A1. Let \( a \) be a positive integer.
(i) Show that
\[
\sum_{1 \leq m \leq x} \frac{1}{m} \sum_{d|m \atop d|a} \mu(d) = \frac{\varphi(a)}{a} \log x + O_a(1).
\]
(ii) Hence deduce that
\[
\sum_{1 \leq m \leq x} \frac{1}{m} = \frac{\varphi(a)}{a} \log x + O_a(1).
\]

A2. Let \( a \) be a positive integer.
(i) Show that
\[
\sum_{n \geq 1} \mu^2(n) \frac{1}{n \varphi(n)} = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \prod_{p|a} \left(1 - \frac{p}{p^2 - p + 1}\right) \log x + O_a(1).
\]
(ii) Hence deduce that
\[
\frac{\varphi(a)}{a} \sum_{1 \leq d \leq x} \frac{\mu^2(d)}{d \varphi(d)} \log(x/d) = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \left(\prod_{p|a} \left(1 - \frac{p}{p^2 - p + 1}\right)\right) \log x + O_a(1).
\]

B3. Let \( a \) be a non-zero integer.
(i) Prove that
\[
\sum_{1 \leq n \leq x} \frac{1}{\varphi(n)} = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \left(\prod_{p|a} \left(1 - \frac{p}{p^2 - p + 1}\right)\right) \log x + O_a(1).
\]
(ii) Apply the Bombieri-Vinogradov theorem to deduce that
\[
\sum_{p \leq x} \tau(p-a) = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \left(\prod_{p|a} \left(1 - \frac{p}{p^2 - p + 1}\right)\right) x + O_a \left(\frac{x \log \log x}{\log x}\right).
\]

B4. In this question, the parameter \( x \) is a sufficiently large positive integer.
(i) Let \( k \in \mathbb{N} \) satisfy \( k \geq 2 \). Show that for each fixed pair of integers \( q \) and \( m \) with \( qm \leq x \), the number of solutions of the equation
\[
p_1^k - p_2^k = qm,
\]
with \( p_1 \) and \( p_2 \) prime numbers, is \( O_\varepsilon(x^\varepsilon) \).

(ii) Prove that when \( k \geq 2 \), one has

\[
\sum_{1 \leq q \leq Q} \sum_{a=1}^{q} \left( \sum_{\substack{p^k \leq x \\ p^k \equiv a \pmod{q}}} \log p \right)^2 \ll Qx^{1/k} \log x + x^{1+\varepsilon}.
\]

(iii) Prove that when \( A > 0 \) is fixed and \( x(\log x)^{-A} \leq Q \leq x \), one has

\[
\sum_{1 \leq q \leq Q} \sum_{a=1}^{q} \left( \theta(x; q, a) - \frac{x}{\phi(q)} \right)^2 \ll Qx \log x.
\]

C5. In this question, you may assume a version of the Elliott-Halberstam Conjecture, namely that for each fixed \( A > 0 \) and \( \varepsilon > 0 \), whenever \( Q \leq x^{1-\varepsilon} \), one has

\[
\sum_{1 \leq q \leq Q} \sup_{1 \leq a \leq q} \max_{y \leq x} \left| \frac{\pi(x; q, a) - \text{li}(x)}{\phi(q)} \right| \ll x(\log x)^{-A}.
\]

(i) Show that

\[
\sum_{1 \leq p \leq x} \tau_3(p - 1) \ll x \log x.
\]

(ii) Obtain a conditional asymptotic formula for

\[
\sum_{1 \leq p \leq x} \tau_3(p - 1).
\]