All we showed in Corollary 13.4 that \( \pi(s) = 5(1-s) \cdot 2^s \pi^{s-1} \Gamma(1-s) \sin \left( \frac{\pi s}{2} \right) \).

By substituting \( 1-s \) for \( s \), we deduce that 
\[ \pi(1-s) = 5(s) \cdot 2^{1-s} \pi^{-s} \Gamma(s) \sin \left( \frac{\pi s}{2} - \frac{\pi}{2} \right) = 5(s) \cdot 2^{1-s} \pi^{-s} \Gamma(s) \cos \left( \frac{\pi s}{2} \right). \]

Q3) By differentiating the relation from Q1, we obtain 
\[ -5'(1-s) = \frac{d}{ds} \left( 5(s) \cdot 2^{1-s} \pi^{-s} \Gamma(s) \cos \left( \frac{\pi s}{2} \right) \right) = 5(s) \cdot 2^{1-s} \pi^{-s} \Gamma(s) \pi \sin \left( \frac{\pi s}{2} \right). \]

Then, for \( k \in \mathbb{N} \) and \( s = 2k+1 \), we deduce that 
\[ -5'(1-2k) = 5(2k+1) \cdot 2^{-2k} \pi^{-2k-1} \Gamma(2k+1) \pi \cdot (-1)^k = \frac{(-1)^k \cdot (2k)! \cdot 5(2k+1)}{2^{2k+1} \pi^{2k}}. \]

Q3) We apply Theorem 15.5 due to Page, noting that from QA1 from Problem Sheet 3, 
\[ \Theta(x;q,a) = \psi(x;q,a) + O(x^{1/2}). \]

(i) When there is no exceptional zero, we have 
\[ \psi(x; q,a) = \frac{x}{\phi(q)} + O(x \exp(-c\sqrt{\log x})); \]
\[ \Theta(x; q,a) = \frac{x}{\phi(q)} + O(x \exp(-c\sqrt{\log x})). \]

Then by Riemann–Stieltjes integration 
\[ \pi(x; q,a) = \int_{1}^{x} \frac{1}{\log u} \, d \Theta(u; q,a) \]
\[ = \frac{1}{\phi(q)} \int_{1}^{x} \frac{du}{\log u} + \int_{1}^{x} \frac{d}{\log u} \left( \Theta(u; q,a) - \frac{u}{\phi(q)} \right) \]
\[ = \frac{\psi(x)}{\phi(q)} + \left( \frac{\Theta(x; q,a) - u/\phi(q)}{\log u} \right) + \int_{1}^{x} \frac{\Theta(u; q,a) - u/\phi(q)}{u(\log u)^2} \, du \]
\[ = \frac{\psi(x)}{\phi(q)} + O \left( x \exp \left( -c\sqrt{\log x} \right) \right) + O \left( x \exp \left( -c\sqrt{\log x} \right) \right). \]

Thus 
\[ \pi(x; q,a) = \frac{\psi(x)}{\phi(q)} + O \left( x \exp \left( -c\sqrt{\log x} \right) \right). \]

(ii) When there is an exceptional character \( \chi \) modulo \( q \) and \( \beta \) is the associated exceptional zero of \( L(s, \chi) \), then 
\[ \psi(x; q,a) = \frac{x}{\phi(q)} - \frac{\chi(a) x^{\beta}}{\phi(q) \beta} + O(x \exp(-c\sqrt{\log x})). \]
\[ \Theta(x; q,a) = \frac{x}{\phi(q)} - \frac{\chi(a) x^{\beta}}{\phi(q) \beta} + O(x \exp(-c\sqrt{\log x})). \]

Then by Riemann–Stieltjes integration 
\[ \pi(x; q,a) = \int_{1}^{x} \frac{1}{\log u} \, d \Theta(x; q,a) \]
\[\frac{1}{\log n} \sum_{\substack{\rho \in \rho(n) \cap [1, \sqrt{n}]}} \frac{\text{Res}_{\gamma}(\zeta(s))}{\gamma(s)} = \sum_{\substack{\rho \in \rho(n) \cap [1, \sqrt{n}]}} \frac{\text{Res}_{\gamma}(\zeta(s))}{\gamma(s)} \]

