MA598 CNUM Analytic Number Theory, II. Problems 5 – Solutions.

QA1 (i) We have

$$\sum_{\substack{1\le m\le x}} \frac{1}{m}\sum_{\substack{d\mid m\\ d\mid a}}\mu(d) = \sum_{d\mid a}\sum_{1\le k\le x/d}\frac{1}{kd}\mu(d), \quad \text{by putting } m=kd$$

$$= \sum_{d\mid a}\frac{\mu(d)}{d}\left(\log\left(\frac{x}{d}\right)+O(1)\right) = (\log x)\sum_{d\mid a}\frac{\mu(d)}{d} + O\left(\sum_{d\mid a}\left(\frac{\log d}{d}+1\right)\right)$$

$$= \frac{\varphi(a)}{a}\log x + O(\tau(a)). = \frac{\varphi(a)}{a}\log x + O_a(1). \square$$

(ii) One has

$$\sum_{\substack{d\mid m\\ d\mid a}}\mu(d) = \sum_{d\mid (m,a)}\mu(d) = \begin{cases}1, & \text{when } (m,a)=1,\\ 0, & \text{when } (m,a)>1.\end{cases}$$

Thus
$$\sum_{\substack{1\le m\le x\\ (m,a)=1}}\frac{1}{m} = \sum_{1\le m\le x}\frac{1}{m}\sum_{\substack{d\mid m\\ d\mid a}}\mu(d) = \frac{\varphi(a)}{a}\log x + O_a(1), \quad \text{by part (i).} \ \square \ //$$

QA2 (i) Using multiplicativity, one sees that

$$\sum_{\substack{n\ge 1\\ (n,a)=1}}\frac{\mu^2(n)}{n\varphi(n)} = \prod_{p\nmid a}\left(1+\sum_{h=1}^{\infty}\frac{\mu^2(p^h)}{p^h\varphi(p^h)}\right) = \prod_{p\nmid a}\left(1+\frac{1}{p(p-1)}\right)$$

$$= \prod_p\Bigg(\underbrace{\frac{(p^2-p+1)(p+1)(p^3-1)}{p(p^2-1)(p^3-1)}}_{\substack{\parallel\\ \frac{p^6-1}{p(p^2-1)(p^3-1)}}}\Bigg)\cdot\prod_{p\mid a}\frac{p(p-1)}{p^2-p+1}$$

$$= \prod_p\frac{1-p^{-6}}{(1-p^{-2})(1-p^{-3})}\cdot\prod_{p\mid a}\frac{p(p-1)}{p^2-p+1}$$

$$= \frac{\zeta(2)\zeta(3)}{\zeta(6)}\prod_{p\mid a}\frac{p(p-1)}{p^2-p+1}. \quad \square$$

(ii) We have

$$\sum_{\substack{1\le d\le x\\ (d,a)=1}}\frac{\mu^2(d)}{d\varphi(d)}\log\left(\frac{x}{d}\right) = (\log x)\sum_{\substack{1\le n\le x\\ (n,a)=1}}\frac{\mu^2(n)}{n\varphi(n)} + O\left(\sum_{d=1}^{\infty}\frac{\log d}{d^{2-\varepsilon}}\right)$$

$$= (\log x)\sum_{\substack{n\ge 1\\ (n,a)=1}}\frac{\mu^2(n)}{n\varphi(n)} + O\left(1+\sum_{n>x}\frac{1}{n^{2-\varepsilon}}\right)$$

$$\overset{(i)}{\Longrightarrow} \frac{\varphi(a)}{a}\sum_{\substack{1\le d\le x\\ (d,a)=1}}\frac{\mu^2(d)}{d\varphi(d)}\log\left(\frac{x}{d}\right) = \frac{\zeta(2)\zeta(3)}{\zeta(6)}\left(\prod_{p\mid a}\frac{p(p-1)}{p^2-p+1}\cdot\frac{p-1}{p}\right)\log x + O_a(1)$$

$$= \frac{\zeta(2)\zeta(3)}{\zeta(6)}\left(\prod_{p\mid a}\left(1-\frac{p}{p^2-p+1}\right)\right)(\log x) + O_a(1). \ \square \ //$$

