MA 598 CNUM Analytic Number Theory, II. Problems 4 – Solutions.

Q1 Since $(a,q)=1$, one has

$$\sum_{n=M+1}^{M+N} \chi(an+b) = \sum_{n=M+1}^{M+N} \chi(a)\chi(n+ba^{-1}) = \chi(a) \sum_{n=M+ba^{-1}+1}^{M+N+ba^{-1}} \chi(n)$$

$$\ll |\chi(a)| \cdot q^{1/2}\log q \qquad \text{(by Polyá – Vinogradov)}$$

$$= \sqrt{q}\,\log q. \;//$$

Q2 By Burgess' inequality, for $r \in \mathbb{N}$ one has

$$\sum_{n=M+1}^{M+N} \chi(n) \ll N\left(\frac{p^{\frac{r+1}{4r}}}{N}\right)^{\frac{1}{r}} (\log p).$$

Let $r = \lceil 1/2\delta \rceil$. Then we see ~~that~~ $p^{\frac{r+1}{4r}} \leqslant p^{\frac{1}{4}+\frac{\delta}{2}} < Np^{-\delta/2}$,

whence
$$\sum_{n=1}^{N} \chi(n) \ll N p^{-\frac{\delta}{2r}} \leqslant Np^{-\delta^2}.$$

So the desired conclusion holds with $\tau = \delta^2$. //

Q3 (i) One has

$$\Sigma(M,N;p) = \frac{1}{q}\sum_{d|q} \varphi(q/d)\mu(d) \sum_{\substack{\chi \\ \mathrm{ord}\,\chi = d}} \sum_{n=M+1}^{M+N} \chi(n)$$

$$= \frac{\varphi(p-1)}{p} N + O(1) + O\Bigg(\frac{1}{q} \sum_{\substack{d|q \\ d>1}} \varphi(q/d)|\mu(d)| \sum_{\substack{\chi \\ \mathrm{ord}\,\chi=d}} \Big| \sum_{n=M+1} \chi(n) \Big| \Bigg),$$

where $q = \prod_{\substack{\pi | p-1 \\ \pi \text{ prime}}} \pi$. Thus, by applying Burgess' inequality, we see ~~that~~

$$\Sigma(M,N;p) = \frac{\varphi(p-1)}{p} N + O(1) + O_r\Bigg(\frac{1}{q} \sum_{\substack{d|q \\ d>1}} \varphi\left(\tfrac{q}{d}\right)\varphi(d)\, N^{1-\frac{1}{r}} p^{\frac{r+1}{4r^2}} \log p \Bigg)$$

$$= \frac{\varphi(p-1)}{p} N + O_r\left(\frac{\varphi(q)}{q}\, 2^{\omega(p-1)} N^{1-\frac{1}{r}} p^{\frac{r+1}{4r^2}} \log p \right)$$

$$= \frac{\varphi(p-1)}{p} N + O_{\varepsilon,r}\left(N^{1-1/r} p^{\frac{r+1}{4r^2}+\varepsilon} \right). \;\square$$

(ii) Let $\delta > 0$ and put $N = p^{1/4+\delta}$. Then on taking $r = \lceil 1/2\delta \rceil$,

 we see that
$$N^{1-1/r}\, p^{\frac{r+1}{4r^2}} = N\left(\frac{p^{\frac{r+1}{4r}}}{N}\right)^{\frac{1}{r}} \le N p^{-\delta^2},$$
whence for ε sufficiently small one has
$$N^{1-1/r}\, p^{\frac{r+1}{4r^2}+\varepsilon} \le N p^{-\delta^2+\varepsilon} < N p^{-\frac{1}{2}\delta^2} < N(\log\log p)^{-2}$$
$$\ll N\frac{\varphi(p-1)}{p}(\log\log p)^{-1}.$$
Thus, for p large enough,
$$\Sigma(M,N;p) = \frac{\varphi(p-1)}{p}N + O_{\varepsilon,r}\left(N^{1-1/r}\, p^{\varepsilon+\frac{r+1}{4r^2}}\right)$$
$$\ge \frac{\varphi(p-1)}{p}N\left(1 - O\left(\frac{1}{\log\log p}\right)\right) \ge 1.$$
So there is always a primitive root modulo p in any interval of integers of length exceeding $p^{1/4+\varepsilon}$. //

