① MA598 CNUM Analytic Number Theory, II. Problems 1 — Solutions.

Q1 (i) If $6k+1$ is prime then $\pi \nmid 6k+1$ for any prime π with $\pi < 6k+1$. Thus $6k+1 \not\equiv 0 \pmod{\pi} \Rightarrow k \not\equiv -6^{-1} \pmod{\pi}$. Similarly, one has $k \not\equiv -12^{-1} \pmod{\pi}$ and $k \not\equiv -18^{-1} \pmod{\pi}$ if $12k+1$ and $18k+1$ are also to be prime numbers. If these congruence classes were not distinct, then one would have $12^{-1} \equiv 6^{-1} \pmod{\pi} \Rightarrow 1 \equiv 2 \pmod{\pi}$ ※ or $6^{-1} \equiv 18^{-1} \pmod{\pi} \Rightarrow 3 \equiv 1 \pmod{\pi}$ ※ or $12^{-1} \equiv 18^{-1} \pmod{\pi} \Rightarrow 3 \equiv 2 \pmod{\pi}$ ※ (since $\pi > 3$). Then (provided that $p < 6k+1$), one has $k \not\equiv \omega \pmod{p}$ for the 3 distinct residue classes $6^{-1}, 12^{-1}, 18^{-1} \pmod{p}$. □

(ii) Let $\mathcal{N} \subseteq [\sqrt{N}, N]$ denote the set of integers k for which $6k+1$, $12k+1$, $18k+1$ are all primes. For each prime number p with $3 < p \le \sqrt{N}$, the number $\delta(p)$ of residue classes modulo p not represented by any such $k \in \mathcal{N}$ is at least 3. We apply Theorem 3.5 (Montgomery's sieve) with $Q = \sqrt{N}$. Then

$$\operatorname{card}(\mathcal{N}) =: Z \le \frac{(N - \sqrt{N}) + Q^2}{L},$$

where

$$L = \sum_{1 \le q \le \sqrt{N}} \mu^2(q) \prod_{\substack{p \mid q \\ p > 3}} \frac{\delta(p)}{p - \delta(p)}$$

$$\geqslant \sum_{1 \le q \le \sqrt{N}} \mu^2(q) \prod_{\substack{p \mid q \\ p > 3}} \frac{3}{p-3}.$$

Noting that the right hand side here is certainly $O(\sqrt{N})$, it follows that

$$C(N) \le \sqrt{N} + Z \ll N \Bigg(\sum_{1 \le q \le \sqrt{N}} \mu^2(q) \prod_{\substack{p \mid q \\ p > 3}} \frac{3}{p-3} \Bigg)^{-1}. \quad □$$

Q2 (i) One has $\displaystyle \prod_{\substack{p \mid q \\ p > 3}} \frac{3}{p-3} \gg \frac{2}{3} \prod_{p \mid q} \frac{3}{p} \ge \frac{2}{3} \frac{3^{\omega(q)}}{q}$. Hence

②

$$\sum_{1\le q\le\sqrt{N}} \mu^2(q)\prod_{\substack{p|q\\p>3}}\frac{3}{p-3} \gg \sum_{1\le q\le\sqrt{N}} \mu^2(q)\,3^{\omega(q)}q^{-1} \gg (\log\sqrt{N})^3.$$

Then by QA1(iii) we obtain $C(N) \ll N/(\log\sqrt{N})^3 \ll N/(\log N)^3$. □

(ii) If $n=(6k+1)(12k+1)(18k+1)$ satisfies $n\le X$, ~~then~~ $k\le X^{1/3}$. Then if $6k+1$, $12k+1$, $18k+1$ are all primes, it follows ~~that the~~ number of choices for k with $k\le X^{1/3}$ is at most $C(X^{1/3}) \ll X^{1/3}/(\log X^{1/3})^3$. Consequently, there are $O(X^{1/3}(\log X)^{-3})$ Carmichael numbers n of the shape $n=(6k+1)(12k+1)(18k+1)$ with $6k+1$, $12k+1$, $18k+1$ all prime and $n\le X$.