QB3 (i) Apply the relation $\frac{n}{\varphi(n)} = \sum_{d|n} \frac{\mu(d)^2}{\varphi(d)}$. Thus

$$\sum_{\substack{1\le n\le x\\ (a,n)=1}} \frac{1}{\varphi(n)} = \sum_{\substack{1\le n\le x\\ (a,n)=1}} \frac{1}{n}\sum_{d|n}\frac{\mu^2(d)}{\varphi(d)} = \sum_{\substack{1\le d\le x\\ (d,a)=1}} \frac{\mu^2(d)}{\varphi(d)} \sum_{\substack{1\le m\le x/d\\ (m,a)=1}} \frac{1}{md}$$

$$= \sum_{\substack{1\le d\le x\\ (d,a)=1}} \frac{\mu^2(d)}{d\varphi(d)}\left(\frac{\varphi(a)}{a}\log\left(\frac{x}{d}\right) + O_a(1)\right) \quad \text{by A1(ii)}$$

$$= \frac{\varphi(a)}{a}\sum_{\substack{1\le d\le x\\ (d,a)=1}} \frac{\mu^2(d)}{d\varphi(d)}\log\left(\frac{x}{d}\right) + O_a\left(\sum_{1\le d\le x}\frac{1}{d^{2-\varepsilon}}\right)$$

$$= \frac{\zeta(2)\zeta(3)}{\zeta(6)}\left(\prod_{p|a}\left(1-\frac{p}{p^2-p+1}\right)\right)(\log x) + O_a(1), \text{ using QA2 (ii)}. \quad \square$$

(ii) One has

$$\sum_{p\le x}\tau(p-a) = 2\sum_{\substack{1\le d\le x^{1/2}(\log x)^{-A}\\ (d,a)=1}} \pi(x;d,a) + \sum_{\substack{x^{1/2}(\log x)^{-A}\le d\le x^{1/2}(\log x)^{A}\\ (d,a)=1}} \pi(x;d,a) - \sum_{\substack{1\le d\le x^{1/2}(\log x)^{A}\\ (d,a)=1}} \pi(dx^{\frac{1}{2}}(\log x)^{-A};d,a) + O_a(1)$$

$$\overset{\text{Lemma 12.3}}{=} 2\sum_{\substack{1\le d\le x^{\frac{1}{2}}(\log x)^{-A}\\ (d,a)=1}} \pi(x;d,a) + O_a\left(\sum_{x^{1/2}(\log x)^{-A}\le d\le x^{\frac{1}{2}}(\log x)^{A}} \frac{x}{\varphi(d)\log(x/d)} + \sum_{1\le d\le x^{\frac{1}{2}}(\log x)^{A}} \frac{dx^{\frac{1}{2}}(\log x)^{-A}}{\varphi(d)\log(x^{1/2}(\log x)^{-A})}\right)$$

$$= 2\sum_{\substack{1\le d\le x^{\frac{1}{2}}(\log x)^{-A}\\ (d,a)=1}} \pi(x;d,a) + O_a\left(\frac{x}{\log x}\sum_{x^{\frac{1}{2}}(\log x)^{-A}\le d\le x^{\frac{1}{2}}(\log x)^{A}} \frac{1}{\varphi(d)} + x^{\frac{1}{2}}(\log x)^{-A-1}\sum_{1\le d\le x^{\frac{1}{2}}(\log x)^{A}} \log\log d\right)$$

$$\overset{\text{Lemma 12.4}}{=} 2\sum_{\substack{1\le d\le x^{\frac{1}{2}}(\log x)^{-A}\\ (d,a)=1}} \pi(x;d,a) + O_a\left(\frac{x}{\log x}\log\left(\frac{x^{\frac{1}{2}}(\log x)^{A}}{x^{\frac{1}{2}}(\log x)^{-A}}\right) + \frac{x}{\log x}\log\log x\right)$$

$$= 2\sum_{\substack{1\le d\le x^{\frac{1}{2}}(\log x)^{-A}\\ (d,a)=1}} \pi(x;d,a) + O_a\left(x\,\frac{\log\log x}{\log x}\right).$$

