MA59800AHA ARITHMETIC HARMONIC ANALYSIS. PROBLEMS 2

TO BE HANDED IN BY WEDNESDAY 12TH FEBRUARY 2020

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

A1. By applying Hölder’s inequality to the relation
\[ \int_0^1 \left| \sum_{1 \leq x \leq X} e(\alpha x^k) \right|^2 \, d\alpha = \lfloor X \rfloor, \]
prove that for each real number \( s \) with \( s \geq 1 \), one has
\[ \int_0^1 \left| \sum_{1 \leq x \leq X} e(\alpha x^k) \right|^{2s} \, d\alpha \gg X^s. \]

A2. (i) Show that when \( 0 \leq \theta \leq \frac{1}{8} \), one has \( \text{Re}(e^{2\pi i \theta}) \geq 1/\sqrt{2} \). Hence deduce that whenever \( 0 \leq \alpha \leq \frac{1}{8}X^{-k} \), one has the lower bound
\[ \left| \sum_{1 \leq x \leq X} e(\alpha x^k) \right| \geq \lfloor X \rfloor / \sqrt{2}. \]

(ii) Prove that when \( s \) and \( k \) are positive integers, one has
\[ \int_0^1 \left| \sum_{1 \leq x \leq X} e(\alpha x^k) \right|^{2s} \, d\alpha \gg X^{2s-k}. \]

B3. Let \( X \geq 1 \) and \( Y \geq 1 \). Suppose that \( \alpha \in \mathbb{R} \) satisfies the condition that there exist \( a \in \mathbb{Z} \) and \( q \in \mathbb{N} \) with \((a, q) = 1\) and \( |\alpha - a/q| \leq q^{-2} \).

(i) Suppose that \( j \) is a natural number for which \( 2^{1-j}X \geq q \). By applying Lemma 3.6, show that
\[ \sum_{2^{-j}X < x \leq 2^{1-j}X} \min \left\{ XY/x, \|\alpha x\|^{-1} \right\} \ll XY \left( q^{-1} + 2^{-j}Y^{-1} + q(XY)^{-1} \right) \log(2q), \]
and hence deduce that
\[ \sum_{q/2 < x \leq X} \min \left\{ XY/x, \|\alpha x\|^{-1} \right\} \ll XY \left( q^{-1} + Y^{-1} + q(XY)^{-1} \right) \left( \log(2qX) \right)^2. \]

(ii) Show that when \( 1 \leq x \leq q/2 \), one has \( \|\alpha x\| \geq 1/(2q) \). Hence deduce that
\[ \sum_{1 \leq x \leq q/2} \|\alpha x\|^{-1} \ll q \log(2q). \]

(iii) Combine the conclusions of (i) and (ii) to conclude that
\[ \sum_{1 \leq x \leq X} \min \left\{ XY/x, \|\alpha x\|^{-1} \right\} \ll XY \left( q^{-1} + Y^{-1} + q(XY)^{-1} \right) \left( \log(2qX) \right)^2. \]
B4. (i) By applying Hölder’s inequality in the shape
\[ \int_0^1 |f(\alpha)|^2 \, d\alpha \leq \left( \int_0^1 |f(\alpha)| \, d\alpha \right)^{2/3} \left( \int_0^1 |f(\alpha)|^4 \, d\alpha \right)^{1/3}, \]
and then applying Parseval’s identity and Hua’s lemma, deduce that for each integer \( k \geq 2 \), one has
\[ \int_0^1 \left| \sum_{1 \leq x \leq X} e(\alpha x^k) \right| \, d\alpha \gg X^{1/2 - \epsilon}. \]
(ii) Prove that for all positive real numbers \( s \), one has
\[ X^{-\epsilon} (X^{s/2} + X^{s-2}) \ll \int_0^1 \left| \sum_{1 \leq x \leq X} e(\alpha x^2) \right|^s \, d\alpha \ll X^{\epsilon} (X^{s/2} + X^{s-2}). \]

C5. (i) Define
\[ S(\alpha) = \sum_{x,y \in \mathbb{N}, 1 \leq xy^2 \leq X} e(\alpha xy^2). \]
Show that
\[ X \ll \int_0^1 |S(\alpha)|^2 \, d\alpha \ll X^{1+\epsilon} \quad \text{and} \quad \int_0^1 |S(\alpha)| \, d\alpha \ll X^{1/3+\epsilon}. \]
(ii) Write \( \tau(n) \) for the number of positive divisors of the natural number \( n \). Show that, whenever \( a \in \mathbb{Z} \) and \( q \in \mathbb{N} \) satisfy \( (a, q) = 1 \), then one has
\[ \sum_{1 \leq n \leq X} \tau(n)e(n\alpha) \ll X^{1+\epsilon} \left( \frac{1}{q + X|q \alpha - a|} + X^{-1/2} + \frac{q + X|q \alpha - a|}{X} \right). \]