[Note: It is conjectured that there are $\gg X^{1-\varepsilon}$ Carmichael numbers not exceeding X.]. □

Q3 (i) One has

$$\sum_{n=M+1}^{M+N}\Big|\sum_{r=1}^{R} c_r e(nx_r)\Big|^2 = \sum_{n=M+1}^{M+N}\sum_{r_1=1}^{R}\sum_{r_2=1}^{R} c_{r_1}\bar{c}_{r_2}\, e(n(x_{r_1}-x_{r_2}))$$

$$= \sum_{n=M+1}^{M+N}\underbrace{\sum_{r=1}^{R}|c_r|^2}_{\uparrow\ ``r_1=r_2"} + \sum_{\substack{1\le r_1,r_2\le R\\ r_1\ne r_2}} c_{r_1}\bar{c}_{r_2}\sum_{n=M+1}^{M+N} e(n(x_{r_1}-x_{r_2}))$$

$$= N\sum_{r=1}^{R}|c_r|^2 + O\Big(\sum_{\substack{1\le r_1,r_2\le R\\ r_1\ne r_2}} |c_{r_1}\bar{c}_{r_2}|\min\{N, \|x_{r_1}-x_{r_2}\|^{-1}\}\Big)$$

$$\le N\sum_{r=1}^{R}|c_r|^2 + O\Big(\sum_{1\le r<s\le R} |c_r\bar{c}_s|\,\|x_r-x_s\|^{-1}\Big).$$ □

(ii) Since the elements x_r are δ-spaced modulo 1, we may rearrange the sequence $(x_r)_1^R$ so ~~that~~ $0\le x_1<x_2<\dots<x_R\le 1$ and $x_j\ge (j-1)\delta$ with $\|x_R-x_1\|\ge\delta$. Thus, dividing the x_i into two groups depending on whether $x_i\in[0,\tfrac12)$ or $x_i\in[\tfrac12,1)$, ~~modulo~~ 1, we see that

$$\sum_{1\le r<s\le R} |c_r \bar{c}_s|\, \|x_r - x_s\|^{-1} \le \frac{1}{2} \sum_{1\le r<s\le R} \left(|c_r|^2 + |c_s|^2\right) \|x_r - x_s\|^{-1}$$

$$\le \sum_{r=1}^{R} |c_r|^2 \cdot \sum_{\substack{1\le s\le R \\ s\ne r}} \|x_r - x_s\|^{-1}$$

$$\le 2\left(\sum_{r=1}^{R} |c_r|^2\right) \sum_{1\le j\le \delta^{-1}} \frac{1}{j\delta}$$

$$\ll \left(\sum_{r=1}^{R} |c_r|^2\right) \frac{1}{\delta} \log\left(\frac{1}{\delta}\right).$$

On combining this estimate with ~~that~~ of part (i), we conclude ~~that~~

$$\sum_{n=M+1}^{M+N} \left|\sum_{r=1}^{R} c_r e(nx_r)\right|^2 \le \left(N + O\left(\frac{1}{\delta}\log\left(\frac{1}{\delta}\right)\right)\right) \sum_{r=1}^{R} |c_r|^2. \quad \square$$

Q4 (i) By applying orthogonality,

$$\int_0^1 f(\alpha) g(-\alpha) K(-\alpha)\, d\alpha = \sum_{r=1}^{\infty} x_r \sum_{s=1}^{\infty} y_s \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{k} \underbrace{\int_0^1 e((r-s-k)\alpha)\, d\alpha}_{=\begin{cases} 0, & \text{when } r-s\ne k \\ 1, & \text{when } r-s=k \end{cases}}$$

$$= \sum_{\substack{r,s\in\mathbb{N} \\ r-s\in\mathbb{Z}\setminus\{0\}}} \frac{x_r y_s}{r-s}. \quad \square$$

$$\downarrow$$
$$r\ne s.$$

(ii) Observe ~~that~~

$$\int_0^1 (\pi - 2\pi\alpha)\, e(-k\alpha)\, d\alpha = \left[\frac{(\pi-2\pi\alpha)e(k\alpha)}{-2\pi i k}\right]_0^1 + \frac{1}{-ik}\int_0^1 e(k\alpha)\, d\alpha$$

$$= \frac{1}{2ik} - \frac{1}{-2ik} + \frac{1}{-ik}\begin{cases} 0, & \text{when } k\ne 0, \\ 1, & \text{when } k=0 \end{cases}$$

$$= \begin{cases} 0, & \text{when } k=0, \\ \dfrac{1}{ik}, & \text{when } k\ne 0. \end{cases}$$

Thus $\displaystyle\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{ik} e(k\alpha) = (\pi - 2\pi\alpha)$ for $0<\alpha<1$, and hence

$$K(\alpha) = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{k} e(k\alpha) = i\,(\pi - 2\pi\alpha) \quad \text{for } 0<\alpha<1. \quad \square$$

