MA598CNUM ANALYTIC NUMBER THEORY, II. PROBLEMS 5

TO BE HANDED IN BY MONDAY 12TH APRIL 2021

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

A1. Let a be a positive integer.
(i) Show that

$$\sum_{1\leqslant m\leqslant x} \frac{1}{m} \sum_{\substack{d\mid m\\ d\mid a}} \mu(d) = \frac{\varphi(a)}{a}\log x + O_a(1).$$

(ii) Hence deduce that

$$\sum_{\substack{1\leqslant m\leqslant x\\ (m,a)=1}} \frac{1}{m} = \frac{\varphi(a)}{a}\log x + O_a(1).$$

A2. Let a be a positive integer.
(i) Show that

$$\sum_{\substack{n\geqslant 1\\ (n,a)=1}} \frac{\mu^2(n)}{n\varphi(n)} = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \prod_{p\mid a} \frac{p(p-1)}{p^2-p+1}.$$

(ii) Hence deduce that

$$\frac{\varphi(a)}{a} \sum_{\substack{1\leqslant d\leqslant x\\ (d,a)=1}} \frac{\mu^2(d)}{d\varphi(d)} \log(x/d) = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \left(\prod_{p\mid a} \left(1 - \frac{p}{p^2-p+1} \right) \right) \log x + O_a(1).$$

B3. Let a be a non-zero integer.
(i) Prove that

$$\sum_{\substack{1\leqslant n\leqslant x\\ (a,n)=1}} \frac{1}{\varphi(n)} = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \left(\prod_{p\mid a} \left(1 - \frac{p}{p^2-p+1} \right) \right) \log x + O_a(1).$$

(ii) Apply the Bombieri-Vinogradov theorem to deduce that

$$\sum_{p\leqslant x} \tau(p-a) = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \left(\prod_{p\mid a} \left(1 - \frac{p}{p^2-p+1} \right) \right) x + O_a\left(x\frac{\log\log x}{\log x} \right).$$

B4. In this question, the parameter x is a sufficiently large positive integer.
(i) Let $k \in \mathbb{N}$ satisfy $k \geqslant 2$. Show that for each fixed pair of integers q and m with $qm \leqslant x$, the number of solutions of the equation

$$p_1^k - p_2^k = qm,$$

with p_1 and p_2 prime numbers, is $O_\varepsilon(x^\varepsilon)$.
(ii) Prove that when $k \geqslant 2$, one has

$$\sum_{1\leqslant q\leqslant Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} \Biggl(\sum_{\substack{p^k\leqslant x\\ p^k\equiv a \pmod{q}}} \log p \Biggr)^2 \ll Qx^{1/k}\log x + x^{1+\varepsilon}.$$

(iii) Prove that when $A>0$ is fixed and $x(\log x)^{-A} \leqslant Q \leqslant x$, one has

$$\sum_{1\leqslant q\leqslant Q} \sum_{\substack{a=1\\(a,q)=1}}^{q} \left(\theta(x;q,a) - \frac{x}{\phi(q)} \right)^2 \ll Qx\log x.$$

C5. In this question, you may assume a version of the Elliott-Halberstam Conjecture, namely that for each fixed $A>0$ and $\varepsilon>0$, whenever $Q\leqslant x^{1-\varepsilon}$, one has

$$\sum_{1\leqslant q\leqslant Q} \sup_{y\leqslant x} \max_{\substack{1\leqslant a\leqslant q\\(a,q)=1}} \left| \pi(x;q,a) - \frac{\mathrm{li}(x)}{\phi(q)} \right| \ll x(\log x)^{-A}.$$

(i) Show that

$$\sum_{1\leqslant p\leqslant x} \tau_3(p-1) \ll x\log x.$$

(ii) Obtain a conditional asymptotic formula for

$$\sum_{1\leqslant p\leqslant x} \tau_3(p-1).$$