Solutions to problem sheet 6.

Q1 We showed in Corollary 13·4 that $\zeta(s) = \zeta(1-s)\cdot 2^s \pi^{s-1}\Gamma(1-s)\sin(\frac{1}{2}\pi s)$.
By substituting $1-s$ for s, we deduce that $\zeta(1-s) = \zeta(s)\,2^{1-s}\pi^{-s}\Gamma(s)\sin(\frac{1}{2}\pi - \frac{1}{2}\pi s)$
$$= \zeta(s)\,2^{1-s}\pi^{-s}\Gamma(s)\cos(\pi s/2). \quad //$$

Q2 By differentiating the relation from Q1, we obtain
$$-\zeta'(1-s) = \frac{d}{ds}\left(\zeta(s)\,2^{1-s}\pi^{-s}\Gamma(s)\right)\cos\left(\frac{\pi s}{2}\right) - \zeta(s)\,2^{1-s}\pi^{-s}\Gamma(s)\frac{\pi}{2}\sin\left(\frac{\pi s}{2}\right).$$
Then for $k\in\mathbb{N}$ and $s = 2k+1$, we deduce that
$$-\zeta'(-2k) = -\zeta(2k+1)\,2^{-2k}\pi^{-2k-1}\Gamma(2k+1)\,\frac{\pi}{2}\cdot(-1)^k = \frac{(-1)^k(2k)!\,\zeta(2k+1)}{2^{2k+1}\pi^{2k}}. \quad //$$

Q3 We apply Theorem 15·5 due to Page, noting that from QA1 from Problem Sheet 3,
$$\theta(x;q,a) = \psi(x;q,a) + O(x^{1/2}).$$
(i) When there is no exceptional zero, we have
$$\psi(x;q,a) = \frac{x}{\varphi(q)} + O(x\exp(-c\sqrt{\log x})) \Rightarrow \theta(x;q,a) = \frac{x}{\varphi(q)} + O(x\exp(-c\sqrt{\log x})).$$
Then by Riemann - Stieltjes integration
$$\begin{aligned}\pi(x;q,a) &= \int_{2^-}^{x}\frac{1}{\log u}\,d\,\theta(u;q,a)\\
&= \frac{1}{\varphi(q)}\int_{2^-}^{x}\frac{du}{\log u} + \int_{2^-}^{x}\frac{1}{\log u}\,d\left(\theta(u;q,a) - \frac{u}{\varphi(q)}\right)\\
&= \frac{li(x)}{\varphi(q)} + \left[\frac{\theta(u;q,a) - u/\varphi(q)}{\log u}\right]_{2^-}^{x} + \int_{2^-}^{x}\frac{\theta(u;q,a) - \frac{u}{\varphi(q)}}{u(\log u)^2}\,du\\
&= \frac{li(x)}{\varphi(q)} + O\left(x\exp(-c\sqrt{\log x})\right) + O\left(x\exp(-c\sqrt{\log x})\left|\int_{2^-}^{x}\frac{du}{u(\log u)^2}\right|\right)\end{aligned}$$
Thus $\pi(x;q,a) = \dfrac{li(x)}{\varphi(q)} + O(x\exp(-c\sqrt{\log x}))$. □

(ii) When there is an exceptional character χ_1 modulo q and β_1 is the associated exceptional zero of $L(s,\chi_1)$, then
$$\psi(x;q,a) = \frac{x}{\varphi(q)} - \frac{\chi_1(a)x^{\beta_1}}{\varphi(q)\beta_1} + O(x\exp(-c\sqrt{\log x}))$$
$$\Downarrow$$
$$\theta(x;q,a) = \frac{x}{\varphi(q)} - \frac{\chi_1(a)x^{\beta_1}}{\varphi(q)\beta_1} + O(x\exp(-c\sqrt{\log x})).$$
Then by Riemann - Stieltjes integration
$$\pi(x;q,a) = \int_{2^-}^{x}\frac{1}{\log u}\,d\,\theta(u;q,a)$$

② $$= \frac{1}{\varphi(q)}\int_{2^-}^{x} \frac{du}{\log u} - \frac{\chi_1(a)}{\varphi(q)\beta_1}\int_{2^-}^{x}\frac{d(u^{\beta_1})}{\log u} + \int_{2^-}^{x}\frac{1}{\log u}\, d\Big(\underbrace{\theta(u;q,a) - \frac{u}{\varphi(q)} + \frac{\chi_1(a)u^{\beta_1}}{\varphi(q)\beta_1}}_{= O(u\exp(-c\sqrt{\log u}))}\Big).$$

