MA598 CNUM Analytic Number Theory, II. Problems 3 – Solutions.

Q1 It follows from what we did in class that

$$E(X) = \operatorname{card}\{n \le X : \ 2n \text{ is } \underline{\text{not}} \text{ represented as the sum of two primes in two or more ways}\}$$

satisfies $E(X) \ll_A X(\log X)^{-A}$, for $A>0$. Take $A=2$. Then for all but $O(X(\log X)^{-2})$ primes p with $p \le X$, there is a representation $2p = p_1+p_2$, with p_1, p_2 primes and $p_1 \ne p_2$. Then

$$\#\{p_i \le X \ (i=1,2,3) : \quad p_1 - 2p_2 + p_3 = 0 \ \ \& \ \ p_1 \ne p_2\}$$

$$\geqslant \pi(X) - O(X(\log X)^{-2}) \gg X/(\log X) \to \infty \text{ as } X \to \infty.$$

So there are infinitely many 3-term progressions in primes of the shape $p_1 - 2p_2 + p_3 = 0$ with $p_1 \ne p_2$. //

Q2 Suppose that $\alpha \in \mathfrak{m}$. By Dirichlet's approximation theorem, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q)=1$, $0 \le a \le q \le N/Q$ such that $|\alpha - a/q| \le q^{-1}QN^{-1} \le QN^{-1}$. If one were to have $q \le Q$, then $\alpha \in \mathfrak{M}_*$, so $q > Q$. But then

$$\sup_{\alpha \in \mathfrak{m}} \Big| \sum_{1 \le n \le N} \mu(n)\, e(n\alpha) \Big| \ll (\log N)^3 \left(NQ^{-\frac{1}{2}} + N^{\frac{4}{5}+\varepsilon} + N^{\frac{1}{2}}(N/Q)^{\frac{1}{2}} \right)$$

$$\ll (\log N)^3 \cdot N(\log N)^{-B/2} \ll N(\log N)^{3-B/2}. //$$

Note that a Siegel–Walfisz theorem applies to the Möbius function in a similar manner to the von Mangoldt function. Using this result, one may show that

$$\sup_{\alpha \in \mathfrak{M}} \Big| \sum_{1 \le n \le N} \mu(n)\, e(n\alpha) \Big| \ll_A N(\log N)^{-A},$$

whence

$$\sup_{\alpha \in [0,1)} \Big| \sum_{1 \le n \le N} \mu(n)\, e(n\alpha) \Big| \ll_A N(\log N)^{-A}, \quad \text{for any } A > 0.$$

Q3 (i) From Lemma 8.1 one sees that $\sup_{\alpha \in \mathfrak{m}} |f(\alpha)| \ll N(\log N)^{(5-B)/2}$.

Hence, by Schwarz's inequality and orthogonality,

$$\int_{\mathfrak{m}} f(\alpha)^2 f(-2\alpha)\, d\alpha \le \Big(\sup_{\alpha \in \mathfrak{m}} |f(\alpha)|\Big) \Big(\int_0^1 |f(\alpha)|^2 d\alpha\Big)^{\frac{1}{2}} \Big(\int_0^1 |f(2\alpha)|^2 d\alpha\Big)^{\frac{1}{2}}$$

$$\ll N(\log N)^{(5-B)/2} \sum_{p \le N} (\log p)^2$$

$$\overset{\text{PNT}}{\ll} N(\log N)^{(5-B)/2}\, N \log N = N^2 (\log N)^{(7-B)/2}. \quad \square$$

② (ii) By applying Lemma 8.3, we see that $\exists c>0$ s.t. when $\alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$,

$$f(\alpha) = \frac{\mu(q)}{\varphi(q)} v(\alpha - a/q) + O(N\exp(-c\sqrt{\log N})), \qquad v(\beta) = \sum_{n\le N} e(\beta n).$$

Noting that under the same condition on α, one has

$$|2\alpha - 2a/q| \le 2Q/N,$$

and that $\frac{2a}{q} = \frac{a'}{q'}$, where $(a',q')=1$, $a' = 2a/(q,2)$, $q'=q/(q,2)$,

we similarly find that

$$f(-2\alpha) = \frac{\mu(q/(q,2))}{\varphi(q/(q,2))} v(-2\alpha + 2a/q) + O(N\exp(-c\sqrt{\log N})).$$

Hence

$$f(\alpha)^2 f(-2\alpha) = \frac{\mu(q)^2\mu(q/(q,2))}{\varphi(q)^2\varphi(q/(q,2))} v(\alpha - a/q)^2 v(-2\alpha + 2a/q) + O(N^3\exp(-c\sqrt{\log N})).$$

