MA598CNUM ANALYTIC NUMBER THEORY, II. PROBLEMS 4

TO BE HANDED IN BY MONDAY 22ND MARCH 2021

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

A1. Suppose that χ is a non-principal character modulo q and that $(a,q)=1$. Apply the Pólya-Vinogradov inequality to show that, whenever M and N are integers with $N>0$ and $(a,q)=1$, one has

$$\sum_{n=M+1}^{M+N} \chi(an+b) \ll \sqrt{q}\log q.$$

A2. Suppose that p is a large prime and χ is a non-principal character modulo p. Suppose also that $\delta>0$ and $N>p^{\frac{1}{4}+\delta}$. By applying Burgess' inequality, prove that there is a positive number τ, depending at most on δ, such that

$$\sum_{1\leqslant n\leqslant N} \chi(n) \ll Np^{-\tau}.$$

B3. Let p be a large prime number, and write $\Xi(M,N;p)$ for the number of primitive roots modulo p in the interval $[M+1,M+N]$.
(i) By substituting Burgess' inequality for the Pólya-Vinogradov inequality in the argument of the proof of Corollary 9.3, show that whenever $r\in\mathbb{N}$, one has

$$\Xi(M,N;p)=\frac{\phi(p-1)}{p}N+O_{\varepsilon,r}\left(N^{1-1/r}p^{\varepsilon+(r+1)/(4r^2)}\right).$$

(ii) Prove that when $\varepsilon>0$, there is always a primitive root modulo p in any interval of integers of length exceeding $p^{1/4+\varepsilon}$.

B4. Let p be a large prime with $p\equiv 1 \pmod 5$, and let χ be a character modulo p of order 5, so that $\chi\neq\chi_0$ but $\chi^5=\chi_0$.
(i) Prove that when $(n,p)=1$, one has

$$\frac{1}{5}\sum_{j=1}^{5}\chi^j(n)=\begin{cases}1, & \text{when } n \text{ is a fifth power modulo } p,\\ 0, & \text{otherwise.}\end{cases}$$

(ii) Let M and N be integers with $N>0$. Show that the number of integers $n\in[M+1,M+N]$ which are fifth powers modulo p is equal to

$$\frac{1}{5}\sum_{j=1}^{5}\sum_{n=M+1}^{M+N}\chi^j(n).$$

(iii) Apply the Pólya-Vinogradov inequality to deduce that when $\delta>0$, there is a fifth power modulo p in every interval of integers of length exceeding $p^{1/2+\delta}$.

C5. Let p be a large prime number with $p \equiv 1 \pmod 3$, and suppose that χ is a cubic character modulo p, so that $\chi^3 = \chi_0$ with $\chi \neq \chi_0$. Suppose also that $1 \leqslant x < p$ and y is a positive number with $y \leqslant x < y^2$.
(i) Apply the argument of the proof of Corollary 9.2 to show that when $\chi(n) = 1$ for $1 \leqslant n \leqslant y$, one has

$$\left| \sum_{1 \leqslant n \leqslant x} \chi(n) \right| \geqslant \psi(x, y) - \frac{1}{2} \sum_{y < \pi \leqslant x} \left\lfloor \frac{x}{\pi} \right\rfloor .$$

Here, the sum over π is implicitly restricted to prime numbers π.
(ii) Suppose, if possible, that $\chi(n) = 1$ for $1 \leqslant n \leqslant y$. Apply the Pólya-Vinogradov inequality to obtain a contradiction when $x = p^{1/2}(\log p)^2$ and $y = x^\theta$ for any $\theta > e^{-2/3}$.
(iii) Hence deduce that there is a positive integer n with $n \ll p^{1/(2e^{2/3})+\varepsilon}$ with the property that $\chi(n) \neq 1$.
(iv) Prove that there is a positive integer n with $n \ll p^{1/(4e^{2/3})+\varepsilon}$ with the property that $\chi(n) \neq 1$.