Hence, by the Bombieri – Vinogradov theorem, one sees that

$$\sum_{p\le x}\tau(p-a) = 2\sum_{\substack{1\le d\le x^{\frac{1}{2}}(\log x)^{-A}\\ (d,a)=1}} \frac{\mathrm{li}(x)}{\varphi(d)} + O_a\left(\sum_{1\le d\le x^{\frac{1}{2}}(\log x)^{-A}} \tilde{E}^*(x;d) + x\,\frac{\log\log x}{\log x}\right)$$

$$= 2\,\mathrm{li}(x)\sum_{\substack{1\le d\le x^{\frac{1}{2}}(\log x)^{-A}\\ (d,a)=1}} \frac{1}{\varphi(d)} + O_a\left(x^{\frac{1}{2}}\left(x^{\frac{1}{2}}(\log x)^{-A}\right)(\log x)^2 + x\,\frac{\log\log x}{\log x}\right)$$

$$\underset{\text{QB3 (i)}}{\overset{\text{Lemma 12.4}}{=}} 2\,\mathrm{li}(x)\left(\frac{\zeta(2)\zeta(3)}{\zeta(6)}\left(\prod_{p|a}\left(1-\frac{p}{p^2-p+1}\right)\right)\left(\log\frac{x^{\frac{1}{2}}}{(\log x)^{A}}\right) + O_a(1)\right) + O_a\left(x\,\frac{\log\log x}{\log x}\right)$$

$$= \frac{\zeta(2)\zeta(3)}{\zeta(6)}\left(\prod_{p\mid a}\left(1-\frac{p}{p^2-p+1}\right)\right)x + O_a\left(x\frac{\log\log x}{\log x}\right). \quad \square$$

Q34 (i) When $k\geq 2$ and $1\leq qm\leq x$, it follows from $p_1^k-p_2^k=qm$ that
$$(p_1-p_2)(p_1^{k-1}+p_1^{k-2}p_2+\cdots+p_2^{k-1})=qm,$$
whence p_1-p_2 and $p_1^{k-1}+\cdots+p_2^{k-1}$ are both divisors of qm, of which there are $\tau(qm)\ll_\varepsilon x^\varepsilon$. Given any fixed such choice, say
$$d_1=p_1-p_2 \quad\text{and}\quad d_2=p_1^{k-1}+\cdots+p_2^{k-1},$$
one has $\underbrace{(p_2+d_1)^{k-1}+\cdots+p_2^{k-1}}_{=\,d_2} = kp_2^{k-1}+\cdots+d_1^{k-1}$, so that p_2 is fixed by solving a polynomial of degree $k-1$ with coefficients fixed by d_1,d_2. There are at most $k-1$ possible choices for p_2, and then p_1 is fixed. So the total number of solutions is $\ll_\varepsilon x^\varepsilon\cdot k = O_\varepsilon(x^\varepsilon)$. $\square$

(ii) When $k\geq 2$, one has
$$\sum_{1\leq q\leq Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left(\sum_{\substack{p^k\leq x\\ p^k\equiv a\,(\mathrm{mod}\,q)}}\log p\right)\leq \sum_{1\leq q\leq Q}\sum_{p_1^k\leq x}\sum_{\substack{p_2^k\leq x\\ p_1^k\equiv p_2^k\,(\mathrm{mod}\,q)}}(\log p_1)(\log p_2)$$
$$\leq \sum_{1\leq q\leq Q}\sum_{\substack{p^k\leq x\\(p_1=p_2)}}(\log p)^2 + 2\sum_{1\leq qm\leq x}\#\{p_1^k-p_2^k=qm : p_i^k\leq x\}$$
$$\ll_\varepsilon Qx^{\frac1k}\log x + x^\varepsilon\sum_{1\leq qm\leq x}1, \text{ using (i)}$$
$$\ll_\varepsilon Qx^{\frac1k}\log x + x^{1+2\varepsilon}. \ \square$$