Since $\operatorname{mes}(\mathfrak{M}) \ll \sum_{1\le q\le Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} QN^{-1} \ll Q^3 N^{-1}$, it follows that

$$\int_{\mathfrak{M}} f(\alpha)^2 f(-2\alpha)\,d\alpha = \mathfrak{S}(Q)J(N;Q) + O(Q^3N^{-1}\cdot N^3\exp(-c\sqrt{\log N})),$$

where

$$J(N;Q) = \int_{-Q/N}^{Q/N} v(\beta)^2 v(-2\beta)\,d\beta$$

and

$$\mathfrak{S}(Q) = \sum_{1\le q\le Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} \frac{\mu(q)^2\mu(q/(q,2))}{\varphi(q)^2\varphi(q/(q,2))}.$$

Here, since $v(\beta) \ll \min\{N, \|\beta\|^{-1}\}$, we see that

$$\int_{Q/N}^{1/2} |v(\beta)^2 v(-2\beta)|\,d\beta \ll N\int_{Q/N}^{1/2} \beta^{-2}\,d\beta \ll N^2Q^{-1},$$

whence

$$J(N;Q) = J(N) + O(N^2Q^{-1}),$$

where

$$J(N) = \int_{-\frac12}^{\frac12} v(\beta)^2 v(-2\beta)\,d\beta \overset{\text{orthog.}}{=} \#\{n_1 - 2n_2 + n_3 = 0 : 1\le n_i\le N\ (i=1,2,3)\}.$$

If n_1, n_3 are not of the same parity, there are no solutions to this equation, and if n_1, n_3 are either both odd or both even then $n_2 = \frac{n_1+n_3}{2}$.

Thus
$$J(N) = \left\lfloor \frac{N}{2} \right\rfloor^2 + \left\lfloor \frac{N+1}{2} \right\rfloor^2 = \tfrac{1}{2}N^2 + O(N).$$

Meanwhile,
$$\sum_{q>Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \frac{\mu(q)^2 \mu(q/(q,2))}{\varphi(q)^2 \varphi(q/(q,2))} \right| \ll \sum_{q>Q} \frac{1}{\varphi(q)^2} \ll Q^{\varepsilon - 1};$$

so that
$$\mathfrak{S}(Q) = \mathfrak{S} + O(Q^{\varepsilon-1}),$$

where
$$\mathfrak{S} = \sum_{q=1}^{\infty} \frac{\mu(q)^2 \mu(q/(q,2))}{\varphi(q)\, \varphi(q/(q,2))} \ll 1.$$

Collecting terms and recalling that $Q = (\log N)^B$, we see that
$$\int_{\mathfrak{M}} f(\alpha)^2 f(-2\alpha)\, d\alpha = \left(\mathfrak{S} + O((\log N)^{\varepsilon - B})\right)\left(J(N) + O(N^2 (\log N)^{-B})\right)$$
$$+ O\left(N^2 \exp(-\tfrac{1}{2} c \sqrt{\log N})\right)$$

$$= \mathfrak{S}\, J(N) + O(N^2 (\log N)^{-B/2}). \quad \square$$

(iii) We have $J(N) = \frac{1}{2}N^2 + O(N)$ and by multiplicativity,
$$\mathfrak{S} = \prod_p \sigma_p,$$
where
$$\sigma_2 = 1 + \sum_{h=1}^{\infty} \frac{\mu(2^h)^2 \mu(2^{h-1})}{2^{h-1}\, \phi(2^{h-1})} = 1 + 1 = 2$$

and when $p > 2$,
$$\sigma_p = 1 + \sum_{h=1}^{\infty} \frac{\mu(p^h)^3}{\varphi(p^h)^2} = 1 - \frac{1}{(p-1)^2}.$$

Then
$$\mathfrak{S} = 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right),$$

and
$$\int_{\mathfrak{M}} f(\alpha)^2 f(-2\alpha)\, d\alpha = \tfrac{1}{2}N^2 \cdot 2 \prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right) + O(N),$$

Whence
$$\sum_{\substack{1 \le p_1, p_2, p_3 \le N \\ p_1 - 2p_2 + p_3 = 0}} (\log p_1)(\log p_2)(\log p_3) = \int_{\mathfrak{M}} f(\alpha)^2 f(-2\alpha)\, d\alpha + \int_{\mathfrak{m}} f(\alpha)^2 f(-2\alpha)\, d\alpha$$

$$= N^2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) + O\left(N^2 (\log N)^{(7-B)/2}\right)$$

$$= N^2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) + O\left(N^2 (\log N)^{-A}\right),$$

for any $A > 0$ (on taking $B = 7 + 2A$). □ //

Q4 (i) One has

$$\int_{\mathfrak{m}} f(\alpha)^2 K(-\alpha)\, d\alpha = \sum_{n \in \mathcal{Z}(N)} \eta_n \int_{\mathfrak{m}} f(\alpha)^2 e(-2n\alpha)\, d\alpha = \sum_{n \in \mathcal{Z}(N)} \left| \int_{\mathfrak{m}} f(\alpha)^2 e(-2n\alpha)\, d\alpha \right|$$

$$> N (\log N)^{-A} \operatorname{card}(\mathcal{Z}(N)) = Z N (\log N)^{-A}. \ □$$

