26 Mar 2020

**Diminishing Ranges**

Recall: \( X = n^{1/k} \) large

\[ X_0 = \frac{1}{20} X, \quad X_{i+1} = \frac{1}{2} X_i^{1 - 1/k} \quad (i \geq 0) \]

So \( X_i \sim X^{(1 - 1/k)^i} \).

\[ g_j(x) = \sum_{0 \leq j \leq s-1} e(\alpha y^k), \quad x_j < y \leq 2x_j \]

\[ G(x) = \prod_{j=0}^{s-1} g_j(x). \quad \text{NB. } |G(x)| \leq G(0) \lesssim X^{k(1 - (1 - 1/k)^s)} \]

**Lemma 13.1** (Diminishing ranges estimate). One has

\[ \int_0^1 |G(x)|^2 \, dx \ll X^{k(1 - (1 - 1/k)^s)} \]

Saves a factor \( X^k - k(1 - 1/k)^s \) over trivial est.
Proof by orthogonality \( \int_0^1 |G(x)|^2 \, dx \) is equal to

the number of solutions of

\[
y_0^k - z_0^k = \sum_{j=1}^{s-1} (y_j^k - z_j^k),
\]

with \( X_j < y_j, z_j \leq 2X_j \), \( 0 \leq j \leq s-1 \).

Observe that when \( y_0 \neq z_0 \), one has

\[
|y_0^k - z_0^k| = (y_0 - z_0) \cdot |y_0^{k-1} + y_0^{k-2} z_0 + \ldots + z_0^{k-1})
\]

\[
\geq 1 \cdot kX_0^{k-1} = kX_0^{k-1}
\]

Also, since

\[
\left| \sum_{j=1}^{s-1} (y_j^k - z_j^k) \right| \leq (2X_1)^k + o(X_1^k)
\]

\[
< X_0^{k-1} (1 + o(1))
\]

Thus

\[
|y_0^k - z_0^k| > \left| \sum_{j=1}^{s-1} (y_j^k - z_j^k) \right|
\]
Thus, the only solutions counted by \( \int_0^1 |G(x)|^2 \, dx \) are those with \( y_0 = z_0 \). Whence

\[
\int_0^1 |G(x)|^2 \, dx \leq X_0 \cdot \int_0^1 |g_1(x)g_2(x) \cdots g_{s-1}(x)|^2 \, dx
\]

Counts the solutions of

\[
\sum_{j=1}^{s-1} (y_j^k - z_j^k) = 0.
\]

Thus by an inductive argument, one sees that

\[
\int_0^1 |G(x)|^2 \, dx \leq X_0 \cdot X_1 \cdot X_2 \cdots \cdot X_{s-1}
\]

\[
\leq X \cdot (1 + (1 - 1/k) + (1 - 1/k)^2 + \cdots + (1 - 1/k)^{s-1})
\]

\[
\leq X \cdot k - k(1 - 1/k)^s
\]

\[\Box\]
Hardy-Littlewood dissection: Take $\delta > 0$ with $\delta < 1/5$, and define

$$M_{\delta} = \bigcup_{0 \leq a \leq q \leq x^{\delta}, (a,q) = 1} M_{\delta}(q,a)$$

and

$$m_{\delta} = [0,1) \setminus M_{\delta}.$$

Minor arcs: By Weyl's inequality, we have

$$\sup_{\alpha \in m_{\delta}} |f(\alpha)| \ll x^{1-\delta 2^{-1-k}+\epsilon},$$

and

$$\sum_{1 \leq x \leq X} e(\alpha x^k)$$

When $n = x_1^k + \ldots + x_t^k + y_0^k + z_0^k + \ldots + y_{s-1}^k + z_{s-1}^k$.

$$\int_{m_{\delta}} f(\alpha)^t G(\alpha)^2 e(-n\alpha) d\alpha \leq \left( \sup_{\alpha \in m_{\delta}} |f(\alpha)| \right)^t \int |G(\alpha)|^2 d\alpha,$$
\[ k (1 - (1 - \frac{1}{k})^S) \]

\[ G(0)^2 \times (1 - (1 - \frac{1}{k})^S) \]

It follows that

\[ \int f(\alpha)^t G(\alpha)^2 e(-n\alpha) d\alpha \ll G(0)^2 \times \frac{t - k - k'}{k' - 1 - \frac{1}{k}} \]

for some \( t > 0 \), provided

\[ t \delta 2^{1-k} > k (1 - \frac{1}{k})^S \quad (13.2) \]

