MA598CNUM ANALYTIC NUMBER THEORY, II. PROBLEMS 2

TO BE HANDED IN BY MONDAY 22ND FEBRUARY 2021

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

A1. Recall that for Dirichlet characters \( \chi \), we define
\[
\psi(x, \chi) = \sum_{n \leq x} \chi(n) \Lambda(n).
\]

Apply the large sieve inequality to show that
\[
\sum_{1 \leq q \leq Q} q \varphi(q) \sum_{\chi} |\psi(x, \chi)|^2 \ll (x + Q^2)x \log x,
\]
in which the sum is restricted to primitive characters modulo \( q \).

A2. Let \((a_m)\) and \((b_n)\) be complex sequences. Prove that when \( N, M \) and \( Q \) are positive integers, then
\[
\sum_{q \leq Q} \frac{q}{\varphi(q)} \sum_{\chi} \left| \sum_{m=1}^{M} \sum_{n=1}^{N} a_m b_n \chi(mn) \right|^2 \leq (M + Q^2)^{1/2} (N + Q^2)^{1/2} \left( \sum_{m=1}^{M} |a_m|^2 \right)^{1/2} \left( \sum_{n=1}^{N} |b_n|^2 \right)^{1/2}.
\]

B3. Let \( \varphi_2(q) \) denote the number of primitive characters modulo \( q \).

(i) By noting that \( \varphi_2(q) \) is multiplicative, show that
\[
\sum_{d \mid q} \varphi_2(d) = \varphi(q).
\]

(ii) By applying Möbius inversion, conclude that
\[
\varphi_2(q) = q \prod_{p \mid q} \left( 1 - \frac{2}{p} \right) \prod_{p^2 \mid q} \left( 1 - \frac{1}{p} \right)^2.
\]

(iii) Deduce that \( \varphi_2(q) = 0 \) when \( q \equiv 2 \pmod{4} \), and otherwise
\[
\varphi_2(q) \gg q \prod_{p \mid q} \left( 1 - 1/p \right)^2,
\]
whence
\[
\varphi_2(q) \gg q (\log \log q)^{-2}.
\]

(iv) Write \( \mathcal{X}(Q) = \{(q, \chi) : 1 \leq q \leq Q \text{ and } \chi \text{ is a primitive character modulo } q\} \). Conclude from question A1 that, when \( A > 0 \) and \( x^{1/2}(\log x)^{-A} \leq Q \leq x^{1/2} \), then for a proportion \( 1 - o(1) \) of the pairs \((q, \chi) \in \mathcal{X}(Q)\), one has
\[
|\psi(x, \chi)| \ll x^{1/2} (\log x)^{A+1/2+\varepsilon}.
\]
B4. (i) Let 

\[ G(s) = \sum_{k \leq V} \mu(k) k^{-s}. \]

(i) Confirm that

\[ \frac{1}{\zeta(s)} = 2G(s) - G(s)^2 \zeta(s) + \left( \frac{1}{\zeta(s)} - G(s) \right) (1 - \zeta(s)G(s)). \]

(ii) Show that

\[ \mu(n) = a_1(n) + a_2(n) + a_3(n), \]

where

\[ a_1(n) = \begin{cases} 2\mu(n), & \text{when } n \leq V, \\ 0, & \text{when } n > V. \end{cases} \]

\[ a_2(n) = -\sum_{d,e \mid n \atop d \leq V, e \leq V} \mu(d)\mu(e), \]

\[ a_3(n) = -\sum_{d,k \mid n \atop d > V, k > V} \sum_{e \mid k \atop e \leq V} \mu(e). \]

C5. Prove that when \( N \) is large and \( \alpha \in \mathbb{R} \), one has

\[ \sum_{1 \leq n \leq N} \mu(n)e(n\alpha) = T_1 + T_2 + T_3, \]

where:

(i) one has

\[ T_1 = 2 \sum_{1 \leq n \leq V} \mu(n)e(n\alpha), \]

(ii) one has

\[ T_2 = -\sum_{m \leq V^2} \left( \sum_{d,e \mid m \atop d \leq V, e \leq V} \mu(d)\mu(e) \right) \sum_{1 \leq n \leq N/m} e(nma), \]

(iii) one has

\[ T_3 = -\sum_{V < m \leq N/V} \sum_{V < n \leq N/m} \mu(m) \left( \sum_{d,n \mid m} \mu(d) \right) e(mna). \]

(iv) Suppose that \( q \in \mathbb{N} \) and \( a \in \mathbb{Z} \) satisfy \( |\alpha - a/q| \leq q^{-2} \). Deduce that for each \( \varepsilon > 0 \), one has

\[ \sum_{1 \leq n \leq N} \mu(n)e(n\alpha) \ll N^\varepsilon \left( Nq^{-1/2} + N^{4/5} + N^{1/2}q^{1/2} \right). \]