MA 598 CNUM Analytic Number Theory, II. Problems 2 - Solutions.

Q1| By the large sieve inequality for characters, one has

$$\sum_{1\le q\le Q}\frac{q}{\varphi(q)}\sum_{\chi}{}^{*}|\psi(x,\chi)|^2=\sum_{1\le q\le Q}\frac{q}{\varphi(q)}\sum_{\chi}{}^{*}\Big|\sum_{n\le x}\chi(n)\Lambda(n)\Big|^2\ll(x+Q^2)\sum_{n\le x}\Lambda(n)^2$$

$$\ll(x+Q^2)(\log x)\sum_{n\le x}\Lambda(n)\ll(x+Q^2)x\log x \quad\text{by PNT.}$$

Q2| By Cauchy's inequality in combination with the large sieve inequality for characters, one has

$$\sum_{q\le Q}\frac{q}{\varphi(q)}\sum_{\chi}{}^{*}\Big|\sum_{m=1}^{M}\sum_{n=1}^{N}a_m b_n\chi(mn)\Big|=\sum_{q\le Q}\frac{q}{\varphi(q)}\sum_{\chi}{}^{*}\Big|\sum_{m=1}^{M}a_m\chi(m)\Big|\Big|\sum_{n=1}^{N}b_n\chi(n)\Big|$$

$$\le\Big(\sum_{q\le Q}\frac{q}{\varphi(q)}\sum_{\chi}{}^{*}\Big|\sum_{m=1}^{M}a_m\chi(m)\Big|^2\Big)^{\frac12}\Big(\sum_{q\le Q}\frac{q}{\varphi(q)}\sum_{\chi}{}^{*}\Big|\sum_{n=1}^{N}b_n\chi(n)\Big|^2\Big)^{\frac12}$$

$$\ll(M+Q^2)^{1/2}\Big(\sum_{m=1}^{M}|a_m|^2\Big)^{\frac12}\cdot(N+Q^2)^{1/2}\Big(\sum_{n=1}^{N}|b_n|^2\Big)^{\frac12}.$$

Q3| (i) The simplest argument is to note that there are $\varphi(q)$ Dirichlet characters χ modulo q, each of which is induced by some primitive character χ^* modulo d for some $d|q$. Thus

$$\varphi(q)=\sum_{d|q}\sum_{\chi\,(\mathrm{mod}\,d)}{}^{*}1=\sum_{d|q}\varphi_2(d).\quad\square$$

Alternatively, since $\varphi_2(q)$ is multiplicative, it suffices to verify the identity when $q=p^h$ for primes p. When p is odd, the primitive characters modulo p^h are given by

$$\chi(n)=e\Big(\frac{k\,\mathrm{ind}_g n}{\varphi(p^h)}\Big)\quad\text{where}\ \begin{cases}(p-1)\nmid k, & \text{when } h=1,\\ p\nmid k, & \text{when } h>1.\end{cases}$$

Thus

$$\varphi_2(p^h)=\begin{cases}p-2, & \text{when } h=1\\ \varphi(p^h)-\frac1p\varphi(p^h)=p^{h-2}(p-1)^2, & \text{when } h>1.\end{cases}$$

Hence

$$\sum_{d|p^h}\varphi_2(d)=\sum_{l=0}^{h}\varphi_2(p^l)=\begin{cases}1+p-2=p-1, & \text{when } h=1,\\ 1+p-2+(p-1)^2(p^0+\dots+p^{h-2})=p-1+(p^{h-1}-1)(p-1)\\ \quad=p^{h-1}(p-1), & \text{when } h>1.\end{cases}$$

Thus $\displaystyle\sum_{d|p^h}\varphi_2(d)=\varphi(p^h)$.

When $p=2$ the primitive characters are given by $\pmod{2^h}$

$$\chi(n)=e\Big(\frac{j\mu}{2}+\frac{k\nu}{2^{h-2}}\Big),$$

when $n\equiv(-1)^\mu 5^\nu \pmod{2^h}$, with $j\in\{0,1\}$ and k odd. Thus when $h\ge3$ we have $\varphi_2(2^h)=2\cdot2^{h-3}=2^{h-2}$. Also $\varphi_2(2^2)=1$ and $\varphi_2(2)=0$.

