Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

A1. Let $\theta$ be a parameter satisfying $0 \leq \theta \leq 1/3$, and let $S(X, \theta)$ denote the number of solutions of the Diophantine equation
$$x^3 - y^3 = u_1^3 + u_2^3 - u_3^3 - u_4^3,$$
with $X < x, y \leq 2X$, and $1 \leq u_i \leq X^{1-\theta}$. Show that for any solution of the above equation with $x \neq y$, we have $|x - y| \leq X^{1-3\theta}$.

A2. Using your answer to question A1, and making use of the symmetry between $x$ and $y$ together with the substitution $y = x + h$, establish the upper bound
$$S(X, \theta) \leq XS_0 + 2S_1,$$
where $S_0$ denotes the number of solutions of the equation
$$u_1^3 + u_2^3 = u_3^3 + u_4^3,$$
with $1 \leq u_i \leq X^{1-\theta}$, and $S_1$ denotes the number of solutions of the equation
$$h(3x^2 + 3xh + h^2) = u_1^3 + u_2^3 - u_3^3 - u_4^3,$$
with $1 \leq u_i \leq X^{1-\theta}$, $1 \leq h \leq X^{1-3\theta}$ and $X < x \leq 2X$.

B3. (a) Let $X$ and $H$ be large real numbers, and define
$$F(\alpha) = \sum_{X < x \leq 2X} \sum_{1 \leq h \leq H} e(\alpha h(3x^2 + 3xh + h^2)).$$

By using a modification of Hua’s lemma, show that
$$\int_0^1 |F(\alpha)|^4 d\alpha \ll H^{3+\epsilon} X^{2+\epsilon}.$$

(b) Recall the notation of question A2 and put $H = X^{1-3\theta}$ and $Q = X^{1-\theta}$. Also, write
$$g(\alpha) = \sum_{1 \leq u \leq Q} e(\alpha u^3).$$

Show that
$$S_1 = \int_0^1 F(\alpha)|g(\alpha)|^4 d\alpha,$$
and hence deduce that
$$S_1 \ll X^{\epsilon}(H^3X^2)^{1/4}Q^{9/4}.$$

B4. (a) Combine your answers to questions A2 and B3 to deduce that
$$S(X, \theta) \ll X^\epsilon(X^{3-2\theta} + X^{(7-9\theta)/2}),$$
and hence deduce that $S(X, 1/5) \ll X^{13/5+\epsilon}$.1
(b) An old problem in additive number theory concerns the density of integers represented as the sum of 3 cubes of natural numbers. Define

\[ \mathcal{N}(N) = \text{card} \{ 1 \leq n \leq N : n = x^3 + y^3 + z^3, \ x, y, z \in \mathbb{N} \}. \]

Also, when \( 1 \leq n \leq N \), let \( R(n) \) denote the number of solutions of the equation \( n = x^3 + y^3 + z^3 \) in positive integers \( x, y, z \) with \( X < x \leq 2X \) and \( 1 \leq y, z \leq X^{1-\theta} \), where \( X = \frac{1}{3} N^{1/3} \).

Use Cauchy’s inequality to show that

\[ \left( \sum_{1 \leq n \leq N} R(n) \right)^2 \leq \left( \frac{1}{\sum_{1 \leq n \leq N} R(n) > 0} \right) \left( \sum_{1 \leq n \leq N} R(n)^2 \right). \]

Hence deduce that \( \mathcal{N}(N) \gg N^{13/15 - \varepsilon} \).

C5. (a) By observing that

\[ (x_1 + x_2 - x_3)^2 - (x_1^2 + x_2^2 - x_3^2) = 2(x_1 - x_3)(x_2 - x_3), \]

prove that \( J_{3,2}(X) \ll X^{3 + \varepsilon} \). Hence deduce the Main Conjecture in Vinogradov’s mean value theorem for \( k = 2 \).

(b) Write \( s_j(x) = x_1^j + x_2^j \) (\( 1 \leq j \leq 3 \)), and note that the polynomials \( s_1(x), s_2(x) \) and \( s_3(x) \) are algebraically dependent. Find a non-zero polynomial \( \Psi(t_1, t_2, t_3) \in \mathbb{Z}[t_1, t_2, t_3] \) having the property that \( \Psi(s_1(x), s_2(x), s_3(x)) = 0 \). Hence deduce that the polynomial

\[ \Psi(x_1 + x_2 + x_3 - x_4, x_1^2 + x_2^2 + x_3^2 - x_4^2, x_1^3 + x_2^3 + x_3^3 - x_4^3) \]

is divisible by \( (x_1 - x_4)(x_2 - x_4)(x_3 - x_4) \).

(c) Use your answer to part (b) to deduce that \( J_{4,3}(X) \ll X^{4 + \varepsilon} \).