MA598AANT ANALYTIC NUMBER THEORY. PROBLEMS 6

TO BE HANDED IN BY FRIDAY 4TH DECEMBER 2020

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

A1. By making use of the functional equation for the Riemann zeta function, show that
\[ \zeta(1 - s) = \zeta(s)2^{1-s}\pi^{-s}\Gamma(s)\cos(\pi s/2). \]

A2. Show that when \( k \in \mathbb{N} \), one has
\[ \zeta'(-2k) = (-1)^k(2k)!\zeta(2k+1)
\]
\[ 2^{2k+1}\pi^{2k}. \]

B3. Show that there is a positive constant \( c \) having the following property. Suppose that \( a \in \mathbb{Z} \) and \( q \in \mathbb{N} \) satisfy \((a, q) = 1\). Then:
(i) when there is no exceptional character modulo \( q \), then
\[ \pi(x; q, a) = \text{li}(x) + O(x \exp(-c\sqrt{\log x})); \]
(ii) when there is an exceptional character \( \chi_1 \) modulo \( q \), and \( \beta_1 \) is the associated exceptional zero of \( L(s, \chi_1) \), then
\[ \pi(x; q, a) = \text{li}(x) - \frac{\chi_1(a)\text{li}(x^{\beta_1})}{\phi(q)} + O(x \exp(-c\sqrt{\log x})). \]

B4. Recall from Landau’s theorem that there is a positive constant \( c \) with the following property. Whenever \( \chi_i \) is a quadratic character modulo \( q_i \) for \( i = 1, 2 \), and \( \chi_1\chi_2 \) is non-principal, then \( L(s, \chi_1)L(s, \chi_2) \) has at most one real zero \( \beta \) such that \( 1 - \beta < c/\log(q_1q_2) \).
(i) Suppose that \( A > 2 \). Show that if \( L(s, \chi_i) \) has a zero \( \beta_i \) satisfying
\[ 1 - \beta_i < \frac{c}{A\log q_i}, \]
for \( i = 1 \) and \( i = 2 \), then either \( q_2 > q_1^{A-1} \) or \( q_1 > q_2^{A-1} \).
(ii) Deduce that if \( (q_i)_{i=1}^{\infty} \) is a strictly increasing sequence of natural numbers having the property that for each \( i \), there is a primitive quadratic character \( \chi_i \) modulo \( q_i \) for which \( L(s, \chi_i) \) has an exceptional real zero \( \beta_i \) with
\[ 1 - \beta_i < \frac{c}{A\log q_i}, \]
then \( q_{i+1} > q_i^{A-1} \) for each \( i \). [Hint: You may assume that \( \chi_i\chi_{i+1} \) is non-principal for each \( i \).]
(iii) Show that when \( Q \) is large, there are at most \( O(\log \log Q) \) moduli \( q \), with \( q \leq Q \), having the property that there is a primitive character \( \chi \) modulo \( q \) for which \( L(s, \chi) \) has
an exceptional real zero $\beta$ with 

$$1 - \beta < \frac{c}{3 \log q}.$$ 

**C5.** Let $r(n)$ denote the number of representations of the integer $n$ as the sum of a prime and a $k$-free integer (i.e. an integer with the property that $p^k \nmid n$ for all primes $p$). We suppose throughout that $k \geq 2$.

(i) Show that

$$r(n) = \sum_{d < n} \sum_{d^k \mid (n-p)} \mu(d).$$

(ii) Deduce that

$$r(n) = \sum_{d \leq n^{1/k}} \mu(d) \pi(n - 1; d^k, n).$$

(iii) By noting that the contribution in this sum from those integers $d$ with $(d, n) > 1$ is $O(n^{1/k})$, show that

$$r(n) = r_1(n) + r_2(n) + O(n^{1/k}),$$

where

$$r_1(n) = \sum_{1 \leq d \leq \log n \leq 2020} \mu(d) \pi(n - 1; d^k, n)$$

and

$$r_2(n) = \sum_{3020 < d \leq n^{1/k}} \mu(d) \pi(n - 1; d^k, n).$$

(iv) Apply the Siegel-Walfisz theorem to show that there is a constant $c > 0$ for which

$$r_1(n) = \text{li}(n) \sum_{1 \leq d \leq \log n \leq 2020} \frac{\mu(d)}{d^{k-1} \phi(d)} + O(n \exp(-c \sqrt{\log n})).$$

(v) By completing the sum in (iv) and applying multiplicativity, deduce that

$$r_1(n) = \text{li}(n) \prod_{(p, n) = 1} \left(1 - \frac{1}{p^{k-1}(p - 1)}\right) + O(n (\log n)^{-2020}).$$

(vi) Show that $r_2(n) \ll n (\log n)^{-2020}$, and hence conclude that

$$r(n) = c(n) \text{li}(n) + O(n (\log n)^{-2020}),$$

where

$$c(n) = \left(\prod_{p \nmid n} \left(1 + \frac{1}{p^k - p^{k-1} - 1}\right)\right) \left(\prod_p \left(1 - \frac{1}{p^{k-1}(p - 1)}\right)\right).$$

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