(iii) We conclude that

$$\left|\sum_{\substack{r,s\in\mathbb{N}\\ r\neq s}} \frac{x_r y_s}{r-s}\right| = \left|\int_0^1 f(\alpha) g(-\alpha)(\pi - 2\pi\alpha)\, d\alpha\right|$$

$$\leq \sup_{\alpha\in(0,1)} |\pi - 2\pi\alpha| \left(\int_0^1 |f(\alpha)|^2 d\alpha\right)^{\frac{1}{2}} \left(\int_0^1 |g(\alpha)|^2 d\alpha\right)^{\frac{1}{2}}$$

$$= \pi \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}} \left(\sum_{m=1}^{\infty} |y_n|^2\right)^{\frac{1}{2}},$$ using orthogonality

to see that $\int_0^1 |f(\alpha)|^2 d\alpha = \int_0^1 \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} x_n \overline{x}_m e((n-m)\alpha)\, d\alpha$

$$= \sum_{n=1}^{\infty} x_n \overline{x}_n,$$

and similarly for $\int_0^1 |g(\alpha)|^2 d\alpha$. □.

Q5/ (i) One has

$$b_k(n) = \mu^2(n) \prod_{\substack{p|n\\ p>k}} \frac{k}{p-k} \gg \mu^2(n) \left(\prod_{\substack{p\leq k\\ p|n}} \frac{k}{p}\cdot\frac{p}{k}\right)\left(\prod_{\substack{p|n\\ p>k}} \frac{k}{p}\right)$$

$$\geq \left(\mu^2(n) \prod_{p|n} \frac{k}{p}\right) \cdot \prod_{p\leq k} \frac{p}{k} \gg_k \mu^2(n) \frac{k^{\omega(n)}}{n}.$$

Thus $b_k(n) \gg c_k(n)$. □

(ii) Proceed by using Euler products. One has

$$\sum_{n=1}^{\infty} c_k(n) n^{-s} = \sum_{n=1}^{\infty} \mu^2(n) \frac{k^{\omega(n)}}{n^{s+1}} = \prod_p \left(1 + \frac{k}{p^{s+1}}\right),$$

whence,

$$\frac{1}{\zeta(s+1)^k} \sum_{n=1}^{\infty} c_k(n) n^{-s} = \left(\prod_p \left(1 - \frac{1}{p^{s+1}}\right)\right)^k \left(\prod_p \left(1 + \frac{k}{p^{s+1}}\right)\right)$$

$$= \prod_p \left(1 - \frac{k}{p^{s+1}} + \frac{\binom{k}{2}}{p^{2s+2}} - \cdots\right)\left(1 + \frac{k}{p^{s+1}}\right)$$

$$= \prod_p \left(1 + \frac{\binom{k}{2} - k^2}{p^{2s+2}} + \frac{-\binom{k}{3} + k\binom{k}{2}}{p^{3s+3}} + \cdots\right)$$

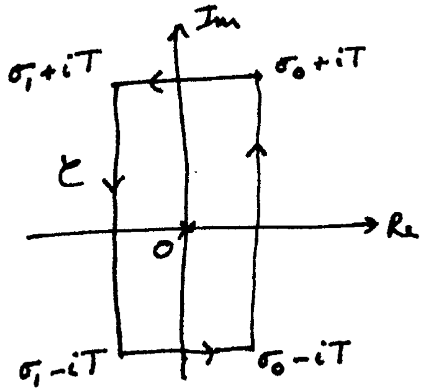
$$= \prod_p \left(1 + \frac{\gamma_{2,k}}{p^{2s+2}} + \frac{\gamma_{3,k}}{p^{3s+3}} + \cdots\right),$$

where $\gamma_{r,k} = (-1)^r\left(\binom{k}{r} - k\binom{k}{r-1}\right)$. By considering logarithms, one sees

⑤ that this is an absolutely convergent Euler product for $\mathrm{Re}(s) > -\frac{1}{2}$, since then $2s+2>1$, and hence it converges to a Dirichlet series $A_k(s)$ also absolutely convergent for $\mathrm{Re}(s) > -\frac{1}{2}$. Hence

$$\sum_{n=1}^{\infty} c_k(n) n^{-s} = A_k(s)\,\zeta(s+1)^k. \quad \square$$

(iii) Apply Perron's formula using the contour:



$$\sigma_0 = 1/\log x$$
$$\sigma_1 = -b/\log T$$

$$\sum_{1\le n\le x} c_k(n) = \frac{1}{2\pi i}\int_{\mathcal{C}} A_k(s)\,\zeta(s+1)^k \frac{x^s}{s}\,ds$$
$$+ O\Bigg(\sum_{\frac{x}{2}<n<2x} |c_k(n)| \min\Big\{1, \frac{x}{T|x-n|}\Big\}\Bigg)$$
$$+ O\Bigg(\frac{4^{\sigma_0}+x^{\sigma_0}}{T}\sum_{n=1}^{\infty}\frac{|c_k(n)|}{n^{\sigma_0}}\Bigg).$$