② Hence
$$\sum_{d|2^h} \varphi_2(d) = \sum_{l=0}^{h} \varphi_2(2^l) = \begin{cases} 1+0, & \text{when } h=1 \\ 1+0+1=2, & \text{when } h=2 \\ 1+0+1+\dots+2^{h-2} = 2^{h-1}, & \text{when } h\geq 3. \end{cases}$$
Thus $\displaystyle\sum_{d|2^h} \varphi_2(d) = \varphi(2^h)$. □

(ii) By Möbius inversion, one has
$$\varphi_2(q) = \sum_{d|q} \mu(q/d)\,\varphi(d).$$
Again, by multiplicativity it suffices to examine prime powers — and here in fact we did all of the work in part (i) (the long-winded version). We have
$$\varphi_2(p^h) = \sum_{l=0}^{h} \mu(p^{h-l})\,\varphi(p^l) = \begin{cases} \varphi(p^h) - \varphi(p^{h-1}) \\ \quad = p^h(1-1/p)^2, & \text{when } h\geqslant 2, \\ \varphi(p) - 1 = p-2, & \text{when } h=1. \end{cases}$$
Thus
$$\varphi_2(q) = \prod_{p^h\|q} \varphi_2(p^h) = \prod_{p\|q}(p-2)\cdot \prod_{\substack{p^2|q \\ p^h\|q}} p^h(1-1/p)^2$$
$$= q\prod_{p\|q}\left(1-\frac{2}{p}\right)\cdot \prod_{p^2|q}(1-1/p)^2. \quad □$$

(iii) When $q\equiv 2 \pmod 4$ one has $2\|q$ and hence
$$\prod_{p\|q}(1-2/p) = \left(1-\frac{2}{2}\right)\prod_{\substack{p\|q \\ p>2}}(1-2/p) = 0.$$
Thus $\varphi_2(q)=0$ in this case. Otherwise,
$$\varphi_2(q) = q\prod_{p\|q}\left(\frac{1-2/p}{1-2/p+1/p^2}\right)\cdot \prod_{p|q}\left(1-\frac{1}{p}\right)^2$$
$$= \frac{\varphi(q)^2}{q}\prod_{p\|q}\left(1+\frac{1}{p(p-2)}\right)^{-1}.$$
On noting that
$$\prod_{\substack{p\|q \\ p>2}}\left(1+\frac{1}{p(p-2)}\right) \leqslant \prod_{p\|q}\left(1+\frac{3}{p^2}\right) \leq \prod_{p}\left(1+\frac{1}{p^2}\right)^3 \leq \zeta(2)^3,$$
we see that $\varphi_2(q) \gg \dfrac{\varphi(q)^2}{q} = q\prod_{p|q}(1-1/p)^2 \gg q(\log\log q)^{-2}$ (using our standard estimates for $\varphi(q)$).

(iv) Observe that
$$\operatorname{card}(\mathcal{X}(Q)) = \sum_{1\leq q\leq Q}\varphi_2(q) \gg \sum_{\frac{Q}{2}\leq q\leq Q} q(\log\log q)^{-2} \gg \frac{Q^2}{(\log\log Q)^{+2}}.$$

(2)

Let $\mathcal{E}_\delta$ denote the set of pairs $(q,\chi) \in \mathfrak{X}(Q)$ for which
$$|\psi(x,\chi)| > x^{1/2} (\log x)^{A+1/2+\delta},$$
for any fixed $\delta > 0$. Then one has

$$\sum_{1\le q\le Q} \frac{q}{\varphi(q)} \sum_{\chi}{}^{*} |\psi(x,\chi)|^2 \geqslant \sum_{\substack{1\le q\le Q \\ (q,\chi)\in\mathcal{E}_\delta}} \sum_{\chi}{}^{*} |\psi(x,\chi)|^2$$

