Theorem 14.1 (Vinogradov's mean value theorem) Suppose that \( k \geq 3 \), and that \( s = rk \) for some \( r \in \mathbb{N} \). Then for all large values of \( X \), one has
\[
J_{s,k}(X) \ll_{s,k} X^{2s - \frac{1}{2}k(k+1)} + \Delta_{s,k},
\]
where
\[
\Delta_{s,k} = \frac{1}{2} k^2 (1 - 1/k)^r.
\]

Proof. (so far) Classified solutions of
\[
\sum_{i=1}^{s} x_i^j = \sum_{i=1}^{s} y_i^j \quad (1 \leq j \leq k)
\]
for \( 1 \leq x, y \leq X \)

I_1: \( x_i = x_h \) for some \( 1 \leq i < h \leq k \). If \( I_1 \) dominates, then
\[
J_{s,k}(X) \ll X^s.
\]
$I_2$: \( x_i = x_h \) for \( n \leq 1 \leq i < h \leq k \).

- Can find a prime \( p \) with \( X^{1/k} < p \leq 2X^{1/k} \) for which \( x_i \equiv x_h \pmod{p} \) for \( n \leq 1 \leq i < h \leq k \).

- Classify solutions into congruence classes and apply Hölder. If these solutions dominate, then there is a prime \( p \) with \( X^{1/k} < p \leq 2X^{1/k} \), and a residue \( \eta \) with \( 0 \leq \eta < p \) such that

\[
J_{s,k}(X) \ll P^{2s-2k} \int_{[0,1)^k} \left| \sum_{x \in \mathcal{X}} g(x; \xi_1) \ldots g(x; \xi_k) \right|^2 \, \text{d}x,
\]

where
\[
g(x; \xi) = \sum_{1 \leq x \leq X} \frac{e(\xi_1 x + \ldots + \xi_k x^k)}{X \equiv \xi \pmod{p}}.
\]
and $A$ is a set of $(\xi_1, \ldots, \xi_k)$ with $0 \leq \xi_i \leq p$ and $\xi_i = \xi_h$ for no $i$ and $h$ with $1 \leq i < h \leq k$.

Integral here counts solutions of

$$
\sum_{i=1}^{k} (x_i^j - y_i^j) = \sum_{h=1}^{s-k} ((p u_h + \eta)^j - (p v_h + \eta)^j) \\
(1 \leq j \leq \ell)
$$

TDI

\[1 \leq x_i, y_i, p u_h + \eta, p v_h + \eta \leq X\]
\[x_1, \ldots, x_k \} \text{ distinct mod } p\]

$$
\sum_{i=1}^{k} ((x_i - \eta)^j - (y_i - \eta)^j) = p^j \sum_{h=1}^{s-k} (u_h^j - v_h^j) \\
(1 \leq j \leq k)
$$

$$
\sum_{i=1}^{k} (x_i - \eta)^j \equiv \sum_{i=1}^{k} (y_i - \eta)^j \pmod{p^j} \\
(1 \leq j \leq k).
$$
If we know \( x_i \equiv y_i \pmod{p^k} \),
then we know \( x_i, y_i \) as integers.
\( p^k > X \geq \max \{ x_i, y_i \} \).

\[ \downarrow \]
There exist integers \( h_1, \ldots, h_k \) with \( 1 \leq h_j \leq p^{k-j} \)
\((1 \leq j \leq k)\) such that
\[ \sum_{i=1}^{k} (x_i - \eta)^j \equiv h_j p^j + \sum_{i=1}^{k} (y_i - \eta)^j \pmod{p^k} \]
\((1 \leq j \leq k)\)

Consider fixed choice for \( h, y \). There are
at most \( (p^{k-1} \cdot p^{k-2} \cdots p \cdot 1) \cdot X^k \)
\[ y \sim p^{\frac{1}{2}k(k-1)} \cdot X^k. \]
Then, for some integers \( \ell_j = \ell_j(y, \eta) \), we have

\[
\sum_{i=1}^{k} (x_i - \eta)^j \equiv \ell_j \pmod{p^k} \quad (1 \leq j \leq k).
\]

If \( x \) and \( x' \) are two solutions of \((14.6)\), then by the Newton-Girard formula we have

\[
\prod_{i=1}^{k} (t - (x_i - \eta)) \equiv \prod_{i=1}^{k} (t - (x'_i - \eta)) \pmod{p^k}.
\]

(as a poly in \( t \)).

Put \( t = x'_j - \eta \) for some \( 1 \leq j \leq k \). Then

\[
\prod_{i=1}^{k} (x'_j - x_i) \equiv 0 \pmod{p^k}.
\]
Recall that $x_1, \ldots, x_k$ are distinct mod $p$, so that at most one of $x'_j - x_i$ can be divisible by $p$. Thus $x'_j \equiv x_i \pmod{p^k}$ for some $1 \leq i \leq k$.

An inductive approach shows from here that

$\{x_1, \ldots, x_k\} \equiv \{x'_1, \ldots, x'_k\} \pmod{p^k}$.

Further, since $1 \leq x_i, x'_i \leq X$ and $p^k > X$, this forces

$\{x_1, \ldots, x_k\} = \{x'_1, \ldots, x'_k\}$.

Whence there are at most $k!$ choices for $x_1, \ldots, x_k$. 

