MA598CNUM ANALYTIC NUMBER THEORY, II. PROBLEMS 2

TO BE HANDED IN BY MONDAY 22ND FEBRUARY 2021

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

A1. Recall that for Dirichlet characters χ, we define

$$\psi(x,\chi)=\sum_{n\leqslant x}\chi(n)\Lambda(n).$$

Apply the large sieve inequality to show that

$$\sum_{1\leqslant q\leqslant Q}\frac{q}{\phi(q)}\sideset{}{^*}\sum_{\chi}|\psi(x,\chi)|^2\ll (x+Q^2)x\log x,$$

in which the sum is restricted to primitive characters modulo q.

A2. Let (a_m) and (b_n) be complex sequences. Prove that when N, M and Q are positive integers, then

$$\sum_{q\leqslant Q}\frac{q}{\varphi(q)}\sideset{}{^*}\sum_{\chi}\Bigl|\sum_{m=1}^{M}\sum_{n=1}^{N}a_mb_n\chi(mn)\Bigr|\leqslant (M+Q^2)^{1/2}(N+Q^2)^{1/2}\Bigl(\sum_{m=1}^{M}|a_m|^2\Bigr)^{1/2}\Bigl(\sum_{n=1}^{N}|b_n|^2\Bigr)^{1/2}.$$

B3. Let $\varphi_2(q)$ denote the number of primitive characters modulo q.
(i) By noting that $\varphi_2(q)$ is multiplicative, show that

$$\sum_{d|q}\varphi_2(d)=\varphi(q).$$

(ii) By applying Möbius inversion, conclude that

$$\varphi_2(q)=q\prod_{p\| q}\left(1-\frac{2}{p}\right)\prod_{p^2|q}\left(1-\frac{1}{p}\right)^2.$$

(iii) Deduce that $\varphi_2(q)=0$ when $q\equiv 2\pmod 4$, and otherwise

$$\varphi_2(q)\gg q\prod_{p|q}(1-1/p)^2,$$

whence

$$\varphi_2(q)\gg q(\log\log q)^{-2}.$$

(iv) Write $\mathfrak{X}(Q)=\{(q,\chi):1\leqslant q\leqslant Q \text{ and } \chi \text{ is a primitive character modulo } q\}$. Conclude from question A1 that, when $A>0$ and $x^{1/2}(\log x)^{-A}\leqslant Q\leqslant x^{1/2}$, then for a proportion $1-o(1)$ of the pairs $(q,\chi)\in\mathfrak{X}(Q)$, one has

$$|\psi(x,\chi)|\ll x^{1/2}(\log x)^{A+1/2+\varepsilon}.$$

B4. (i) Let
$$G(s)=\sum_{k\leqslant V}\mu(k)k^{-s}.$$
(i) Confirm that
$$\frac{1}{\zeta(s)}=2G(s)-G(s)^2\zeta(s)+\left(\frac{1}{\zeta(s)}-G(s)\right)(1-\zeta(s)G(s)).$$
(ii) Show that
$$\mu(n)=a_1(n)+a_2(n)+a_3(n),$$
where
$$a_1(n)=\begin{cases}2\mu(n), & \text{when } n\leqslant V,\\ 0, & \text{when } n>V.\end{cases},$$
$$a_2(n)=-\sum_{\substack{def=n\\ d\leqslant V\\ e\leqslant V}}\mu(d)\mu(e),$$
$$a_3(n)=-\sum_{\substack{dk=n\\ d>V\\ k>V}}\mu(d)\sum_{\substack{e|k\\ e\leqslant V}}\mu(e).$$

C5. Prove that when N is large and $\alpha\in\mathbb{R}$, one has
$$\sum_{1\leqslant n\leqslant N}\mu(n)e(n\alpha)=T_1+T_2+T_3,$$
where:
(i) one has
$$T_1=2\sum_{1\leqslant n\leqslant V}\mu(n)e(n\alpha),$$
(ii) one has
$$T_2=-\sum_{m\leqslant V^2}\left(\sum_{\substack{de=m\\ d\leqslant V\\ e\leqslant V}}\mu(d)\mu(e)\right)\sum_{1\leqslant n\leqslant N/m}e(nm\alpha),$$
(iii) one has
$$T_3=-\sum_{V<m\leqslant N/V}\sum_{V<n\leqslant N/m}\mu(m)\left(\sum_{\substack{d|n\\ d\leqslant V}}\mu(d)\right)e(mn\alpha).$$
(iv) Suppose that $q\in\mathbb{N}$ and $a\in\mathbb{Z}$ satisfy $|\alpha-a/q|\leqslant q^{-2}$. Deduce that for each $\varepsilon>0$, one has
$$\sum_{1\leqslant n\leqslant N}\mu(n)e(n\alpha)\ll N^{\varepsilon}\left(Nq^{-1/2}+N^{4/5}+N^{1/2}q^{1/2}\right).$$

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