The last term may be handled as in part (i), and the second via the substitution $v = u^{\beta_1}$. Thus we deduce that

$$\pi(x;q,a) = \frac{li(x)}{\varphi(q)} - \frac{\chi_1(a)}{\varphi(q)}\, li(x^{\beta_1}) + O(x\exp(-c\sqrt{\log x})). \square$$

Q4 (i) Suppose that $L(s,\chi_i)$ has a zero β_i satisfying $1-\beta_i < c/A\log q_i$ for $i=1$ and 2. By Landau's theorem, if $1-\beta_1 < c/\log(q_1q_2)$ then $1-\beta_2 \geq c/\log(q_1q_2)$. Thus

$$\frac{c}{\log(q_1q_2)} \leq 1-\beta_2 < \frac{c}{A\log q_2}, \quad \text{whence } \log(q_1q_2) > A\log q_2 \Rightarrow q_1 > q_2^{A-1}.$$

If instead $1-\beta_2 < c/\log(q_1q_2)$ then similarly $q_2 > q_1^{A-1}$. Further, if $1-\beta_i \geq c/\log(q_1q_2)$ $(i=1,2)$, then we deduce similarly that $q_1 > q_2^{A-1}$ and $q_2 > q_1^{A-1}$.

So either $q_2 > q_1^{A-1}$ or $q_1 > q_2^{A-1}$ (or both). $\square$

(ii) Suppose that $(q_i)_{i=1}^{\infty}$ is a strictly increasing sequence of natural numbers with the stated property. Since for each i one has $\chi_i\chi_{i+1}$ non-principal, it follows from part (i) that $q_i > q_{i+1}^{A-1}$ or $q_{i+1} > q_i^{A-1}$. Since $A>2$, we see that the latter must hold. $\square$

(iii) Put $A=3$, and suppose that χ is a primitive character modulo q for which $L(s,\chi)$ has an exceptional real zero β with $1-\beta < c/3\log q$. Then if χ' is another primitive character modulo q' for which $L(s,\chi')$ has an exceptional real zero β with $1-\beta' < c/3\log q'$, we deduce from (ii) that when $q'>q$, then $q' > q^{A-1} = q^2$. [Notice that $\chi\chi'$ is non-principal owing to the primitivity of χ (mod q) and χ' (mod q')]. Let $2 \leq q_1 < q_2 < \dots < q_n \leq Q$ be the moduli associated with primitive characters $\chi_1,\dots,\chi_n$ modulo $q_1,\dots,q_n$, respectively, for which $L(s,\chi_j)$ has an exceptional real zero β_j with $1-\beta_j < c/3\log q_j$ $(1\leq j\leq n)$. Then we have

$$q_j > q_{j-1}^2 > \dots > q_1^{2^{j-1}} \geq 2^{2^{j-1}} \quad (1\leq j \leq n),$$ whence we deduce that if $q_n \leq Q$, then

$$2^{2^{n-1}} \leq Q \quad \Rightarrow \quad n \ll \log\log Q.$$

So there are at most $O(\log\log Q)$ such exceptional moduli. $\square$

Q5 (i) We have $r(n) = \#\{n = p+x : p \text{ prime } \& \ x \ k\text{-free}\}$

$$= \sum_{\substack{p<n \\ n-p\ k\text{-free}}} 1 = \sum_{p<n} \sum_{d^k \mid (n-p)} \mu(d)$$

Here we use that if $D \geq 1$ and D^k is the largest k-th power dividing $n-p$, then
$$\sum_{d^k \mid (n-p)} \mu(d) = \sum_{d \mid D} \mu(d) = \begin{cases} 1, & D=1 \\ 0, & D>1 \end{cases}. \qquad \square$$

(ii) Thus
$$r(n) = \sum_{d \leq n^{1/k}} \mu(d) \sum_{\substack{p<n \\ n-p \equiv 0 \pmod{d^k}}} 1 = \sum_{d \leq n^{1/k}} \mu(d)\, \pi(n-1; d^k, n). \ \square$$

(iii) When $(d,n)>1$, any prime p counted by $\pi(n-1; d^k, n)$ satisfies $p \equiv n \pmod{d^k}$, whence $(n,d) \mid p$, so that $p = (n,d)$. Thus the contribution to the sum from those terms with $(d,n)>1$ is at most $\sum_{d \leq n^{1/k}} |\mu(d)| \leq n^{1/k}$. Hence
$$r(n) = \overbrace{\sum_{\substack{1 \leq d \leq (\log n)^{2020} \\ (d,n)=1}} \mu(d)\, \pi(n-1; d^k, n)}^{r_1(n)} + \overbrace{\sum_{\substack{(\log n)^{2020} \leq d \leq n^{1/k} \\ (d,n)=1}} \mu(d)\, \pi(n-1; d^k, n)}^{r_2(n)} + O(n^{1/k}). \square$$

