MA598AANT ANALYTIC NUMBER THEORY. PROBLEMS 6

TO BE HANDED IN BY FRIDAY 4TH DECEMBER 2020

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

A1. By making use of the functional equation for the Riemann zeta function, show that

$$\zeta(1-s) = \zeta(s)2^{1-s}\pi^{-s}\Gamma(s)\cos(\pi s/2).$$

A2. Show that when $k \in \mathbb{N}$, one has

$$\zeta'(-2k) = \frac{(-1)^k(2k)!\zeta(2k+1)}{2^{2k+1}\pi^{2k}}.$$

B3. Show that there is a positive constant c having the following property. Suppose that $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$. Then:
(i) when there is no exceptional character modulo q, then

$$\pi(x; q, a) = \frac{\mathrm{li}(x)}{\phi(q)} + O(x\exp(-c\sqrt{\log x}));$$

(ii) when there is an exceptional character χ_1 modulo q, and β_1 is the associated exceptional zero of $L(s, \chi_1)$, then

$$\pi(x; q, a) = \frac{\mathrm{li}(x)}{\phi(q)} - \frac{\chi_1(a)\mathrm{li}(x^{\beta_1})}{\phi(q)} + O(x\exp(-c\sqrt{\log x})).$$

B4. Recall from Landau's theorem that there is a positive constant c with the following property. Whenever χ_i is a quadratic character modulo q_i for $i = 1, 2$, and $\chi_1\chi_2$ is non-principal, then $L(s, \chi_1)L(s, \chi_2)$ has at most one real zero β such that $1-\beta < c/\log(q_1q_2)$.
(i) Suppose that $A > 2$. Show that if $L(s, \chi_i)$ has a zero β_i satisfying

$$1 - \beta_i < \frac{c}{A\log q_i},$$

for $i = 1$ and $i = 2$, then either $q_2 > q_1^{A-1}$ or $q_1 > q_2^{A-1}$.
(ii) Deduce that if $(q_i)_{i=1}^{\infty}$ is a strictly increasing sequence of natural numbers having the property that for each i, there is a primitive quadratic character χ_i modulo q_i for which $L(s, \chi_i)$ has an exceptional real zero β_i with

$$1 - \beta_i < \frac{c}{A\log q_i},$$

then $q_{i+1} > q_i^{A-1}$ for each i.
[Hint: You may assume that $\chi_i\chi_{i+1}$ is non-principal for each i.]
(iii) Show that when Q is large, there are at most $O(\log\log Q)$ moduli q, with $q \leqslant Q$, having the property that there is a primitive character χ modulo q for which $L(s, \chi)$ has

an exceptional real zero β with
$$1-\beta<\frac{c}{3\log q}.$$

C5. Let $r(n)$ denote the number of representations of the integer n as the sum of a prime and a k-free integer (i.e. an integer with the property that $p^k\nmid n$ for all primes p). We suppose throughout that $k\geqslant 2$.

(i) Show that
$$r(n)=\sum_{p<n}\sum_{d^k|(n-p)}\mu(d).$$

(ii) Deduce that
$$r(n)=\sum_{d\leqslant n^{1/k}}\mu(d)\pi(n-1;d^k,n).$$

(iii) By noting that the contribution in this sum from those integers d with $(d,n)>1$ is $O(n^{1/k})$, show that
$$r(n)=r_1(n)+r_2(n)+O(n^{1/k}),$$
where
$$r_1(n)=\sum_{\substack{1\leqslant d\leqslant(\log n)^{2020}\\(d,n)=1}}\mu(d)\pi(n-1;d^k,n)$$
and
$$r_2(n)=\sum_{\substack{(\log n)^{2020}<d\leqslant n^{1/k}\\(d,n)=1}}\mu(d)\pi(n-1;d^k,n).$$

(iv) Apply the Siegel-Walfisz theorem to show that there is a constant $c>0$ for which
$$r_1(n)=\mathrm{li}(n)\sum_{\substack{1\leqslant d\leqslant(\log n)^{2020}\\(d,n)=1}}\frac{\mu(d)}{d^{k-1}\phi(d)}+O(n\exp(-c\sqrt{\log n})).$$

(v) By completing the sum in (iv) and applying multiplicativity, deduce that
$$r_1(n)=\mathrm{li}(n)\prod_{(p,n)=1}\left(1-\frac{1}{p^{k-1}(p-1)}\right)+O(n(\log n)^{-2020}).$$

(vi) Show that $r_2(n)\ll n(\log n)^{-2020}$, and hence conclude that
$$r(n)=c(n)\mathrm{li}(n)+O(n(\log n)^{-2020}),$$
where
$$c(n)=\left(\prod_{p|n}\left(1+\frac{1}{p^k-p^{k-1}-1}\right)\right)\left(\prod_{p}\left(1-\frac{1}{p^{k-1}(p-1)}\right)\right).$$