(ii) By Schwarz's inequality and orthogonality,

$$\int_{\mathfrak{m}} f(\alpha)^2 K(-\alpha)\, d\alpha \le \left(\int_{\mathfrak{m}} |f(\alpha)|^4\, d\alpha\right)^{1/2} \Bigg(\underbrace{\int_0^1 |K(\alpha)|^2\, d\alpha}_{\displaystyle \sum_{n \in \mathcal{Z}(N)} |\eta_n|^2 = \operatorname{card}(\mathcal{Z}(N))}\Bigg)^{\frac{1}{2}}$$

$$= Z^{1/2} \left(\int_{\mathfrak{m}} |f(\alpha)|^4\, d\alpha\right)^{\frac{1}{2}}. \ □$$

(iii) Hence

$$Z N (\log N)^{-A} < \int_{\mathfrak{m}} f(\alpha)^2 K(-\alpha)\, d\alpha \le Z^{\frac{1}{2}} \left(N^3 (\log N)^{6-B}\right)^{\frac{1}{2}},$$

since

$$\int_{\mathfrak{m}} |f(\alpha)|^4\, d\alpha \le \left(\sup_{\alpha \in \mathfrak{m}} |f(\alpha)|\right)^2 \int_0^1 |f(\alpha)|^2\, d\alpha$$

$$\ll \left(N (\log N)^{(5-B)/2}\right)^2 \sum_{p \le N} (\log p)^2$$

$$\ll N^3 (\log N)^{6-B}.$$

Hence $\quad Z N (\log N)^{-A} \ll Z^{\frac{1}{2}} \left(N^3 (\log N)^{6-B}\right)^{\frac{1}{2}}$. □

(iv). From this last statement, we see that $Z^{\frac{1}{2}} \ll N^{\frac{1}{2}} (\log N)^{\frac{6-B}{2} + A}$, whence $Z \ll N (\log N)^{6-B+2A}$. By taking $A = B/4$, it follows that with at most $O\left(N (\log N)^{6-B/2}\right)$ exceptions, one has $\int_{\mathfrak{m}} f(\alpha)^2 e(-2n\alpha)\, d\alpha \ll N (\log N)^{-A} = N (\log N)^{-B/4}$. □ //

⑤

Q5| We may follow the argument of Q3, applying the triangle inequality where necessary. Thus, putting $N = n$,

$$\sum_{\substack{p_1, p_2, p_3 \text{ prime} \\ p_1 + p_2 + 2p_3 = n}} (\log p_1)(\log p_2)(\log p_3) = \int_0^1 f(\alpha)^2 f(2\alpha)\, e(-n\alpha)\, d\alpha$$

$$= \underbrace{\int_{\mathfrak{M}} f(\alpha)^2 f(2\alpha) e(-n\alpha) d\alpha}_{I(\mathfrak{M})} + \underbrace{\int_{\mathfrak{m}} f(\alpha)^2 f(2\alpha) e(-n\alpha) d\alpha}_{I(\mathfrak{m})}.$$

But

$$I(\mathfrak{m}) \le \int_{\mathfrak{m}} |f(\alpha)^2 f(2\alpha)|\, d\alpha \le \sup_{\alpha \in \mathfrak{m}} |f(\alpha)| \left(\int_0^1 |f(\alpha)|^2 d\alpha \right)^{\frac{1}{2}} \left(\int_0^1 |f(2\alpha)|^2 d\alpha \right)^{\frac{1}{2}}$$

$$\ll N^2 (\log N)^{(7-B)/2}.$$

Also,

$$f(\alpha)^2 f(2\alpha)\, e(-n\alpha) = e\left(\frac{na}{q}\right) \frac{\mu(q)^2 \mu(q/(q,2))}{\varphi(q)^2 \varphi(q/(q,2))} \, v(\alpha - a/q)^2 \, v(2\alpha - 2a/q) \times e(-n(\alpha - a/q)),$$

for $\alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$, whence

$$I(\mathfrak{M}) = \mathfrak{S}'(Q)\, J'(N;Q) + O\left(N^2 \exp\left(-\tfrac{1}{2} c \sqrt{\log N}\right)\right),$$