The last term may be handled as in part (ii), and the second via the substitution

\[v = u^{\beta_1}. \text{ Thus we deduce that} \]

\[\pi(x; 9, 1) = \frac{\pi(x; 9, 1)}{\varphi(q)} = \frac{\pi(x; 9, 1)}{\varphi(q)} + O(x \exp(-c \sqrt{\log x})). \]

(i) Suppose that \(L(s, \chi)\) has a zero \(\beta_i\) satisfying \(1 - \beta_i < c / \log(q)q^2\), then \(1 - \beta_i < c / \log(q)q^2\). Thus \(x \approx \log x\). Whence \(\log(q)q^2 > A \log q \Rightarrow q > q_{2A}^{-1}\).

If instead \(1 - \beta_i < c / \log(q)q^2\) then similarly \(q > q_{2A}^{-1}\). Further, if \(1 - \beta_i < c / \log(q)q^2\), then we deduce similarly that \(q > q_{2A}^{-1}\) and \(q > q_{2A}^{-1}\).

So either \(q > q_{2A}^{-1}\) or \(q > q_{2A}^{-1}\) (or both).

(ii) Suppose that \((q_i)_{i=1}^\infty\) is a strictly increasing sequence of natural numbers with the stated property. Since for each \(i\) one has \(\chi; \chi + i\) non-principal, it follows from part (i) that \(q_i > q_{2A}^{-1}\) or \(q_{i+1} > q_{2A}^{-1}\). Since \(A > 2\), we see that the latter must hold.

(iii) Put \(A = 3\), and suppose that \(\chi^\prime\) is a primitive character modulo \(q\) for which \(L(s, \chi)\) has an exceptional real zero \(\beta\) with \(1 - \beta < c / 3 \log q\). Then \(\chi^\prime\) is another primitive character modulo \(q^\prime\) for which \(L(s, \chi^\prime)\) has an exceptional real zero \(\beta\) with \(1 - \beta < c / 3 \log q^\prime\), we deduce from (ii) that when \(q > q_{2A}^{-1}\), then \(q > q_{2A}^{-1}\). [Notice that \(\chi \chi^\prime\) is non-principal owing to the primality of \(\chi\) (mod \(q\)) and \(\chi^\prime\) (mod \(q^\prime\)).] Let \(2 \leq q < q_{2 \cdots 2} \leq q_{2 \cdots 2} \leq q_{n} \leq q\). Be the moduli associated with primitive characters \(\chi_1, \ldots, \chi_n\) modulo \(q_1, \ldots, q_n\), respectively, for which \(L(s, \chi_j)\) has an exceptional real zero \(\beta_j\) with \(1 - \beta_j < c / 3 \log q_j \leq q_{n} \leq q\). Then we have

\[q_2 > q_{2A}^{-1} > \ldots > q_{2A}^{-1} \geq 2^{-A} \leq q_{2A}^{-1}, \text{whence we deduce that if } q_{n} \leq q_{2A}^{-1}, \text{then}\]

\[2^{n-1} \leq q_{2A}^{-1} \leq n < \log n. \]

So there are at most \(O \left( \log \log n \right)\) such exceptional moduli.

(i) We have \(r(n) = \# \{n = p + x: p \text{ prime, } x \text{ k-free} \}
\]

\[= \sum_{p \leq n} \sum_{d \mid (n-p)} \mu(d) \sum_{k \leq n-p} = \sum_{p \leq n} \sum_{d \mid (n-p)} \mu(d) \sum_{k \leq n-p} \mu(k)\]
Here we use that if $D \geq 1$ and $D^k$ is the largest $k$-th power dividing $n - p$, then
\[ \sum_{d | (n - p)^k} \mu(d) = \sum_{d | D^k} \mu(d) = \begin{cases} 1, & d = 1 \\ 0, & d > 1 \end{cases}. \]

(ii) Thus
\[ r(n) = \sum_{d \leq n^{1/k}} \mu(d) \sum_{p < n \atop n - p \equiv 0 \pmod{d^k}} 1 = \sum_{d \leq n^{1/k}} \mu(d) \pi(n - 1; d^k, n). \]

