MA598CNUM ANALYTIC NUMBER THEORY, II. PROBLEMS 3

TO BE HANDED IN BY MONDAY 8TH MARCH 2021

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

A1. Let $E(X) = \text{card}\{n \leqslant X : 2n \text{ is not equal to the sum of two distinct primes}\}$. By recalling that for any $A > 0$, one has $E(X) \ll X(\log X)^{-A}$, show that there are infinitely many 3-term progressions in prime numbers of the shape

$$p_1 - 2p_2 + p_3 = 0 \quad (p_1 \neq p_2).$$

A2. Assume a refinement to the conclusion of problem C5 on problem set 2. Thus, suppose that whenever $\alpha \in \mathbb{R}$, $q \in \mathbb{N}$ and $a \in \mathbb{Z}$ satisfy $|\alpha - a/q| \leqslant q^{-2}$, then for each $\varepsilon > 0$ one has

$$\sum_{1 \leqslant n \leqslant N} \mu(n)e(n\alpha) \ll (\log N)^3 (Nq^{-1/2} + N^{4/5+\varepsilon} + N^{1/2}q^{1/2}).$$

Let $\mathfrak{M}$ denote the union of the intervals $\mathfrak{M}(q,a) = \{\alpha \in [0,1) : |\alpha - a/q| \leqslant QN^{-1}\}$, with $0 \leqslant a \leqslant q \leqslant Q$ and $(a,q) = 1$, and put $\mathfrak{m} = [0,1) \setminus \mathfrak{M}$. Show that when $Q = (\log N)^B$ with $B > 0$, then one has

$$\sup_{\alpha \in \mathfrak{m}} \left| \sum_{1 \leqslant n \leqslant N} \mu(n)e(n\alpha) \right| \ll N(\log N)^{3-B/2}.$$

B3. This question derives an asymptotic formula for the (weighted) number of 3-term arithmetic progressions in prime numbers of size at most N. Write

$$f(\alpha) = \sum_{p \leqslant N} (\log p)e(p\alpha).$$

Also, define the major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$ as in question A2.
(i) Prove that

$$\int_{\mathfrak{m}} f(\alpha)^2 f(-2\alpha)\, \mathrm{d}\alpha \ll N^2 (\log N)^{(7-B)/2}.$$

(ii) Show that

$$\int_{\mathfrak{M}} f(\alpha)^2 f(-2\alpha)\, \mathrm{d}\alpha = \mathfrak{S} J(N) + O(N^2(\log N)^{-B/2}),$$

where

$$J(N) = \text{card}\{\mathbf{n} \in [1,N]^3 \cap \mathbb{Z}^3 : n_1 - 2n_2 + n_3 = 0\}$$

and

$$\mathfrak{S} = \sum_{q=1}^{\infty} \frac{\mu(q)^2 \mu(q/(q,2))}{\phi(q)\phi(q/(q,2))}.$$

(iii) Hence conclude that for any $A>0$, one has

$$\sum_{\substack{1\leqslant p_1,p_2,p_3\leqslant N\\ p_1-2p_2+p_3=0}} (\log p_1)(\log p_2)(\log p_3) = N^2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)+O(N^2(\log N)^{-A}).$$

B4. Recall the definitions of $f(\alpha)$, $\mathfrak{M}$ and $\mathfrak{m}$ from question B3. Let $\mathcal{Z}(N)$ denote the set of integers n with $1\leqslant n\leqslant N$ for which

$$\left|\int_{\mathfrak{m}} f(\alpha)^2 e(-2n\alpha)\,\mathrm{d}\alpha\right| > N(\log N)^{-A}.$$

Also, write $Z=\mathrm{card}(\mathcal{Z}(N))$ and define the unimodular number η_n by putting

$$\eta_n\int_{\mathfrak{m}} f(\alpha)^2 e(-2n\alpha)\,\mathrm{d}\alpha = \left|\int_{\mathfrak{m}} f(\alpha)^2 e(-2n\alpha)\,\mathrm{d}\alpha\right|,$$

when the right hand side is non-zero, and otherwise put $\eta_n=0$.
(i) Define

$$K(\alpha)=\sum_{n\in\mathcal{Z}(N)}\eta_n e(2n\alpha).$$

Explain why one has

$$\int_{\mathfrak{m}} f(\alpha)^2 K(-\alpha)\,\mathrm{d}\alpha > ZN(\log N)^{-A}.$$

(ii) Show that

$$\int_{\mathfrak{m}} f(\alpha)^2 K(-\alpha)\,\mathrm{d}\alpha \leqslant \left(\int_{\mathfrak{m}} |f(\alpha)|^4\,\mathrm{d}\alpha\right)^{1/2} Z^{1/2}.$$

(iii) Hence deduce that

$$ZN(\log N)^{-A} \ll Z^{1/2}\left(N^3(\log N)^{6-B}\right)^{1/2}.$$

(iv) Conclude that

$$\int_{\mathfrak{m}} f(\alpha)^2 e(-2n\alpha)\,\mathrm{d}\alpha \ll N(\log N)^{-B/4}$$

for all integers with $1\leqslant n\leqslant N$, except possibly for a set of integers of cardinality at most $O(N(\log N)^{6-B/2})$.

C5. Obtain an asymptotic formula for

$$\sum_{\substack{p_1,p_2,p_3\text{ prime}\\ p_1+p_2+2p_3=n}} (\log p_1)(\log p_2)(\log p_3),$$

and hence deduce that all large even integers n may be written in the shape

$$n=p_1+p_2+2p_3,$$

with p_1,p_2,p_3 prime.