(iii) From the Barban-Davenport-Halberstam theorem, one has
$$\sum_{1\leq q\leq Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left(\psi(x;q,a)-\frac{x}{\varphi(q)}\right)^2 \ll Qx\log x \quad\text{for } x(\log x)^{-A}\leq Q\leq x.$$

But
$$\psi(x;q,a) = \theta(x;q,a)+\sum_{\substack{p^2\leq x\\ p^2\equiv a\,(\mathrm{mod}\,q)}}\log p+\sum_{\substack{p^3\leq x\\ p^3\equiv a\,(\mathrm{mod}\,q)}}\log p+\cdots+\sum_{\substack{p^k\leq x\\ p^k\equiv a\,(\mathrm{mod}\,q)}}\log p,$$
where $k=\left\lceil\frac{\log x}{\log 2}\right\rceil$. Thus,

$$\left(\theta(x;q,a)-\frac{x}{\varphi(q)}\right)^2 \ll \left(\psi(x;q,a)-\frac{x}{\varphi(q)}\right)^2 + \left(\sum_{l=2}^{k}\sum_{\substack{p^l\le x\\ p^l\equiv a \,(\mathrm{mod}\, q)}}\log p\right)^2$$

$$\ll \left(\psi(x;q,a)-\frac{x}{\varphi(q)}\right)^2 + k\sum_{l=2}^{k}\left(\sum_{\substack{p^l\le x\\ p^l\equiv a\,(\mathrm{mod}\, q)}}\log p\right)^2.$$

We therefore deduce that

$$\sum_{1\le q\le Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left(\theta(x;q,a)-\frac{x}{\varphi(q)}\right)^2 \ll \sum_{1\le q\le Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left(\psi(x;q,a)-\frac{x}{\varphi(q)}\right)^2$$

$$+(\log x)\sum_{l=2}^{k}\left(\sum_{1\le q\le Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left(\sum_{\substack{p^l\le x\\ p^l\equiv a\,(\mathrm{mod}\, q)}}\log p\right)^2\right)$$

$$\ll Qx(\log x) + (\log x)\sum_{l=2}^{k}\left(Qx^{1/l}(\log x)+x^{1+\varepsilon}\right)$$

$$\ll Qx(\log x) + Qx^{1/2}(\log x)^2 + x^{1+\varepsilon}(\log x)^2.$$

But $Q \geq x(\log x)^{-A}$, and thus

$$\sum_{1\le q\le Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left(\theta(x;q,a)-\frac{x}{\varphi(q)}\right)^2 \ll Qx(\log x). \quad \square$$

QC5 (i) In this first part, we can be crude regarding the combinatorics. Observe that if $d_1d_2d_3=p-1$, then one of d_1,d_2,d_3 is at least $(p-1)^{1/3}$, and wlog d_3. Then one of d_1 and d_2 is at least $\left(\frac{p-1}{d_3}\right)^{1/2}$, wlog d_2. Then $d_1\le \frac{p-1}{d_2d_3}\le\left(\frac{p-1}{d_3}\right)^{1/2}\le (p-1)^{1/3}$. In this way, by relabelling divisors as necessary, one sees that

$$\sum_{1\le p\le x}\tau_3(p-1) = \sum_{1\le p\le x}\sum_{d_1d_2d_3=p-1}1 \ll \sum_{1\le d_1\le x^{1/3}}\sum_{1\le d_2\le (x/d_1)^{1/2}}\sum_{\substack{1\le p\le x\\ p\equiv 1\,(\mathrm{mod}\, d_1d_2)}}1$$

$$\ll \sum_{1\le d_1\le x^{1/3}}\sum_{1\le d_2\le (x/d_1)^{1/2}}\pi(x;1,d_1d_2)$$

$$\ll \sum_{1\le d_1\le x^{1/3}}\sum_{1\le d_2\le (x/d_1)^{1/2}}\frac{li(x)}{\varphi(d_1d_2)} + O\left(\sum_{1\le d_1\le x^{1/3}}\sum_{1\le d_2\le (x/d_1)^{1/2}}\tilde{E}^*(x;d_1d_2)\right).$$