(iii) When $(d, n) > 1$, any prime $p$ counted by $\pi(n - 1; d^k, n)$ satisfies $p \equiv n \pmod{d^k}$, whence $(n, d) | p$, so that $p = (n, d)$.
Thus the contribution to the sum from those terms with $(d, n) > 1$ is at most \[ \sum_{d \leq n^{1/k}} |\mu(d)| \leq n^{1/k}. \] Hence
\[ r(n) = \sum_{1 \leq d \leq (\log n)^{2020} \atop (d, n) = 1} \mu(d) \pi(n - 1; d^k, n) + \sum_{(\log n)^{2020} \leq d \leq n^{1/k} \atop (d, n) = 1} \mu(d) \pi(n - 1; d^k, n) + o(n^{1/k}). \]

(iv) By the Siegel–Walfisz theorem, when $d \leq (\log n)^{2020}$, one has $d^k \leq (\log n)^{2020k}$, and hence
\[ \pi(n - 1; d^k, n) = \frac{\text{li}(n - 1)}{d^{k-1} \phi(d)} + O_k \left( n \exp (-c n^{1/k} \log n) \right), \]
suitable $c > 0$.
Thus
\[ r_1(n) = \text{li}(n) \sum_{1 \leq d \leq (\log n)^{2020} \atop (d, n) = 1} \frac{\mu(d)}{d^{k-1} \phi(d)} + O_k \left( n \exp (-c n^{1/k} \log n) \right). \]

(v) One has \[ \sum_{d > (\log n)^{2020}} \left| \frac{\mu(d)}{d^{k-1} \phi(d)} \right| \ll \sum_{d > (\log n)^{2020}} \frac{1}{d^{k-1/2}} \ll (\log n)^{-2020}. \]
Thus, using multiplicativity,
\[ \sum_{1 \leq d \leq (\log n)^{2020} \atop (d, n) = 1} \frac{\mu(d)}{d^{k-1} \phi(d)} = \sum_{d = 1}^{\infty} \frac{\mu(d)}{d^{k-1} \phi(d)} + O \left( (\log n)^{-2020} \right) \]
Thus, from part (iv),
\[ r_1(n) = li(n) \prod_{(p,n)=1} \left( 1 - \frac{1}{p^{k-1}(p-1)} \right) + O \left( n \left( \log n \right)^{-2020} \right). \]

(vi) One has
\[ r_2(n) \leq \sum_{d \leq \sqrt{n}} \pi(n^{1/d^k}, n) \leq \sum_{d \mid n} \sum_{d \leq n^{1/k}} \# \{ m \leq n-1 : m \equiv n \left( \text{mod } d^k \right) \} \]
\[ \leq \sum_{d \gg (\log n)^{2020}} \left( \frac{n}{dk} + 1 \right) \ll n \left( \log n \right)^{-2020} + n^{1/k}. \]

Then \( r_2(n) \ll n \left( \log n \right)^{-2020} \), and so from parts (iii) and (v) we conclude that
\[ r(n) = r_1(n) + r_2(n) + O \left( n^{1/k} \right) \]
\[ = li(n) \prod_{(p,n)=1} \left( 1 - \frac{1}{p^{k-1}(p-1)} \right) + O \left( n \left( \log n \right)^{-2020} \right) \]
\[ = li(n) \prod_{p} \left( 1 - \frac{1}{p^{k-1}(p-1)} \right) \left( \prod_{p \mid n} \left( 1 - \frac{1}{p^{k-1}(p-1)} \right) \right)^{-1} + O \left( n \left( \log n \right)^{-2020} \right) \]
\[ = li(n) \left( \prod_{p \mid n} \left( \frac{p^{k-1}(p-1) - 1}{p^{k-1}(p-1)} \right) \right)^{-1} \prod_{p} \left( 1 - \frac{1}{p^{k-1}(p-1)} \right) + O \left( n \left( \log n \right)^{-2020} \right) \]
\[ = li(n) \prod_{p \mid n} \left( 1 + \frac{1}{p^{k-1}(p-1) - 1} \right) \prod_{p} \left( 1 - \frac{1}{p^{k-1}(p-1)} \right) + O \left( n \left( \log n \right)^{-2020} \right). \]

Thus \( r(n) = c(n) li(n) + O \left( n \left( \log n \right)^{-2020} \right). \)