$$

We observe that

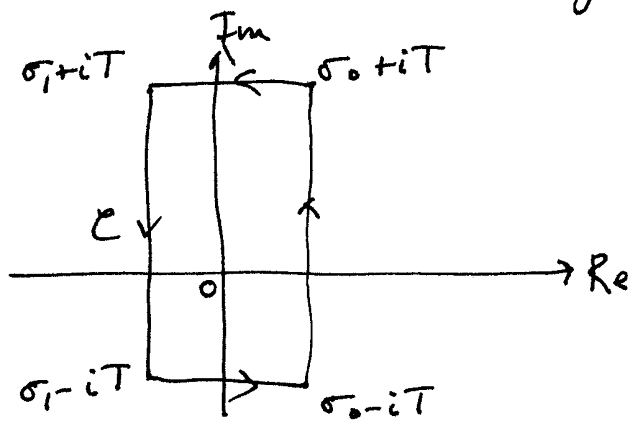
$$\frac{1}{\zeta(s+1+it')} = \sum_{n=1}^{\infty}\left(\mu(n)n^{-1-it'}\right)n^{-s}$$

is absolutely convergent for $\sigma>0$. We may therefore apply the method of proof of Theorem 11·1, noting that with

$$\sigma_1 = -b/\log T,$$

where $b = b(t')$ is any positive constant for which $\zeta(s+1+it')$ has no zeros with $\sigma > -2b/\log t$, then we have $1/\zeta(s+1+it') \ll \log T$

on both horizontal segments and on the left hand vertical segment of $\mathcal{C}$ (overleaf). Thus the contributions from these segments to $\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{1}{\zeta(s+1+it')}\frac{x^s}{s}\,ds$ are, as in the proof of Theorem 11·1,



$$\sigma_0 = 1/\log x$$
$$\sigma_1 = -b/\log T$$

$$\ll \frac{x^{\sigma_0}}{T}(\sigma_0-\sigma_1)\cdot\log T + \log T \cdot x^{\sigma_1}\int_{-T}^{T}\frac{dt}{\frac{1}{\log T}+|t|} \ll x^{\sigma_1}(\log T)^3 + \frac{x^{\sigma_0}}{T}.$$

Hence

$$\sum_{1\le n\le x}\mu(n)\,n^{-1-it'} = \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{1}{\zeta(s+1+it')}\frac{x^s}{s}\,ds + O\left((\log T)^3\left(\frac{1}{T}+x^{-b/\log T}\right)\right)$$
$$= \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{1}{\zeta(s+1+it')}\frac{x^s}{s}\,ds + O\left(\exp(-c\sqrt{\log x})\right),$$

some $c>0$, on taking $T=\exp(\sqrt{\log x})$.

The argument of the integral $\frac{1}{\zeta(s+1)}\frac{x^s}{s}$ is analytic inside and on $\mathcal{C}$, (except for checking at $s=0$ that $s\zeta(s+1)$ has no zero on account of the simple pole of $\zeta(s)$ at $s=1$. ✓). Thus the integral here is 0 when $t'=0$, and one concludes for (ii) that

$$\sum_{1\le n\le x}\mu(n)\,n^{-1} \ll \exp(-c\sqrt{\log x}).$$

When $t'\ne 0$, the argument of the integral $\frac{1}{\zeta(s+1+it')}\frac{x^s}{s}$ is analytic inside and on $\mathcal{C}$, except for a simple pole at $s=0$ with residue $\frac{1}{\zeta(1+it')}$. Thus $\sum_{1\le n\le x}\mu(n)\,n^{-1-it'} = \frac{1}{\zeta(1+it')} + O(\exp(-c\sqrt{\log x}))$.

The conclusion $\sum_{n=1}^{\infty}\mu(n)\,n^{-1-it} = \frac{1}{\zeta(1+it)}$ follows on taking $x\to\infty$. //