MA598CNUM ANALYTIC NUMBER THEORY, II. PROBLEMS 1

TO BE HANDED IN BY MONDAY 8TH FEBRUARY 2021

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

A1. (i) Let N be large, and let p be a prime number with $3 < p \leqslant \sqrt{N}$. Show that if $6k+1$, $12k+1$ and $18k+1$ are all prime numbers, then $k \not\equiv \omega \pmod{p}$ for 3 distinct residue classes ω modulo p.
(ii) Apply Montgomery's version of the large sieve inequality to deduce that the number $C(N)$ of integers k with $1 \leqslant k \leqslant N$ for which $6k+1$, $12k+1$ and $18k+1$ are all simultaneously primes, satisfies

$$C(N) \ll N \left(\sum_{1 \leqslant q \leqslant \sqrt{N}} \mu^2(q) \prod_{\substack{p \mid q \\ p > 3}} \frac{3}{p-3} \right)^{-1}.$$

A2. (i) Assuming the conclusion of question A1 and the lower bound

$$\sum_{1 \leqslant n \leqslant x} \mu^2(n) 3^{\omega(n)} n^{-1} \gg (\log x)^3,$$

show that $C(N) \ll N/(\log N)^3$.
(ii) The integer $n = (6k+1)(12k+1)(18k+1)$ is known to be a *Carmichael* number whenever $6k+1$, $12k+1$ and $18k+1$ are all prime numbers. Such a number satisfies the relation $a^{n-1} \equiv 1 \pmod{n}$ for each integer a with $(a,n)=1$, so is a pseudo-prime with respect to Fermat's Little Theorem. Show that the number of Carmichael numbers not exceeding X of this special form is $O(X^{1/3}(\log X)^{-3})$.

B3. Let M and N be integers with $N \geqslant 1$, and let x_r $(1 \leqslant r \leqslant R)$ be δ-spaced real numbers modulo 1.
(i) Prove that

$$\sum_{n=M+1}^{M+N} \left| \sum_{r=1}^{R} c_r e(nx_r) \right|^2 \leqslant N \sum_{r=1}^{R} |c_r|^2 + O(\Xi(\mathbf{c}, \mathbf{x})),$$

where

$$\Xi(\mathbf{c}, \mathbf{x}) = \sum_{1 \leqslant r < s \leqslant R} |c_r \overline{c_s}| \|x_r - x_s\|^{-1}.$$

Here, you may find it useful to recall that $\sum_{1 \leqslant n \leqslant X} e(n\alpha) \ll \min\{N, \|\alpha\|^{-1}\}$.
(ii) Prove the *dual form* of the large sieve inequality in the form

$$\sum_{n=M+1}^{M+N} \left| \sum_{r=1}^{R} c_r e(nx_r) \right|^2 \leqslant \big(N + O(\delta^{-1} \log(1/\delta)) \big) \sum_{r=1}^{R} |c_r|^2.$$

B4. Let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ denote sequences of complex numbers with $\sum_{n\in\mathbb{N}}|x_n|^2<\infty$ and $\sum_{n\in\mathbb{N}}|y_n|^2<\infty$, and define the series

$$f(\alpha)=\sum_{n=1}^\infty x_n e(n\alpha)\quad\text{and}\quad g(\alpha)=\sum_{n=1}^\infty y_n e(n\alpha).$$

Also, put

$$K(\alpha)=\sum_{k\in\mathbb{Z}\setminus\{0\}} k^{-1}e(k\alpha).$$

(i) Show that

$$\sum_{\substack{r,s\in\mathbb{N}\\ r\neq s}}\frac{x_r y_s}{r-s}=\int_0^1 f(\alpha)g(-\alpha)K(-\alpha)\,\mathrm{d}\alpha.$$

(ii) Show that $K(\alpha)=i(\pi-2\pi\alpha)$ for $0<\alpha<1$.
(iii) Obtain Hilbert's inequality in the form

$$\left|\sum_{\substack{r,s\in\mathbb{N}\\ r\neq s}}\frac{x_r y_s}{r-s}\right|\leqslant\pi\left(\sum_{n=1}^\infty|x_n|^2\right)^{1/2}\left(\sum_{n=1}^\infty|y_n|^2\right)^{1/2}.$$

C5. When $k\in\mathbb{N}$, define the arithmetic functions

$$b_k(n)=\mu^2(n)\prod_{\substack{p|n\\ p>k}}\frac{k}{p-k}\quad\text{and}\quad c_k(n)=\mu^2(n)k^{\omega(n)}n^{-1}.$$

(i) Show that for each natural number n, one has $b_k(n)\gg c_k(n)$.
(ii) By applying multiplicativity, show that there is a Dirichlet series $A_k(s)$ for which, when $\mathrm{Re}(s)>0$,

$$\sum_{n=1}^\infty c_k(n)n^{-s}=A_k(s)\zeta(s+1)^k,$$

and having the property that $A_k(s)$ is absolutely convergent for $\mathrm{Re}(s)>-\frac{1}{2}$.
(iii) Deduce (via Perron's formula) that when x is large, one has

$$\sum_{1\leqslant n\leqslant x}c_k(n)\sim\frac{A_k(0)}{k!}(\log x)^k,$$

and hence infer that

$$\sum_{1\leqslant n\leqslant x}b_k(n)\gg(\log x)^k.$$