Fix any one such choice of $\xi$ and $\eta$, and substitute into

$$
\sum_{i=1}^{k} (\xi_i - \eta)^j - \sum_{i=1}^{k} (\eta_i - \eta)^j = p^j \sum_{h=1}^{s-k} (u_h^j - v_h^j) 
$$

By orthogonality, the number of solutions for $u$ and $v$ is

$$
\int \left| \int_{\Gamma_0} \left| 1 + f_k (\alpha; (X-\eta)/p) \right| e^{-n_1 - \ldots - \alpha u_h n_k} du \right|^{2s-2k} \text{d} \alpha 
$$

with $u_h = 0$ for $h = 0, 1, \ldots, k$.

$$
\sum e(\alpha, u + \ldots + \alpha u^k) - \eta/p < u \leq (X-\eta)/p
$$
Triangle ineq.
\[
\leq \int_{[0,1]^k} \left| 1 + f_u(x; (X-\eta)/p) \right|^{2s-2k} \, dx
\]
\[
\leq 1 + \int_{[0,1]^k} \left| f_u(x; (X-\eta)/p) \right|^{2s-2k} \, dx
\]
\[
\leq J_{s-k,k}( (x-\eta)/p )
\]

At this point, we have shown:
\[
J_{s,k}(X) \leq p^{2s-2k} \left( \frac{1}{2} \right)^{k(k-1)/2} \frac{k!}{k!} \cdot (1 + J_{s-k,k}(X))
\]
# k, y # x

Recall that \( X^{1/k} < p \leq 2X^{1/k} \). Thus
\[ J_{s,k}(X) \lessapprox (X^{\frac{1}{k}})^{\frac{2s-2k}{2}} \cdot X^k \]

Ind. Hyp.

\[ \lessapprox (X^{\frac{1}{k}})^{\frac{2s-2k}{2}} \cdot X \]

\[ (X^{1-\frac{1}{k}})^{\frac{2s-2k}{2}} - \frac{1}{2} \frac{k(k+1)}{s-\frac{1}{k}} \]

(Where \( \Delta_{s-k,k} = \frac{1}{2} k^2 (1 - \frac{1}{k})^{r-1} \))

\[ s = rk \]

\[ \lessapprox X^{(2s-2k) + k - \frac{1}{2} k(k+1)} \cdot X^{\frac{1}{2} (k+1) + \frac{1}{2} (k-1)} \]

\[ \cdot (X^{1-\frac{1}{k}})^{\Delta_{s-k,k}} \]
Then
\[ J_{s,u}(X) \leq X^{2s - \frac{1}{2} k(k+1) + \Delta_{s-k,u}(1-k)} \]

\[ \leq X^{2s - \frac{1}{2} k(k+1) + \Delta_{s,u}} \]

where
\[ \Delta_{s,u} = \Delta_{s-k,u}(1-k) \]

\[ = \frac{1}{2} k^2 (1-k)^{r-1} (1-k) \]

\[ = \frac{1}{2} k^2 (1-k)^r. \]

So inductive hypothesis holds with \( r \) in place of \( r-1 \).

\[ s \leq \frac{1}{2} k(k+1) - \frac{1}{3} k + o(k) \]

\[ s \leq \frac{1}{2} k(k+1) \]
§15. An analogue of Hua's Lemma.

\[ I_{s,u}(X) := \int_0^1 \left| \sum_{1 \leq x \leq X} e(\alpha x^u) \right|^2 \, d\alpha \]

\[ J_{s,u}(X) := \int_0^1 \left| \sum_{\ell=1}^s e(\alpha_1 x + \ldots + \alpha_u x^u) \right|^2 \, d\alpha \]

Observe that by orthogonality \( I_{s,u}(X) \) counts integral solutions of

\[ x_1^u + \ldots + x_s^u = y_1^u + \ldots + y_s^u \]

with \( 1 \leq x, y \leq X \).

This is equal to the number of solutions of

\[ \sum_{i=1}^s (x_i^u - y_i^u) = 0 \]
\[ \sum_{i=1}^{\frac{s}{\lambda}} (x_i^j - y_i^j) = h_j \quad (1 \leq j \leq k-1), \]

with \( 1 \leq x_i, y_i \leq X \) and \( |h_j| \leq sX^j \) \( (1 \leq j \leq k-1) \).

[The \( h_j \)'s are large enough so (15.1) imposes no constraint.] Then by orthogonality,

\[
\mathcal{I}_{s,h} (X) \leq \sum \ldots \sum_{1 \leq h_1 \leq sX} \ldots \sum_{1 \leq h_{k-1} \leq sX^k} \int_{[0,1)^{k-1}} \left| \int_{[0,1]^{k-1}} f(x_j;X) e^{-\alpha_1 h_1 - \ldots - \alpha_{k-1} h_{k-1}} \, d\alpha \right|^{\frac{2s}{\lambda}} \, d\alpha
\]

\( \Delta \neq 0 \)

\[
\ll X^{1 \ldots (k-1)} \int_{[0,1]^{k-1}} \left| \int_{[0,1]^{k-1}} f(x_j;X) \right|^{\frac{2s}{\lambda}} \, d\alpha \\mathcal{J}_{s,h} (X).
\]
Theorem 15.1. One has
\[
\int_1^X \sum_{1 \leq x \leq X} e(x^{1/4}) \, dx \ll X^{1/2} \frac{k(k-1)}{s^{1/2}} J_{s,k}(X).
\]

Thus, when \( s = rk \) with \( r \in \mathbb{N} \), one has
\[
\int_1^X \sum_{1 \leq x \leq X} e(x^{1/4}) \, dx \ll X \frac{1}{s} \left( \frac{1}{2} k^2 (1-\frac{1}{4k}) \right)^r.
\]