Q4 (i) There is a positive integer u and a primitive root g modulo p so ~~that~~
$$\chi(n) = e\left(\frac{u\,\mathrm{ord}_g(n)}{p-1}\right).$$
Since χ has order 5, we have for $(n,p)=1$ ~~that~~
$$1 = \chi_0(n) = \chi(n)^5 = \chi(n^5) = e\left(\frac{5u\,\mathrm{ord}_g(n)}{p-1}\right),$$
whence in particular $e\left(\frac{5u\,\mathrm{ord}_g(g)}{p-1}\right) = e\left(\frac{5u}{p-1}\right) = 1$. So $\frac{p-1}{5}\,\Big|\,u$,
and we see ~~that~~ for some ~~integer~~ k we have
$$\chi(n) = e\left(\frac{k}{5}\,\mathrm{ord}_g(n)\right).$$
Since $\chi \ne \chi_0$, we must have $5 \nmid k$.

When n is a 5^{th} power modulo p, we have $n \equiv g^{5m} \pmod p$ for some $m \in \mathbb{Z}$, whence $\chi(n) = e(km) = 1$, so
$$\frac{1}{5}\sum_{j=1}^{5}\chi^j(n) = \frac{1}{5}\sum_{j=1}^{5} 1 = 1.$$
Meanwhile, when n is not a 5^{th} power modulo p, we have $n \equiv g^{5m+l} \pmod p$ for some $l \in \{1,2,3,4\}$. Thus $\chi(n) = e(km + kl/5) = e(kl/5) \ne 1$,
whence
$$\frac{1}{5}\sum_{j=1}^{5}\chi^j(n) = \frac{1}{5}\sum_{j=1}^{5} e(jkl/5) = \frac{1-e(kl)}{1-e(kl/5)}\cdot e(kl/5) = 0.$$

③

Thus $\frac{1}{5}\sum_{j=1}^{5}\chi^j(n) = \begin{cases} 1, & \text{when } n \text{ is a fifth power modulo } p, \\ 0, & \text{when } n \text{ is \underline{not} a fifth power modulo } p. \end{cases}$ □

(ii) One ~~therefore sees that~~

$$\sum_{\substack{n\in[M+1,M+N] \\ n \text{ is a } 5^{\text{th}} \text{ power} \\ \text{modulo } p}} 1 = \sum_{n=M+1}^{M+N}\left(\frac{1}{5}\sum_{j=1}^{5}\chi^j(n)\right) = \frac{1}{5}\sum_{j=1}^{5}\sum_{n=M+1}^{M+N}\chi^j(n). \quad □$$

(iii) The right hand side here is

$$\frac{1}{5}\sum_{j=1}^{4}\sum_{n=M+1}^{M+N}\chi^j(n) + \frac{1}{5}\sum_{n=M+1}^{M+N}\chi_0(n)$$

$$= \frac{1}{5}\sum_{\substack{n=M+1 \\ (n,p)=1}}^{M+N} 1 + O\left(\frac{1}{5}\sum_{j=1}^{4} p^{1/2}\log p\right) \quad \text{by Polya-Vinogradov}$$

$$= \frac{1}{5} N\frac{\varphi(p-1)}{p-1} + O(p^{1/2}\log p)$$

$$\geq \frac{1}{5}N\frac{\varphi(p-1)}{p-1}\left(1 - O\left(\frac{p^{1/2}\log p}{N/\log\log p}\right)\right).$$

Then, provided ~~that~~ $N > Ap^{1/2}\log p\log\log p$ and A is sufficiently large, one must have ~~that~~

$$\sum_{\substack{n\in[M+1,M+N] \\ n \text{ is a } 5^{\text{th}} \text{ power} \\ \text{modulo } p}} 1 > \frac{1}{10}N\frac{\varphi(p-1)}{p-1} \geqslant 1,$$

and so ~~there~~ is a 5^{th} power modulo p in every interval of integers of length exceeding $p^{1/2+\delta}$, for any $\delta > 0$ (p large enough). //