Making use of the Elliott – Halberstam theorem, we see that

$$\sum_{1\le d_1\le x^{1/3}} \ \sum_{1\le d_2\le (x/d_1)^{1/2}} \tilde{E}^*(x;d_1 d_2) \ \le \ \sum_{1\le d\le x^{2/3}} \tau(d)\,\tilde{E}^*(x;d)$$

$$\overset{\text{Cauchy-Schwarz}}{\le} \left(\sum_{1\le d\le x^{2/3}} \tau(d)^2\,\tilde{E}^*(x;d)\right)^{\frac{1}{2}}\left(\sum_{1\le d\le x^{2/3}} \tilde{E}^*(x;d)\right)^{\frac{1}{2}}.$$

$$\ll \left(\sum_{1\le d\le x^{2/3}} \tau(d)^2\,\frac{x}{\varphi(d)\log(x/d)}\right)^{\frac{1}{2}} \left(x(\log x)^{-A}\right)^{\frac{1}{2}}, \quad \text{any } A>0.$$

$$\ll x^{\frac{1}{2}}\left(\sum_{1\le d\le x^{2/3}} \frac{\tau(d)^2}{d}\right)^{\frac{1}{2}} \left(x(\log x)^{-A}\right)^{\frac{1}{2}}$$

$$\ll x\left((\log x)^4\right)^{\frac{1}{2}}\left((\log x)^{-A}\right)^{\frac{1}{2}} = o(x).$$

Meanwhile, in similar fashion, the main term gives

$$\sum_{1\le d_1\le x^{1/3}} \ \sum_{1\le d_2\le (x/d_1)^{1/2}} \frac{li(x)}{\varphi(d_1 d_2)} \ll li(x)\sum_{1\le d\le x^{2/3}} \frac{\tau(d)}{\varphi(d)}.$$

The evaluation of the last sum may be tackled in several ways. For example, noting that $n/\varphi(n) = \sum_{d|n} \frac{\mu(d)^2}{\varphi(d)}$, we find that

$$\sum_{1\le n\le x^{2/3}} \frac{\tau(n)}{\varphi(n)} = \sum_{1\le n\le x^{2/3}} \frac{\tau(n)}{n}\sum_{d|n}\frac{\mu(d)^2}{\varphi(d)} = \sum_{1\le d\le x^{2/3}} \frac{\mu(d)^2}{\varphi(d)}\sum_{1\le m\le x/d}\frac{\tau(md)}{md}$$

$$\le \sum_{1\le d\le x^{2/3}} \frac{\tau(d)}{d\,\varphi(d)} \sum_{1\le m\le x/d} \frac{\tau(m)}{m}.$$

Standard techniques (see ANT 1) show that $\sum_{1\le m\le x/d} \frac{\tau(m)}{m} \ll \log\left(\frac{x}{d}\right)^2$, and thus

$$\sum_{1\le n\le x^{2/3}} \frac{\tau(n)}{\varphi(n)} \ll (\log x)^2 \sum_{1\le d\le x^{2/3}} \frac{\tau(d)}{d\varphi(d)} + O\left(\log x \sum_{1\le d\le x^{2/3}} \frac{\tau(d)\log d}{d\,\varphi(d)}\right)$$

$$\ll (\log x)^2 \sum_{1\le d\le x^{2/3}} d^{\varepsilon-2} + O\left((\log x)\sum_{1\le d\le x^{2/3}} d^{\varepsilon-2}\right)$$

$$\ll (\log x)^2.$$

Hence the main term is $\ll li(x)\cdot(\log x)^2 \ll x\log x$, and we conclude that $\sum_{1\le p\le x} \tau_3(p-1) \ll x\log x$. □

(ii) To obtain an asymptotic formula, make all these terms explicit. Fun, Fun, Fun!