$\boxed{QA1}$ $\Uparrow$

$$(x+Q^2)\, x \log x \qquad\qquad > \left(x^{1/2}(\log x)^{A+\frac{1}{2}+\delta}\right)^2 |\mathcal{E}_\delta|.$$

Thus
$$|\mathcal{E}_\delta| \ll \frac{(x+Q^2)\, x\log x}{x(\log x)^{1+2A+2\delta}} = (x+Q^2)(\log x)^{-2A-2\delta}$$
$$\ll x(\log x)^{-2A-2\delta}; \quad \text{when} \quad x^{1/2}(\log x)^{-A} \le Q \le x^{1/2}.$$

But
$$\operatorname{card}(\mathfrak{X}(Q)) \gg \frac{Q^2}{(\log\log Q)^2} \gg x(\log x)^{-2A}(\log\log x)^{-2}.$$

Hence
$$\frac{|\mathcal{E}_\delta|}{\operatorname{card}(\mathfrak{X}(Q))} \ll \frac{x(\log x)^{-2A-2\delta}}{x(\log x)^{-2A}(\log\log x)^{-2}} \ll (\log x)^{-\delta} \to 0 \text{ as } x\to\infty.$$

Then a proportion $1-o_\delta(1)$ of the pairs $(q,\chi)\in\mathfrak{X}(Q)$ satisfy
$$|\psi(x,\chi)| \ll x^{1/2}(\log x)^{A+1/2+\delta},$$
for any fixed $\delta>0$. □

Q4 (i) One has
$$\left(\frac{1}{\zeta(s)} - G(s)\right)\left(1-\zeta(s)G(s)\right) = \frac{1}{\zeta(s)} - \left(2G(s) - G(s)^2\zeta(s)\right),$$
hence result. □

(ii) Examine Dirichlet series expansions of left and right hand sides. Thus
$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \qquad 2G(s) = \sum_{k\le V} \frac{2\mu(k)}{k^{s}} = \sum_{n=1}^{\infty} \frac{a_1(n)}{n^s},$$
$$-G(s)^2\zeta(s) = -\sum_{n=1}^{\infty} \sum_{\substack{k_1k_2 l = n \\ k_1\le V,\, k_2\le V}} \frac{\mu(k_1)\mu(k_2)}{(k_1k_2 l)^s} = \sum_{n=1}^{\infty} \frac{a_2(n)}{n^s},$$

$$\left(\frac{1}{\zeta(s)} - G(s)\right)\left(1-\zeta(s)G(s)\right) = \left(\sum_{k>V}\mu(k)k^{-s}\right)\left(1-\sum_{n=1}^{\infty}\Big(\sum_{\substack{k'd=n\\k'\le V}}\mu(k')\Big)n^{-s}\right)$$

Note that when $n\le V$, one has

$$\sum_{\substack{k'd=n\\k'\le V}}\mu(k') = \sum_{k'|n}\mu(k') = \begin{cases}1, & \text{when } n=1,\\ 0, & \text{when } n>1.\end{cases}$$

Thus

$$\left(\frac{1}{\zeta(s)} - G(s)\right)\left(1-\zeta(s)G(s)\right) = \left(\sum_{k>V}\mu(k)k^{-s}\right)\left(-\sum_{n>V}\ \sum_{\substack{k'd=n \,\leftrightarrow\, k'|n\\k'\le V}}\mu(k')n^{-s}\right)$$

$$= -\sum_{m=1}^{\infty}\Bigg(\sum_{\substack{nk=m\\k>V\\n>V}}\mu(k)\sum_{\substack{k'|n\\k'\le V}}\mu(k')\Bigg)m^{-s}$$

$$= \sum_{n=1}^{\infty} a_3(n)n^{-s}.$$

Then by comparing Dirichlet series coefficients from part (i), we see that

$$\mu(n) = a_1(n) + a_2(n) + a_3(n). \ \square$$

Q5 (i) Apply the conclusion of Q4:

(ii)
(iii)

$$\sum_{1\le n\le N}\mu(n)e(n\alpha) = \sum_{1\le n\le N}\left(a_1(n)+a_2(n)+a_3(n)\right)e(n\alpha)$$

$$= 2\sum_{1\le n\le V}\mu(n)e(n\alpha) + \sum_{1\le n\le N}\Bigg(-\sum_{\substack{def=n\\d\le V\\e\le V}}\mu(d)\mu(e)\Bigg)e(n\alpha)$$

$$+ \sum_{1\le n\le N}\Bigg(-\sum_{\substack{dk=n\\d>V\\k>V}}\mu(d)\sum_{\substack{e|k\\e\le V}}\mu(e)\Bigg)e(n\alpha)$$

$$= \underbrace{2\sum_{1\le n\le V}\mu(n)e(n\alpha)}_{T_1} + \underbrace{-\sum_{m\le V^2}\Bigg(\sum_{\substack{de=m\\d\le V\\e\le V}}\mu(d)\mu(e)\Bigg)\sum_{1\le n\le N/m}e(nm\alpha)}_{T_2}$$

$$+ \underbrace{-\sum_{V<d\le N/V}\mu(d)\sum_{V<k\le N/d}\Bigg(\sum_{\substack{e|k\\e\le V}}\mu(e)\Bigg)e(dk)}_{T_3} \quad \left(\begin{smallmatrix} d & k & e\\ \updownarrow & \updownarrow & \updownarrow\\ m & n & d\end{smallmatrix}\right) \ \square$$

(iv) Observe ~~that~~ by a trivial estimate, one has $T_1 \le 2V$. Also, since

$$\sum_{\substack{de=m\\ d\le V\\ e\le V}} \mu(d)\mu(e) \le \tau(m),$$

we see ~~that~~

$$|T_2| \le \sum_{m\le V^2} \tau(m) \Big| \sum_{1\le n\le N/m} e(nm\alpha) \Big|$$

$$\ll \sum_{m\le V^2} \underset{V^{\varepsilon}}{\underset{\Uparrow}{\tau(m)}} \min\{ N/m, \|m\alpha\|^{-1} \}$$

$$\ll V^{\varepsilon} N^{1+\varepsilon} \left(q^{-1} + (N/V^2)^{-1} + qN^{-1} \right).$$

Finally, since $b(n) := \sum_{\substack{d|n\\ d\le V}} \mu(d) \ll \varepsilon$, we may apply our methods from the treatment of Prime Number sums to confirm ~~that~~

$$|T_3|^2 \ll N^{\varepsilon} \max_{V\le M\le N/V} \Big| \sum_{M<m\le 2M} \mu(m) \sum_{V<k\le N/m} b(k)\, e(mk\alpha) \Big|^2$$

$$\ll N^{\varepsilon} \max_{V\le M\le N/V} \Big(\sum_{M<m\le 2M} |\mu(m)|^2 \Big) \Big(\sum_{M<m\le 2M} \Big| \sum_{V<k\le N/m} b(k)\, e(mk\alpha) \Big|^2 \Big)$$

$$\ll N^{\varepsilon} \max_{V\le M\le N/V} M \Big(\sum_{k\le N/M} |b(k)|^2 \Big) \max_{k_1\le N/M} \sum_{k_2\le N/M} \Big| \sum_{\substack{M<m\le 2M\\ m\le N/k_i\\ (i=1,2)}} e(m(k_2-k_1)\alpha) \Big|$$

$$\ll N^{1+2\varepsilon} \cdot N^{1+\varepsilon} \left(q^{-1} + V^{-1} + qN^{-1} \right),$$

whence

$$T_3 \ll N^{1+2\varepsilon} \left(q^{-1/2} + V^{-1/2} + q^{1/2} N^{-1/2} \right).$$

By combining ~~these estimates~~ and taking $V = N^{2/5}$, we find ~~that~~

$$\sum_{1\le n\le N} \mu(n)\, e(n\alpha) \ll N^{\varepsilon} \left(Nq^{-1/2} + N^{4/5} + q^{\frac12} N^{1/2} \right),$$

as desired. // (The choice $V = N^{2/5}$ is motivated by $V^2/N = V^{-\frac12}$).