Q5 (i) Since $\chi^3 = \chi_0$ with $\chi \neq \chi_0$, one sees ~~that~~ the non-zero values of $\chi(n)$ are cubic roots of unity ω with $\omega \neq 1$ when $\chi(n)\neq 1$. In particular, when $\chi(n) \neq 1$, one has $\text{Re}(\chi(n)) = -\frac{1}{2}$. Thus, when $y \leq x \leq y^2$, one has

$$\sum_{1\leq n\leq x}\chi(n) = \sum_{\substack{1\leq n\leq x \\ \pi | n \Rightarrow \pi\leq y}}\chi(n) + \sum_{y<\pi\leq x}\sum_{\substack{1\leq n\leq x \\ \pi | n}}\chi(n)$$

(assume $\chi(n) = 1$ for $1 \leq n \leq y$)

$$= \psi(x,y) + \sum_{y<\pi\le x} \chi(\pi)\left\lfloor \frac{x}{\pi}\right\rfloor ,$$

so that

$$\left|\sum_{1\le n\le x} \chi(n)\right| \geq \left| \mathrm{Re} \sum_{1\le n\le x} \chi(n)\right|$$

$$\geq \psi(x,y) - \frac{1}{2}\sum_{y<\pi\le x} \left\lfloor \frac{x}{\pi}\right\rfloor . \quad \square$$

(ii) When $\chi^3=\chi_0$ and $\chi\neq\chi_0$, and $\chi(n)=1$ for $1\le n\le y$, it follows that

$$\left|\sum_{1\le n\le x} \chi(n)\right| = \lfloor x\rfloor - \frac{3}{2}\sum_{y<\pi\le x}\left\lfloor \frac{x}{\pi}\right\rfloor$$

$$= x\left(1-\frac{3}{2}\sum_{y<\pi\le x}\frac{1}{\pi}\right) + O\left(\frac{x}{\log x}\right)$$

$$= x\left(1-\frac{3}{2}\log\left(\frac{\log x}{\log y}\right)\right) + O\left(\frac{x}{\log x}\right).$$

We apply the Pólya-Vinogradov inequality with $x = p^{1/2}(\log p)^2$ to see that

$$x\left(1-\frac{3}{2}\log\left(\frac{\log x}{\log y}\right)\right) + O\left(\frac{x}{\log x}\right) \ll p^{\frac{1}{2}}\log p \asymp \frac{x}{\log x},$$

whence $\quad \frac{3}{2}\log\left(\frac{\log x}{\log y}\right) \geq 1 + O\left(\frac{1}{\log x}\right) \Rightarrow \frac{\log x}{\log y} \geq e^{2/3} + O\left(\frac{1}{\log x}\right)$

$$\Rightarrow \quad \log y \leq e^{-2/3}\log x + O(1)$$

$$\Rightarrow \quad y \ll x^{e^{-2/3}} \ll p^{1/2e^{2/3}+\frac{1}{2}\varepsilon}.$$

Then $\chi(n)\neq 1$ for some integer n with $1\le n \ll p^{1/2e^{2/3}+\varepsilon}$. $\square$

(iii) If instead of the Pólya-Vinogradov inequality, we apply Burgess' inequality then we find that for $r\geq 1$,

$$\left|\sum_{1\le n\le x}\chi(n)\right| \ll x^{1-\frac{1}{r}}\, p^{\frac{r+1}{4r^2}}\log p = x\left(\frac{p^{\frac{r+1}{4r}}(\log p)^r}{x}\right)^{\frac{1}{r}}.$$

We take $x = p^{\frac{1}{4}+\frac{1}{2r}}$ and r large. Then

$$x\left(1-\frac{3}{2}\log\left(\frac{\log x}{\log y}\right)\right) + O\left(\frac{x}{\log x}\right) \ll x\left(\frac{p^{\frac{1}{4}+\frac{1}{4r}}(\log p)^r}{p^{\frac{1}{4}+\frac{1}{2r}}}\right)^{\frac{1}{r}} \ll \frac{x}{\log x},$$

so ~~that~~, as before,

$$\log y \leq e^{-2/3} \log x + o(1)$$

$$\Rightarrow \quad y \ll x^{e^{-2/3}} \ll \left(p^{\frac{1}{4}+\frac{1}{2r}}\right)^{e^{-2/3}}.$$

Since r may be chosen large enough in terms of ε, it follows that

$$y \ll p^{\frac{1}{4e^{2/3}} + \varepsilon},$$

and that $\chi(n) \neq 1$ for some integer n with $1 \leq n \ll p^{\frac{1}{4e^{2/3}} + \varepsilon}$. $\square$ //