(iv) By the Siegel-Walfisz theorem, when $d \leq (\log n)^{2020}$, one has $d^k \leq (\log n)^{2020k}$, and hence
$$\pi(n-1; d^k, n) = \frac{li(n-1)}{\varphi(d^k)} + O_k\left(n \exp(-c\sqrt{\log n})\right),$$
suitable $c>0$
$$= \frac{li(n)}{d^{k-1}\varphi(d)} + O_k\left(n \exp(-c\sqrt{\log n})\right).$$

Thus
$$r_1(n) = li(n) \sum_{\substack{1 \leq d \leq (\log n)^{2020} \\ (d,n)=1}} \frac{\mu(d)}{d^{k-1}\varphi(d)} + O_k\left(n \exp(-c\sqrt{\log n})\right). \ \square$$

(v) One has
$$\sum_{d > (\log n)^{2020}} \left| \frac{\mu(d)}{d^{k-1}\varphi(d)} \right| \ll \sum_{d > (\log n)^{2020}} \frac{1}{d^{k-1/2}} \ll (\log n)^{-2020}.$$

Thus, using multiplicativity,
$$\sum_{\substack{1 \leq d \leq (\log n)^{2020} \\ (d,n)=1}} \frac{\mu(d)}{d^{k-1}\varphi(d)} = \sum_{\substack{d=1 \\ (d,n)=1}}^{\infty} \frac{\mu(d)}{d^{k-1}\varphi(d)} + O\left((\log n)^{-2020}\right)$$

④

$$= \prod_{(p,n)=1} \left(\sum_{h=0}^{\infty} \frac{\mu(p^h)}{(p^h)^{k-1}\varphi(p^h)} \right) + O\left((\log n)^{-2020}\right)$$

$$= \prod_{(p,n)=1} \left(1 - \frac{1}{p^{k-1}(p-1)} \right) + O\left((\log n)^{-2020}\right).$$

Thus, from part (iv),

$$r_1(n) = li(n) \prod_{(p,n)=1} \left(1 - \frac{1}{p^{k-1}(p-1)}\right) + O\left(n (\log n)^{-2020}\right). \square$$

(vi) One has

$$r_2(n) \leq \sum_{\substack{d > (\log n)^{2020} \\ d \leq n^{1/k}}} \pi(n-1; d^k, n) \leq \sum_{d > (\log n)^{2020}} \# \{ m \leq n-1 : m \equiv n \pmod{d^k} \}$$

$$\leq \sum_{\substack{d > (\log n)^{2020} \\ d \leq n^{1/k}}} \left(\frac{n}{d^k} + 1 \right) \ll n (\log n)^{-2020} + n^{1/k}.$$

Then $r_2(n) \ll n (\log n)^{-2020}$, and so from parts (iii) and (v) we conclude ~~that~~

$$r(n) = r_1(n) + r_2(n) + O(n^{1/k})$$

$$= li(n) \prod_{(p,n)=1} \left(1 - \frac{1}{p^{k-1}(p-1)}\right) + O\left(n (\log n)^{-2020}\right)$$

$$= li(n) \prod_{p} \left(1 - \frac{1}{p^{k-1}(p-1)}\right) \cdot \left(\prod_{p \mid n} \left(1 - \frac{1}{p^{k-1}(p-1)}\right)\right)^{-1} + O\left(n (\log n)^{-2020}\right)$$

$$= li(n) \left(\prod_{p \mid n} \left(\frac{p^{k-1}(p-1) - 1}{p^{k-1}(p-1)} \right)\right)^{-1} \prod_{p} \left(1 - \frac{1}{p^{k-1}(p-1)}\right) + O\left(n (\log n)^{-2020}\right)$$

$$= li(n) \prod_{p \mid n} \left(1 + \frac{1}{p^{k-1}(p-1) - 1}\right) \cdot \prod_{p} \left(1 - \frac{1}{p^{k-1}(p-1)}\right) + O\left(n (\log n)^{-2020}\right).$$

Thus $r(n) = c(n)\, li(n) + O\left(n (\log n)^{-2020}\right)$. $\square$

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