Shortly we'll show

\[ \int f(\alpha)^t G(\alpha)^2 e(-n\alpha) d\alpha \gg G(0)^2 n \frac{\frac{1}{k} - 1}{\exp \text{ maj. arc. est.}} \]

provided that \( t \geq 4k \). Thus we will deduce
Then
\[ p(n) = \int_{m_5}^{\infty} f(x)^t G(x)^2 e^{\lambda x - \alpha x} \, dx \to G(0)^2 n^{\frac{1}{k} - 1} \] 
\[ \text{as } n \to \infty. \]

So
\[ G(k) \leq \inf_{s \geq 0} \left( 2s + \max \left\{ 4k, \left\lceil \frac{k (1 - 1/k)^s}{\delta 2^{1-k}} \right\rceil \right\} \right) \]

Observe that
\[ (1 - \frac{1}{k})^s \leq e^{-s/k} \]

and we can take \( \delta = 1/8 \), so
\[ 2s + \left\lceil \frac{k (1 - 1/k)^s}{\delta 2^{1-k}} \right\rceil \leq 2s + \left\lceil 8k 2^{k-1} e^{-s/k} \right\rceil \]

Seek to choose \( s \) with (think: diff. wrt \( s \))
\[ 2 = 2^{k+2} e^{-s/k} \iff s = k \log (2^{k+1}). \]
Then put $s = \lceil k^2 \log 2 \rceil > k + 1$ for $k \geq 2$, and we obtain

$$t = \left\lceil \frac{k(1-1/k)^s}{\delta \cdot 2^{1-k}} \right\rceil \leq \left\lceil k2^{k+2} \cdot 2^{-k} \right\rceil = 4k.$$

Thus we see that

$$G(k) \leq 2 \lceil k^2 \log 2 \rceil + 4k$$

$$< (2 \log 2)^k + 4k + 1.$$

Major arcs. We have

$$\sum \int f(x) \overline{G(x)^2} e(-nx) \, dx = \sum \int f(x)^* e(-Nx) \, dx$$
\[ N = n - \sum_{j=0}^{s-1} (y_j^k + z_j^k) \]

and the summation over \( y_j, z_j \) is over
\[ X_j < y_j, z_j \leq 2X_j \quad (0 \leq j \leq s-1). \]

Note that
\[ \sum_{j=0}^{s-1} (y_j^k + z_j^k) \leq 2(2x_0)^k + O(x_1^k) \]
\[ < 10^{1-k} n, \quad x_0 = \frac{1}{20} X \]

whence
\[ N = n - \sum_{j=0}^{s-1} (y_j^k + z_j^k) > \frac{1}{2} n \]

(when \( n \) large).

Since \( s < 1/5 \) and \( t \geq 4k \), our major arc analysis still works (singular integral works...
for \( s \geq k+1 \); singular series required
\( s \geq 2^k+1 \) in class, but \( \alpha_5 \) from HW4 shows
that \( s \geq 4k \) is enough — key estimate
\( S(q,a) \ll q \frac{1}{\log t} \frac{1}{\log (1+\sqrt{t})} \frac{1}{\log (N_{1/2})} \) for \( (q,a)=1 \).

Thus
\[
\int_{M_0} f(x)^t e^{-(N-x)} dx = \frac{\Gamma(1+\sqrt{t})}{\Gamma(t/k)} \sum_{n=1}^{\infty} \frac{\log n}{n^{t/k}}
\]

Hence
\[
\int_{M_0} f(x)^t G(x)^2 e^{-(n-x)} dx \gg n^{t/k-1} \sum_{y=1}^{N \leq \delta n} \frac{1}{y^{t/k-1}} \sum_{\frac{y}{z} \leq N/e} 1
\]

\[
= n^{t/k-1} G(0)^2.
\]
This justifies our earlier claim.

Theorem 13.2 When $k \geq 2$, one has

$$G(k) < 2(\log 2) \cdot k^2 + 4k + 1.$$

Improves

$$G(k) \leq 2^k + 1 \quad \text{for } k \geq 7.$$

Next up: Vinogradov's mean value theorem

~ 1935, ..., 1980s ..., 2010 ...

$\rightarrow$ $G(k) \leq 4k \log k + O(k \log \log k)$.

1947

1959: $G(k) \leq 2k \log k + \ldots$