First the error terms. We have

$$\sum_{n=1}^{\infty}\frac{|c_k(n)|}{n^{\sigma_0}} \le \sum_{n=1}^{\infty}\frac{k^{\omega(n)}}{n^{1+1/\log x}} \le \sum_{n=1}^{\infty}\frac{\tau_k(n)}{n^{1+1/\log x}} = \zeta(1+1/\log x)^k$$
$$\ll (\log x)^k \qquad \Big(\text{using } \zeta(s) \ll \frac{1}{s-1} \text{ for } s \text{ close to } 1\Big).$$

The first error term is more fiddley. One has

$$\sum_{|x-n|\le \frac{x}{T}} |c_k(n)| \min\Big\{1, \frac{x}{T|x-n|}\Big\} \ll \Big(\sum_{|x-n|\le\frac{x}{T}} 1\Big)^{\frac12}\Big(\sum_{x<n\le 2x}|c_k(n)|^2\Big)^{\frac12}$$
$$\ll \Big(\frac{x}{T}\Big)^{\frac12}\cdot\Big(\frac{1}{x^2}\sum_{n\le 2x} k^{2\omega(n)}\Big)^{\frac12} \ll T^{-\frac12}\Big(\frac{1}{x}\sum_{n\le 2x}\tau_{k^2}(n)\Big)^{\frac12}$$
$$\ll T^{-\frac12}(\log x)^{k^2}.$$

Also,

$$\sum_{\substack{|x-n|>\frac{x}{T}\\ \frac{x}{2}<n<2x}} |c_k(n)| \min\Big\{1, \frac{x}{T|x-n|}\Big\} \ll \frac{1}{x}\cdot\frac{x}{T}\Big(\sum_{n\le 2x} k^{2\omega(n)}\Big)^{\frac12}\Big(\sum_{\frac{x}{T}<h\le 2x}\frac{1}{h^2}\Big)^{\frac12}$$
$$\ll T^{-1}\big(x(\log x)^{k^2}\big)^{\frac12}\Big(\frac{T}{x}\Big)^{\frac12}$$
$$\ll T^{-\frac12}(\log x)^{k^2}.$$

Then
$$\sum_{1\le n\le x} c_k(n) = \frac{1}{2\pi i}\int_{\mathcal{C}} A_k(s)\,\zeta(s+1)^k \frac{x^s}{s}\,ds + O\left(T^{-\frac{1}{2}}(\log x)^{k^2}\right),$$
the error in which is harmless, since ultimately we take $T = \exp(c\sqrt{\log x})$ for a suitable $c>0$.

The main term in the contour integral arises from the pole at $s=0$ of order $k+1$ (recall ~~that~~ $\zeta(s+1)$ has a simple pole at $s=0$, so $\zeta(s+1)^k s^{-1}$ has a pole of order $k+1$ at $s=0$). The residue at this pole is
$$\frac{1}{k!}\lim_{s\to 0}\frac{d^k}{ds^k}\left(s^{k+1}\cdot A_k(s)\,\zeta(s+1)^k\frac{x^s}{s}\right)$$
$$= \frac{1}{k!}\lim_{s\to 0}\frac{d^k}{ds^k}\left(A_k(s)\left(1+c_0 s+\dots\right)^k x^s\right)$$
$$= \frac{1}{k!}\left(A_k(0)(\log x)^k + O\left((\log x)^{k-1}\right)\right).$$
It remains to estimate the contribution from top, bottom and left hand segments of $\mathcal{C}$, and here we make use of our standard estimates for $\zeta(s+1)$. These contribute
$$\ll (\log T)^k \frac{x^{\sigma_0}}{T}(\sigma_0-\sigma_1) \ll \frac{(\log T)^{k-1}}{T} + x^{\sigma_1}(\log T)^{k+1}$$
$$+ (\log T)^{k+1}x^{\sigma_1} + x^{\sigma_1}/(1-\sigma_1)$$
$$\ll \frac{(\log T)^{k-1}}{T} + (\log T)^{k+1}\exp\left(-b\frac{\log x}{\log T}\right).$$
Thus, on taking $T = \exp(c\sqrt{\log x})$ for a suitable $c>0$, we see that these additional terms contribute a quantity
$$\ll \exp\left(-\tfrac{1}{2}c\sqrt{\log x}\right).$$
Combining all of these estimates, it is evident that
$$\sum_{1\le n\le x} c_k(n) = \frac{1}{k!}A_k(0)(\log x)^k + O\left((\log x)^{k-1}\right). \square$$
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