where

$$\mathfrak{S}'(Q) = \sum_{1 \le q \le Q} \underbrace{\sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{na}{q}\right)}_{\frac{\mu(q/(q,n))\varphi(q)}{\varphi(q/(q,n))}} \cdot \frac{\mu(q)^2 \mu(q/(q,2))}{\varphi(q)^2 \varphi(q/(q,2))}$$

$$= \sum_{1 \le q \le Q} \frac{\mu(q)^2 \mu(q/(q,2)) \mu(q/(q,n))}{\varphi(q) \varphi(q/(q,2)) \varphi(q/(q,n))},$$

and

$$J'(N;Q) = \int_{-Q/N}^{Q/N} v(\beta)^2 v(2\beta)\, e(-\beta n)\, d\beta$$

$$= J'(N) + O(N^2 Q^{-1}),$$

where

$$J'(N) = \int_{-\frac{1}{2}}^{\frac{1}{2}} v(\beta)^2 v(+2\beta)\, e(-\beta n)\, d\beta.$$

We have, by orthogonality,

$$J'(N) = \#\{ n_1 + n_2 + 2n_3 = n : 1 \le n_i \le N \ (i=1,2,3) \}$$

$$= \sum_{1 \le n_3 \le \frac{n-2}{2}} \sum_{1 \le n_2 \le n - 2n_3 - 1} 1$$

$$= \sum_{1 \le n_3 \le \frac{n-2}{2}} (n - 2n_3 - 1) = n\frac{(n-2)}{2} - 2\cdot\frac{1}{2}\left(\frac{n-2}{2}\right)\left(\frac{n}{2}\right) - \frac{n-2}{2}$$

$$= \frac{1}{4}n^2 + O(n).$$

Also, $\mathfrak{S}'(Q) = \mathfrak{S}' + O(Q^{\varepsilon-1})$, where

$$\mathfrak{S}' = \sum_{q=1}^{\infty} \frac{\mu(q)^2 \mu(q/(q,2))\, \mu(q/(q,n))}{\varphi(q)\varphi(q/(q,2))\,\varphi(q/(q,n))} = \prod_p \sigma_p,$$

where

$$\sigma_2 = 1 + \sum_{h=1}^{\infty} \frac{\mu(2^h)^2 \mu(2^{h-1})\,\mu(2^h/(2^h,n))}{2^{h-1}\varphi(2^{h-1})\,\varphi(2^h/(2^h,n))} = 1 + \frac{\mu(2/(n,2))}{\varphi(2/(n,2))}$$

$$= \begin{cases} 0, & n \text{ odd}, \\ 2, & n \text{ even}, \end{cases}$$

and when $p > 2$,

$$\sigma_p = 1 + \sum_{h=1}^{\infty} \frac{\mu(p^h)^3 \mu(p^h/(p^h,n))}{\varphi(p^h)^2\varphi(p^h/(p^h,n))} = 1 - \frac{\mu(p/(p,n))}{(p-1)^2\varphi(p/(p,n))}$$

$$= \begin{cases} 1 - 1/(p-1)^2, & \text{when } p \mid n, \\ 1 + 1/(p-1)^3, & \text{when } p \nmid n. \end{cases}$$

Thus

$\mathfrak{S}' = 0$ when n is odd, and when n is even,

$$\mathfrak{S}' = 2 \prod_{\substack{p \mid n \\ p>2}} \left(1 - 1/(p-1)^2\right) \prod_{\substack{p \nmid n \\ p>2}} \left(1 + 1/(p-1)^3\right)$$

Hence, for any $A>0$,

$$\sum_{\substack{p_1,p_2,p_3 \text{ prime} \\ p_1+p_2+2p_3=n}} (\log p_1)(\log p_2)(\log p_3) = \begin{cases} \overbrace{\frac{1}{2}n^2 \prod_{p\mid n}\left(1 - 1/(p-1)^2\right)\prod_{p\nmid n}\left(1 + 1/(p-1)^3\right)}^{\gg n^2} + O(n^2(\log n)^{-A}) & \text{when } n \text{ is even} \\ O(n^2(\log n)^{-A}), & \text{when } n \text{ is odd}. \end{cases}$$

$\to \infty$ as n (odd) $\to \infty$.

Notice that when n is odd, ~~then~~ $p_1 + p_2 + 2p_3 = n$ is soluble only when $p_1 = 2$ or $p_2 = 2$, in which case (respectively) $p_2 = n-2-2p_3$ or $p_1 = n - 2 - 2p_3$ (and this is reminiscent of binary Goldbach problem). Then all large even integers are represented as $n \to \infty$. //